ECE3084 L04 - Continuous Time Convolution

Wednesday, January 18, 2017 11:27 PM

Recall: We can construct an arbitrary input signal $x(t)$ as the "sum" (integral) of an infinite number of scaled and shifted delta functions:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

For LTI systems, we can then construct the output signal as the same sum of scaled and shifted impulse responses:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

Convolution Integral

* Can be any two functions, not just x and h!

Properties of Convolution

- Commutativity: $x(t) * h(t) = h(t) * x(t)$

Proof:

$$x(t) * h(t) = \int_{\tau=-\infty}^{\tau=\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Change of Variables: Let $\lambda = t-\tau$, $\tau = t-\lambda$, $d\tau = -d\lambda$

$$= \int_{t-\lambda=-\infty}^{t-\lambda=\infty} x(t-\lambda)\,h(\lambda)\,(-d\lambda)$$

$$= -\int_{\lambda=\cancel{t}+\infty}^{\lambda=\cancel{t}-\infty} x(t-\lambda)\,h(\lambda)\,d\lambda$$

$$= \int_{\lambda=-\infty}^{\lambda=\infty} h(\lambda)\,x(t-\lambda)\,d\lambda$$

$$= h(t) * x(t)$$

Flip the bounds of integration to cancel out the negative

- Associativity: $(x(t) * y(t)) * z(t) = x(t) * (y(t) * z(t))$
- Distributivity: $x(t) * (y(t) + z(t)) = x(t) * y(t) + x(t) * z(t)$
- Time Shift: If: $x(t) * h(t) = y(t)$

  Then: $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$

  Note: Can plug in 0 for either t_0/t_1
- Differentiation: $\dfrac{dx(t)}{dt} * h(t) = x(t) * \dfrac{dh(t)}{dt} = \dfrac{dy(t)}{dt}$
- Convolution with impulse: $x(t) * \delta(t) = x(t)$

  $x(t) * \delta(t-t_0) = x(t-t_0)$ Delay operator

Example: $x(t) = u(t)$ $h(t) = \delta(t+1) - \delta(t-1)$ Find $x(t) * h(t)$.

$$x(t) * h(t) = u(t) * (\delta(t+1) - \delta(t-1))$$
$$= u(t) * \delta(t+1) - u(t) * \delta(t-1)$$
$$= u(t+1) - u(t-1)$$

Distributive property

Convolution w/ Impulse

Example: $x(t) = u(t-1)$ $h(t) = e^{-t}u(t)$ Find $y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} \underbrace{u(\tau-1)}_{\substack{\text{Nonzero when} \\ \tau-1>0 \\ \Rightarrow \tau>1 \text{ lower bound}}} e^{-(t-\tau)} \underbrace{u(t-\tau)}_{\substack{\text{Nonzero when} \\ t-\tau>0 \\ \Rightarrow \tau<t \text{ upper bound}}} d\tau$$

Both bounds must be satisfied for $y(t)$ to be nonzero. $1 < \tau < t$ also implies that we must have $1 < t$ for $y(t)$ to be nonzero.

$$= u(t-1)\int_{1}^{t} e^{-(t-\tau)}\,d\tau$$

Just changing the bounds alone is not enough in this case because one of them is not a constant.

$$= u(t-1)\,e^{-t}\int_{1}^{t} e^{\tau}\,d\tau$$
$$= u(t-1)\,e^{-t}\left(e^{\tau}\right]_{1}^{t} = u(t-1)\,e^{-t}(e^{t}-e)$$
$$= (1-e^{-t+1})\,u(t-1) = (1-e^{-(t-1)})\,u(t-1)$$

The bounds can be confusing & hard to get right when just slogging through the math. It's often easier to get things right if you can visualize what is going on.

Convolution Integral: $\boxed{x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}$

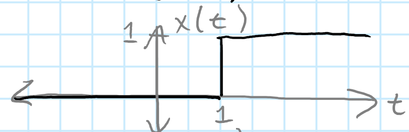
Intuition / Graphical convolution: Let $y(t) = x(t) * h(t)$

To calculate $y(t)$ for a given (fixed) value of t, we have to integrate $x(\tau)h(t-\tau)$ over all values of τ.

