# ECE3084-L06 Fourier Series

Thursday, January 26, 2017 3:38 AM

We will consider two classes of signals:

1. Periodic $\longrightarrow$ Fourier Series
   - signals are infinite in length
2. Not periodic $\longrightarrow$ Fourier Transform
   - can deal with finite-length signals

## Fourier Series (for periodic signals)

Period: $T_0$

Fundamental frequency: $f_0 = \frac{1}{T_0}$ (in Hz)

$\omega_0 = \frac{2\pi}{T_0}$ (in radians)

Fundamental Freq = lowest freq in the sum. All others are harmonics (multiples) of it.

A signal $x(t)$ with fundamental period $T_0$ can be expressed as a sum of complex sinusoids:

Fourier Synthesis Sum:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}}$$

How do you find the $a_k$ coefficients?

1. Can often directly expand terms using Euler's formulas.

   e.g. $x(t) = \sin(\omega_0 t)\cos^2(\omega_0 t)$

   $\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\frac{1}{4}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)^2$

   $\frac{1}{8j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\left(e^{j2\omega_0 t} + 2 + e^{-j2\omega_0 t}\right)$

   $\frac{1}{8j}\left(e^{j3\omega_0 t} + e^{j\omega_0 t} - e^{-j\omega_0 t} - e^{-j3\omega_0 t}\right)$

   $a_3 = a_1 = \frac{1}{8j} \qquad a_{-1} = a_{-3} = -\frac{1}{8j}$

2. Otherwise, via integration:

Fourier Analysis Integral:

$$\boxed{a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\frac{2\pi}{T_0}kt}\,dt = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 kt}\,dt}$$

For $k=0$, this simplifies to: $a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$

If $x(t)$ is real, then the FS coefficients satisfy <u>conjugate symmetry</u>:

$$a_{-k} = a_k^*$$

and it can be written in terms of cosines:

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2a_k\cos\left(\frac{2\pi}{T_0}kt + \angle a_k\right)$$

## System response to periodic signals

If the input to an LTI system can be written as the sum of sinusoids, then the output can be written as the same sum, but with each sinusoid scaled & phase-shifted by the system's frequency response $H(j\omega)$ at that frequency:

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 k t}$$

If $x(t)$ is real: (split the sum into 3 parts: $k=0$, $k>0$, $k<0$)

Apply conjugate symmetry

Apply $H(-j\omega) = H^*(j\omega)$

$$y(t) = a_0 + \sum_{k=1}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 k} + \sum_{k=1}^{\infty} a_{-k} H(-j\omega_0 k) e^{-j\omega_0 k}$$

$$= a_0 + \sum_{k=1}^{\infty} \Big( a_k H(j\omega_0 k) e^{j\omega_0 k} + a_k^* H^*(j\omega_0 k) e^{-j\omega_0 k}$$

$$= a_0 + \sum_{k=1}^{\infty} \Big( |a_k H(j\omega_0 k)| \, e^{j(\omega_0 k + \angle a_k + \angle H(j\omega_0 k))} + |a_k H(j\omega_0 k)| \, e^{-j(\omega_0 k + \angle a_k + \angle H(j\omega_0 k))} \Big)$$

Inverse Euler's

$$= a_0 + \sum_{k=1}^{\infty} 2 |a_k H(j\omega_0 k)| \cos(\omega_0 k + \angle a_k + \angle H(j\omega_0 k))$$

## Properties of Fourier Series

Assume $x(t)$ has FS coefficients $a_k$, $w(t)$ has FS coefficients $b_k$, and that both have the same fundamental period $T_0$

- **Linearity:** If $y(t) = Ax(t) + Bw(t)$, then $y(t)$ also has a period of $T_0$ and its FS coefficients are $c_k = Aa_k + Bb_k$
- **Scaling and Offset:** If $y(t) = Ax(t) + C$ then:

$$c_k = Aa_k \quad \text{for } k \neq 0$$
$$c_0 = Aa_0 + C$$

This is a special case of linearity where one signal is a DC offset.

- **Time Shift:** If $y(t) = x(t - t_d)$, then it also has a period of $T_0$ and its FS coefficients are:

$$c_k = a_k \underbrace{e^{-jk\omega_0 t_d}}$$

Note that $c_0 = a_0$

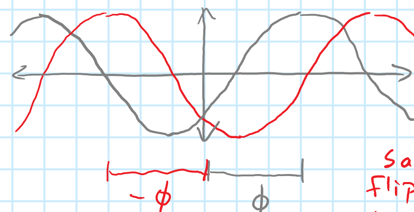
This is just a phase shift by the appropriate amount for each freq. The angle of the shift scales linearly w/ freq.

