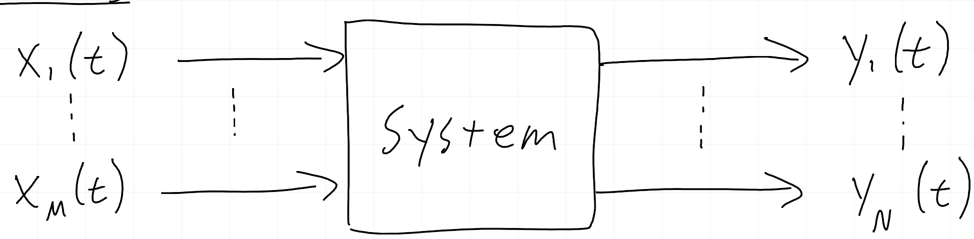
Previously: Signals map a domain to a range

Continuous signal: $\mathbb{R} \rightarrow \mathbb{R}$ (t, $x(t)$) $\mathbb{R}^M \rightarrow \mathbb{R}^N$ multi-dimensional

<u>Systems</u>:



"MIMO" – Multiple Input / Multiple Output

- Systems map one set of signals (or functions) to another set of signals (or functions):

$$(\mathbb{R} \rightarrow \mathbb{R}) \longrightarrow (\mathbb{R} \rightarrow \mathbb{R})$$

(can be multi-dimensional)

- Domains & Ranges can be the same or different

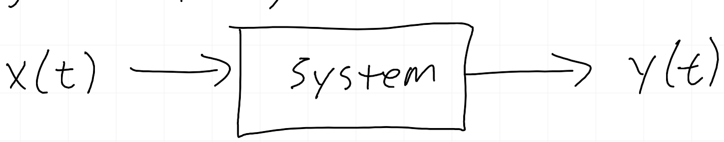
Sampler: $\underset{t \quad\; x(t)}{(\mathbb{R} \rightarrow \mathbb{R})} \longrightarrow \underset{n \quad\; x[n]}{(\mathbb{Z} \rightarrow \mathbb{R})}$

DSP: $(\mathbb{Z} \rightarrow \mathbb{R}) \longrightarrow (\mathbb{Z} \rightarrow \mathbb{R})$

domain / range can be complex too
- communications
- radar

- Sometimes we have to deal with initial conditions
  Example: Resuming cruise control from different speeds

- In this course, we will focus on SISO (Single Input / Single Output)



System Properties

- Linearity: Linear combinations of inputs produce the same linear combinations of outputs.

  If:
  $$x_1(t) \longrightarrow y_1(t)$$
  $$x_2(t) \longrightarrow y_2(t)$$
  Then:
  $$A_1 x_1(t) + A_2 x_2(t) \longrightarrow A_1 y_1(t) + A_2 y_2(t)$$
  For all scalars A_1 & A_2, and for all signals $x_1(t)$ & $x_2(t)$.

- Time invariance (more generally, "shift invariance"): Shifting the input produces the same shift of the output (without causing any other changes).

  If:
  $$x(t) \longrightarrow y(t)$$
  Then:
  $$x(t-t_0) \longrightarrow y(t-t_0)$$
  For all possible scalars t_0 and all possible signals $x(t)$.

- Causality: "Cause and effect." An output cannot be dependent on a future input.

  $\forall t$, the output $y(t)$ is not a function of $x(\tau)$ for any $\tau > t$

Examples:

$y(t) = 3x(t)$ LTI, Causal

$y(t) = 3x(t-3)$ LTI, Causal

$y(t) = 3x(t+3)$ LTI, non-causal

$y(t) = \dfrac{dx(t)}{dt} = \lim_{\epsilon \to 0} \dfrac{x(t) - x(t-\epsilon)}{\epsilon}$ LTI, Causal if left derivative

$y(t) = x\left(\frac{1}{3}t\right)$ Linear, non-TI, non-causal

Can you show/prove these properties?