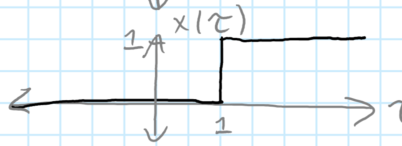
What does that look like?

$x(\tau)$ — looks the same as $x(t)$, but with τ instead.

From the example: $x(t) = u(t-1)$

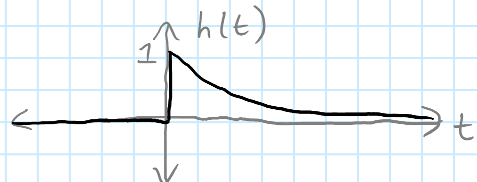


$x(\tau) = u(\tau - 1)$

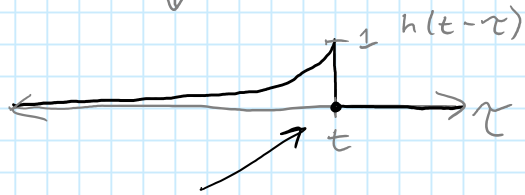


$h(t-\tau) = h(-(\tau - t))$ — It's a version of $h(t)$ that has been flipped and shifted by t.

From the example: $h(t) = e^{-t}u(t)$



$$h(t-\tau) = h(-(\tau - t))$$
$$= e^{\tau - t}u(t-\tau)$$

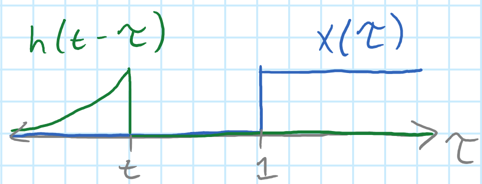


View the spot t on $h(t-\tau)$ as the "handle" you use to slide $h(t-\tau)$ across $x(\tau)$.

Two regions as you slide:

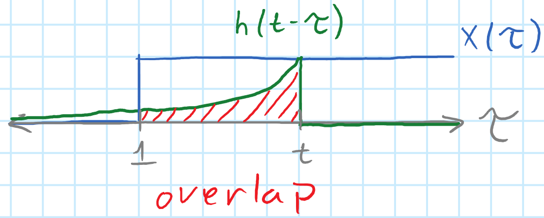
Region 1: $t < 1$

$\Rightarrow$ No overlap, $\therefore$ $y(t) = 0$



Region 2: $t > 1$

$\Rightarrow$ Overlap from $\tau = 1$ to $\tau = t$

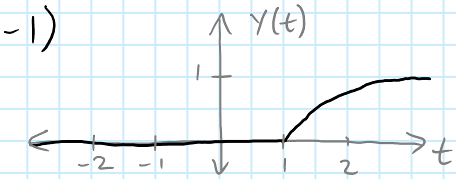


$$y(t) = \int_1^t e^{-(t-\tau)}\,d\tau$$
$$= -e^{-t}\int_1^t e^{\tau}\,d\tau = e^{-t}\left(e^{\tau}\right]_1^t$$
$$= e^{-t}(e^t - e^1) = 1 - e^{-(t-1)}$$

Put the two regions together using appropriate step functions:

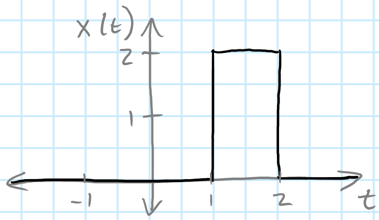
$$y(t) = (0)u(1-t) + \left(1-e^{-(t-1)}\right)u(t-1)$$

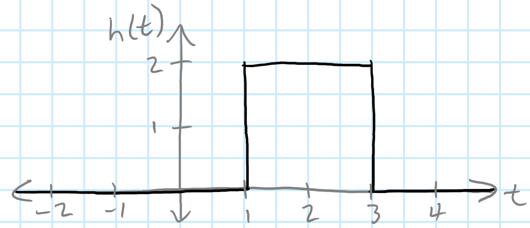
$$y(t) = \left(1-e^{-(t-1)}\right)u(t-1)$$