This can be proven with a change of variables: $\tau = t - t_d$

- **Time Reversal (Flip):** If $x(t)$ is real and $y(t) = x(-t)$, then $y(t)$'s FS coefficients are $c_k = a_k^*$

  This can be proven with the change of variables $\tau = -t$

  Can also think of it as shifting the phase of all the cosines that make up the real signal in the opposite direction — makes sense that this flips it in the time domain:

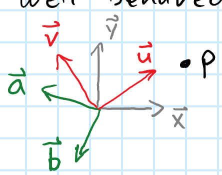



  Same as flipping in time domain

  Also makes sense with conjugate symmetry: $a_k^* = a_{-k}$

## Why do FS work?

All sinusoids that are harmonics of the same fundamental frequency $\omega_0$ are orthogonal $\longrightarrow$ They form an orthogonal basis set for ("well-behaved") signals that are periodic with period $T_0 = \frac{2\pi}{\omega_0}$




Can write the same point x in coodinates of other orthonormal basis sets.

Orthogonality of signals: similar to vectors $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = 0$ (For column vectors)

For complex vectors: $\vec{x} \cdot \vec{y} = \vec{x}^H \vec{y}$ Hermetian: conjugate transpose

Why is the conjugate necessary? Often want $\vec{x} \cdot \vec{x} = |x|^2$

If $\vec{x} = a + bi$ (a 1D complex vector): $|\vec{x}| = a^2 + b^2$

$$(a+bi)(a+bi) = a^2 + 2abi - b^2 \neq a^2 + b^2 \quad \times$$

$$(a+bi)^*(a+bi) = (a-bi)(a+bi) = a^2 + abi - abi + b^2 = a^2 + b^2 \quad \checkmark$$

Inner products are a more general form of dot products.

Typically for periodic signals: (Can be defined in different ways)

$$\langle x(\cdot), y(\cdot) \rangle = \frac{1}{T_0} \int_{T_0} x(t)\, y^*(t)\, dt$$

$\cdot$ means variable name doesn't matter

We're still multiplying corresponding values of x and y together and "adding" (integrating) them all up.

So, $x(t) \perp y(t)$ if $\langle x(t), y(t) \rangle = 0$

To show orthogonality: Start with this integral, then we'll use the result:

$$\frac{1}{T_0}\int_{T_0} e^{\left(j\frac{2\pi}{T_0}kt\right)}dt = \frac{1}{T_0}\left(\frac{1}{j\frac{2\pi}{T_0}k}\right)e^{\left(j\frac{2\pi}{T_0}kt\right)}\Bigg|_{t_0}^{t_0+T_0}$$

$t_0$ is an arbitrary start time

$$= \frac{1}{j2\pi k}\left(e^{j\frac{2\pi}{T_0}k(t_0+T_0)} - e^{j\frac{2\pi}{T_0}kt_0}\right)$$

$$= \frac{1}{j2\pi k}e^{j\frac{2\pi}{T_0}kt_0}\left(\underbrace{e^{j2\pi k}}_{=1 \text{ for integer } k} - 1\right)$$

= 1 for integer $k$, which zeroes out the whole expression*

* EXCEPT for $k=0$ because then we also have division by zero. (Can use L'Hopital's, or just plug in $k=0$ earlier (before evaluating the integral)

For $k=0$: $\displaystyle \frac{1}{T_0}\int_{T_0} e^{\left(j\frac{2\pi}{T_0}0t\right)}dt = \frac{1}{T_0}\int_{T_0} 1\, dt = \frac{T_0}{T_0} = 1$

So, we get 1 when $k=0$, and 0 for $k \neq 0$. That's just a Kronecker delta function! Final result:

$$\frac{1}{T_0}\int_{T_0} e^{\left(j\frac{2\pi}{T_0}kt\right)}dt = \delta[k]$$

Orthogonality of harmonics: given integers $k$ & $\ell$

$$\left\langle e^{j\frac{2\pi}{T_0}kt}, e^{j\frac{2\pi}{T_0}\ell t}\right\rangle = \frac{1}{T_0}\int_{T_0} e^{j\frac{2\pi}{T_0}kt}e^{-j\frac{2\pi}{T_0}\ell t}dt$$

$$= \frac{1}{T_0}\int_{T_0} e^{j\frac{2\pi}{T_0}(k-\ell)t}dt$$

Apply result from above

$$= \delta[k-\ell]$$

Which is zero for all $k \neq \ell$. Thus, all harmonics are orthogonal to eachother (but not to themselves, i.e. when $k=\ell$).

