# ECE3084 L03 - Impulse and frequency response

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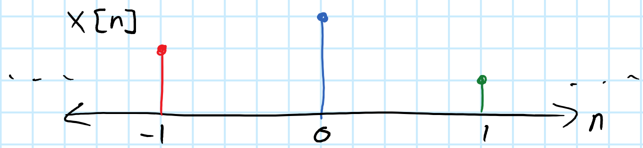
We talked a lot about LTI. But why are these properties important?

## Discrete (ECE 2026) Review:

Can view any signal $x[n]$ as a sum of scaled, shifted delta functions.

It even matches how we plot them (stem plot)




$$\cdots\ x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]\ \cdots$$

$$x[n] = \sum_k x[k]\,\delta[n-k]$$

Define the impulse response $h[n]$ as the output when $x[n] = \delta[n]$ is placed on the input

$$x[n] = \delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n] = h[n]$$

Then the output for any signal $x[n]$ can be written as:

$$y[n] = \underbrace{\sum_k}_{\text{Superposition}} \underbrace{x[k]}_{\text{Scaling}}\, h[\underbrace{n-k}_{\text{Time invariance}}]$$

Scaling, Superposition — Linearity

All we did is replace the $\delta[n-k]$ with $h[n-k]$.

We call this operation "convolution" and use the $*$ symbol to denote it:

$$x[n] * h[n] = \sum_k x[n]\,h[n-k] \qquad = (x * h)[n]$$

Another way it can be written.

The system is characterized by $h[n]$ — It contains <u>all</u> the information about what the system does

(Reminder: this is <u>only</u> for LTI systems)

But, can we do the same thing in continuous time?

Continuous Time:

Can we construct any $x(t)$ in the same way using $\delta(t)$?
Just replace the sum with an integral (the continuous version of summation).

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Does this work?
Not as obvious as in discrete time

$$= \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau \qquad \text{Sampling property}$$

$$= x(t)\underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau}_{1} = x(t)$$

Define continuous time impulse response, $h(t)$, as output when $\delta(t)$ is placed on input.

$$x(t) = \delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = h(t)$$

Like with discrete time, we can now construct the output for any input using $h(t)$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Once again, $h(t)$ characterizes the system
This is continuous time convolution: (Uses the same notation)

$$\boxed{x(t) * h(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

We'll revisit convolution & its properties soon.

# Frequency Response

It's often useful to know how a system responds to different frequencies.
Complex exponentials are elegant/convenient for representing sinusoids.
Euler's formula: $$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Since we care about time and frequency, we split $\theta$ up into $\omega t$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Note: $\underset{\uparrow}{\omega} = 2\pi \underset{\uparrow}{f} = \frac{2\pi}{T}$

Angular freq (rad/time unit) — Freq (cycles/time unit or Hz) — $T$: Period (time units/cycle)

What happens when we feed a sinusoid $e^{j\omega t}$ into an LTI system?
Discrete time review: $x[n] = e^{j\hat{\omega} n}$

$$y[n] = x[n] * h[n] = h[n] * x[n] = \sum_k h[k]\, x[n-k]$$

↳ commutative property – coming soon

$$= \sum_k h[k]\, e^{j\hat{\omega}(n-k)} = \sum_k h[k]\, e^{j\hat{\omega} n} e^{-j\hat{\omega} k}$$

$$= e^{j\hat{\omega} n} \underbrace{\sum_k h[k]\, e^{-j\hat{\omega} k}}$$

we're defining this

$$= \underbrace{e^{j\hat{\omega} n}}_{x[n]} \underbrace{H(e^{j\hat{\omega}})}_{\text{Frequency Response}}$$

Note that there is no "n" in $H(e^{j\hat{\omega}})$. This means $H(e^{j\hat{\omega}})$ is just a complex-valued constant with respect to $n$.

Easiest to interpret $H(e^{j\hat{\omega}})$ in polar form — with a magnitude & a phase.
It can scale and shift the phase of $x[n]$, but it can't change the frequency! Sinusoid in → sinusoid out

A more general sinusoid: $x[n] = \underset{\text{magnitude}}{A}\, \underset{\text{phase shift}}{e^{j\phi}}\, e^{j\hat{\omega} n}$

$$y[n] = A e^{j\phi} e^{j\hat{\omega} n} H(e^{j\hat{\omega}}) = A\,|H(e^{j\hat{\omega}})|\, e^{j(\phi + \angle\{H(e^{j\hat{\omega}})\})} e^{j\hat{\omega} n}$$

$x[n]$ real: $x[n] = A\cos(\hat{\omega} n + \phi)$ $\qquad y[n] = A\,|H(e^{j\hat{\omega}})| \cos(\hat{\omega} n + \phi + \angle\{H(e^{j\hat{\omega}})\})$

# Continuous Time:

Feed in a sinusoid: $x(t) = e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow y(t) = ?$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega t} e^{-j\omega\tau}\, d\tau = e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau}_{\text{define as } H(j\omega)}$$

$$= \underbrace{e^{j\omega t}}_{x(t)} \underbrace{H(j\omega)}_{\text{Freq Response}}$$

No "t"s in definition of $H(j\omega)$ — it's just a complex constant with respect to t. Sinusoid in $\rightarrow$ Sinusoid out (at same freq)

Note that $\omega$ is in $H(j\omega)$ — so it can scale & shift the phases of different frequencies by different amounts.

If you take the real part of the input & output:

$$x(t) = A\cos(\omega t + \phi) \implies y(t) = A|H(j\omega)|\cos(\omega t + \phi + \angle\{H(j\omega)\})$$

# Fourier Transforms:

The steps we took to transform an impulse response into a frequency response can be applied to any signal — not just to impulse responses.

$$\boxed{\begin{aligned} \text{DTFT:}\quad & X(e^{j\omega}) = \sum_n x[n]\, e^{-j\hat{\omega} n} \\ \text{CTFT:}\quad & X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

Note: Can use n and t instead of k and $\tau$ because we no longer have both $x(\cdot)$ and $h(\cdot)$ to keep track of indep vars for.

And their inverses (for completeness):

$$\boxed{\begin{aligned} \text{Inv DTFT:}\quad & x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}\, d\omega \\ \text{Inv CTFT:}\quad & x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \end{aligned}}$$

Similar to the forward transforms. Main difference is the sign in the exponent

The DTFT pair is here for context. We'll be looking at the CTFT pair.

We've assumed the sums & integrals defining freq responses and Fourier transforms exist (i.e. don't blow up or be undefined). The Laplace transform will allow us to analyze signals for which the Fourier transform is undefined. (Z transform does the same for discrete time.)

There are other transforms — you can represent signals as sums of other things as well. Sinusoids are especially important because they are "eigenfunctions" of LTI systems, with the frequency response being analogous to the eigenvalues. Sinusoid in, sinusoid out.

$$A\vec{x} = \lambda \vec{x}$$