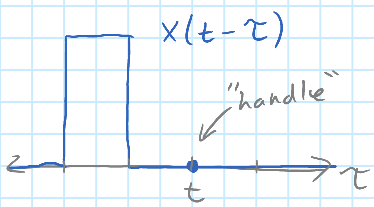
Example: $x(t) = 2\big(u(t-1) - u(t-2)\big)$ $\quad h(t) = 2\big(u(t-1) - u(t-3)\big)$

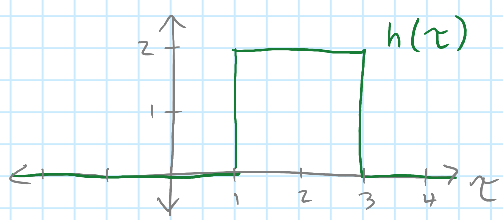
Compute $y(t) = x(t) * h(t)$





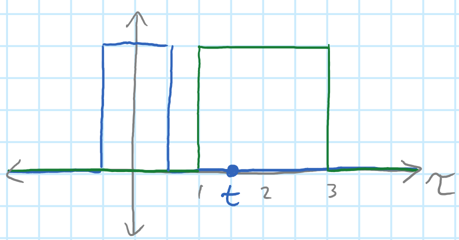
Note: $x(t) * h(t) = h(t) * x(t)$ It's usually easier to let the "simpler" one be "$t-\tau$" (the one you flip & slide around).





5 Regions:

Region 1:



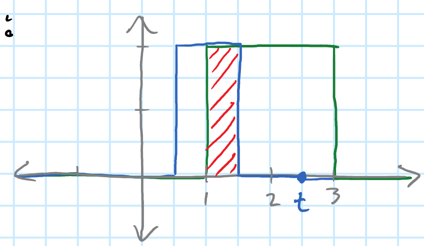
$t-1 < 1$

$\Rightarrow \quad t < 2$

No overlap

$y(t) = 0$

Region 2:



$t-1 > 1$ and $t-2 < 1$

$\Rightarrow \quad 2 < t < 3$

Overlap from $\tau = 1$ to $\tau = t-1$

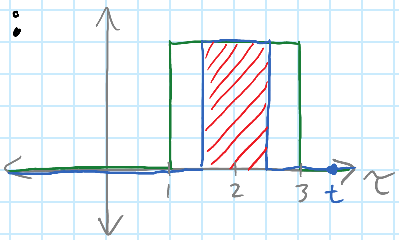
$$y(t) = \int_1^{t-1} (2)(2)\, d\tau$$

$$= 4\left(t\right]_1^{t-1}$$

$$= 4(t-1-1)$$

$$= 4t - 8$$

Region 3:



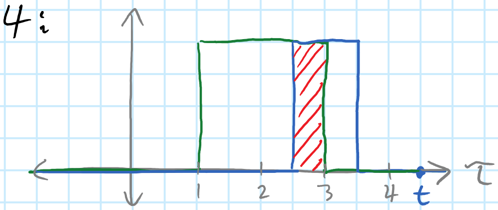
$t-2>1$ and $t-1<3$

$\Rightarrow \quad 3<t<4$

Overlap from $t-2$ to $t-1$

$$y(t)=\int_{t-2}^{t-1}(2)(2)\,d\tau$$
$$=4\left(t\right]_{t-2}^{t-1}$$
$$=4(t-1-t+2)$$
$$=4$$

Region 4:



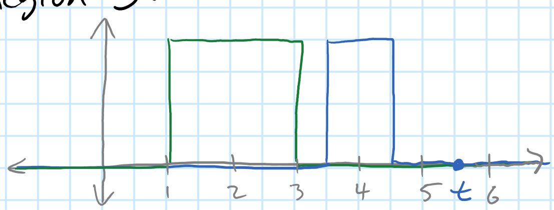
$t-1>3$ and $t-2<3$

$\Rightarrow \quad 4<t<5$

Overlap from $t-2$ to 3

$$y(t)=\int_{t-2}^{3}(2)(2)\,d\tau$$
$$=4\left(t\right]_{t-2}^{3}$$
$$=4(3-t+2)$$
$$=20-4t$$

Region 5:



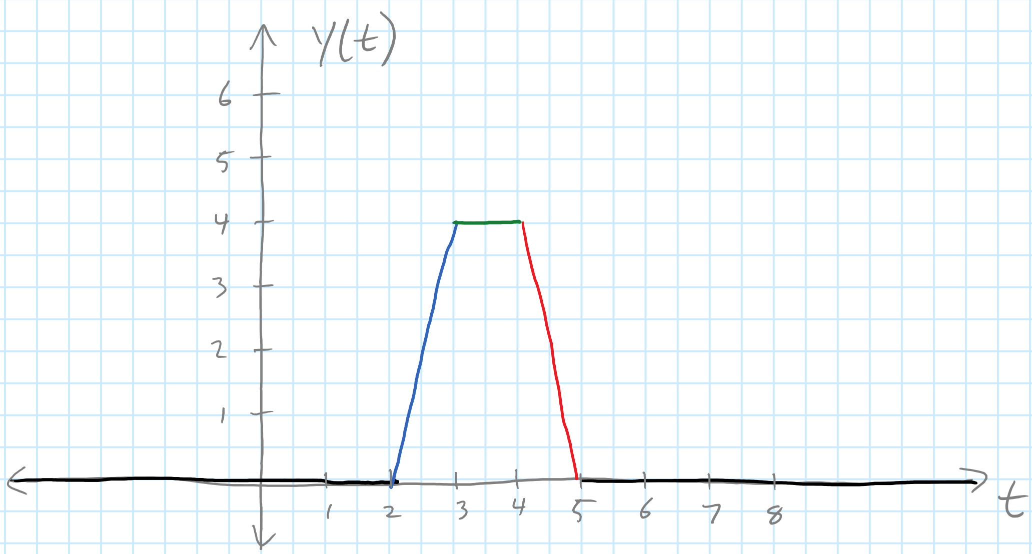
$t-2>3 \quad \Rightarrow \quad t>5$

No overlap

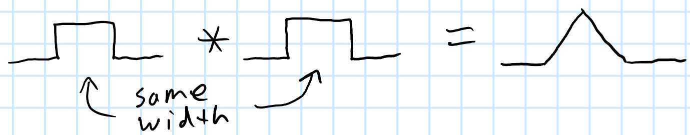
$y(t)=0$

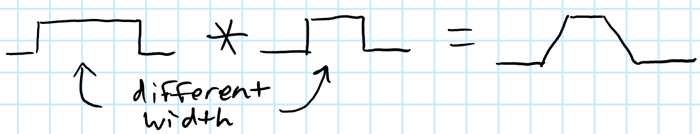
Put them all together:

$$y(t)=\underbrace{(4t-8)\big(u(t-2)-u(t-3)\big)}_{\text{Region 2}}+\underbrace{4\big(u(t-3)-u(t-4)\big)}_{\text{Region 3}}+\underbrace{(20-4t)\big(u(t-4)-u(t-5)\big)}_{\text{Region 4}}$$



Tips:

- The output starts at the sum of the start times of the two inputs, and ends at the sum of the two end times.
- Flip & shift the simpler signal (not shifted to start with, simpler mathematical representation, etc).
- 

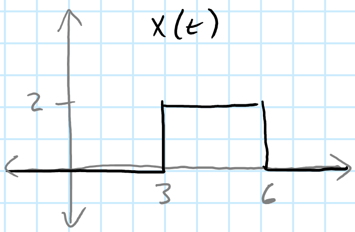
- 

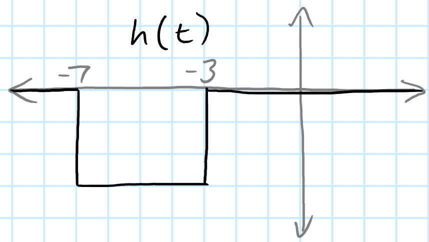
Bonus: Fast convolution of rect pulses

Let $x(t) = 2(u(t-3) - u(t-6))$

$h(t) = -3(u(t+7) - u(t+3))$

Find $y(t) = x(t) * h(t)$





$y(t)$ will be a trapezoid shape:

starts @ sum of start times: $3 + -7 = -4$

ends @ sum of end times: $6 + -3 = 3$

Width of slopes equal to smaller width of the two rects: 3

height equal to product of heights of rects and the smaller width: $(2)(-3)(3) = -18$

height of x

height of h

smaller width (of x in this case)

This is the area of the product of the two rects when they're fully overlapping.