Let $x_1(t)$ and $x_2(t)$ be two input signals such that

$x_1(t) \longrightarrow y_1(t) = x_1\left(\frac{1}{3}t\right)$

$x_2(t) \longrightarrow y_2(t) = x_2\left(\frac{1}{3}t\right)$

Let $x_3(t) = A_1 x_1(t) + A_2 x_2(t)$ be a new input. The output is then:

$$y_3(t) = x_3\left(\tfrac{1}{3}t\right) = A_1 x_1\left(\tfrac{1}{3}t\right) + A_2 x_2\left(\tfrac{1}{3}t\right)$$
$$= A_1 y_1(t) + A_2 y_2(t) \quad \therefore \text{ it's linear}$$

Let $x_4(t) = x_1(t - t_0)$ be the input. The output is then:

$$y_4(t) = x_4\left(\tfrac{1}{3}t\right) = x_1\left(\tfrac{1}{3}t - t_0\right)$$
$$= x_1\left(\tfrac{1}{3}(t - 3t_0)\right)$$
$$= y_1(t - 3t_0)$$

The output got shifted 3 times as much as the input.

$\therefore$ non-TI

When $t < 0$, then $\frac{1}{3}t > t$. Therefore $y(t) = x\left(\frac{1}{3}t\right)$ depends on future values of the input in this region.

$\therefore$ non-causal

$y(t) = 3x(t) + 5$ Nonlinear, TI, causal "Affine"

$$y_3(t) = 3\left(A_1 x_1(t) + A_2 x_2(t)\right) + 5 = 3A_1 x_1(t) + 3A_2 x_2(t) + 5$$

$\neq$

$$A_1 y_1(t) + A_2 y_2(t) = 3A_1 x_1(t) + 5 + 3A_2 x_2(t) + 5 = 3A_1 x_1(t) + 3A_2 x_2(t) + 10$$

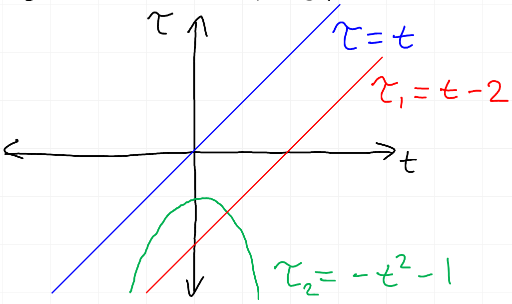
How can you show causality for something that <u>is</u> causal?

$$y(t) = 3x^3\underbrace{(t-2)}_{\tau_1} + x\underbrace{(-t^2-1)}_{\tau_2} + 10$$

The arguments passed into $x(\cdot)$ are what matter. You need to show that they're $\leq t$ for all values of t.

So, $t-2 \leq t$ is pretty obvious

What about $-t^2-1 \leq t$?
Could show by plotting
Everything needs to be below the $\tau = t$ line for it to be causal.



$\therefore$ nonlinear, non-TI, causal

$y(t) = x(-t^2)$	Linear, non-TI, non-causal ↳ when $t \in (-1, 0)$
$y(t) = 3x(t) + t$	Nonlinear, non-TI, causal
$y(t) = x^2(t)$	Nonlinear, TI, causal
$y(t) = t^2 x(t)$	Linear, non-TI, causal
$y(t) = (t+3)^2 x(t)$	Linear, non-TI, causal
$y(t) = x(-t)$	Linear, non-TI, non-causal
$y(t) = x(12)$	Linear, non-TI, non-causal
$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$	LTI, causal

Some rules of thumb: (Try to prove it using the definitions to be sure)

Linearity breaks if: "mess" with the x's. $y(t) = x^2(t)$

TI breaks if: have t's outside of $x(t)$
mess with t's inside $x(t)$ $y(t) = x(\frac{1}{3}t)$
- Adding / subtracting constants is OK though

Causality breaks if: add positive constant to t
otherwise "look into the future"

The rest of this course deals with LTI systems

Linearity & TI are approximations.

Few physical systems are truly linear
- Transistors often operate in a range in which they can be approximated as linear.
- Components fry / explode if you pump too much electricity through them - which is highly non-linear behavior.

Few physical systems are time-invariant
- Things heat up and their behavior changes. Sometimes we have to compensate for that.

But, many important systems are well-approximated by the LTI assumptions!