We can also use the above integral result to check the FS analysis & synthesis equations:

$$a_\ell = \underbrace{\frac{1}{T_0}\int_{T_0}}_{} \Bigg( \underbrace{\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt}}_{\text{synthesis}} \Bigg) \underbrace{e^{-j\frac{2\pi}{T_0}\ell t}\, dt}_{}$$

Analysis

$$= \sum_{k=-\infty}^{\infty} a_k \left( \frac{1}{T_0}\int_{T_0} e^{j\frac{2\pi}{T_0}kt} e^{-j\frac{2\pi}{T_0}\ell t}\, dt \right)$$

Use integral result from before.

$$= \sum_{k=-\infty}^{\infty} a_k\, \delta[k-\ell] = a_\ell \; \checkmark$$

All terms are zero except when $k=\ell$

## Symmetry

Even functions: $f(-x) = f(x)$ e.g. $\cos(-x) = \cos(x)$

Odd Functions: $f(-x) = -f(x)$ e.g. $\sin(-x) = -\sin(x)$

If $f(x)$ is odd, then $\int_{-\infty}^{\infty} f(x)\,dx = 0$ periodic: $\int_{T_0} f(x)\,dx = 0$

even x even = even odd x odd = even even x odd = odd

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-j\omega_0 kt}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t) \underbrace{\cos(-\omega_0 kt)}_{\text{even}}\, dt + \frac{j}{T_0}\int_{T_0} x(t) \underbrace{\sin(-\omega_0 kt)}_{\text{odd}}\, dt$$

If $x(t)$ is <u>even</u> $\Rightarrow$ $a_k$ are even $a_{-k} = a_k$

Because the 2nd integral goes to zero, and the sign of $k$ doesn't matter in $\cos(\cdot)$

$x(t)$ <u>even and real</u> $\Rightarrow$ $a_k$ are real $a_{-k} = a_k = a_{-k}^*$

If $x(t)$ is <u>odd</u> $\Rightarrow$ $a_k$ are odd $a_{-k} = -a_k$

Because 1st integral goes to zero, & the sign of $k$ pops out of $\sin(\cdot)$

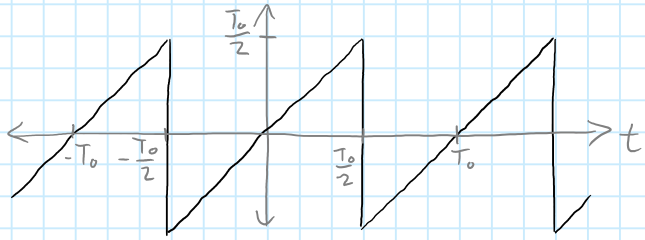
$x(t)$ <u>odd and real</u> $\Rightarrow$ $a_k$ are <u>imaginary</u> $a_{-k} = -a_k = -a_{-k}^*$

Examples:

Sawtooth wave:

$x(t) = t$ for $-\frac{T_0}{2} \le t < \frac{T_0}{2}$

& periodic w/ period $T_0$




$a_0 = 0$ since no DC offset

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} t e^{-j\frac{2\pi}{T_0}kt}\,dt = \text{Integration by parts} = \frac{jT_0(-1)^k}{2\pi k} \quad \text{for } k \neq 0$$

$$\int u\,dv = uv - \int v\,du$$

$u = t$ $\quad dv = e^{-j\frac{2\pi}{T_0}kt}\,dt$

$du = dt$ $\quad v = \frac{1}{-j\frac{2\pi}{T_0}k} e^{-j\frac{2\pi}{T_0}kt}$

$$\Rightarrow = \frac{1}{-j\frac{2\pi}{T_0}k}\left(t e^{-j\frac{2\pi}{T_0}kt} - \int e^{-j\frac{2\pi}{T_0}kt}\,dt\right)$$

$$= -\frac{T_0}{j2\pi k}\left(t e^{-j\frac{2\pi}{T_0}kt} - \frac{1}{-j\frac{2\pi}{T_0}k} e^{-j\frac{2\pi}{T_0}kt}\right)\Bigg|_{t=-\frac{T_0}{2}}^{t=\frac{T_0}{2}}$$

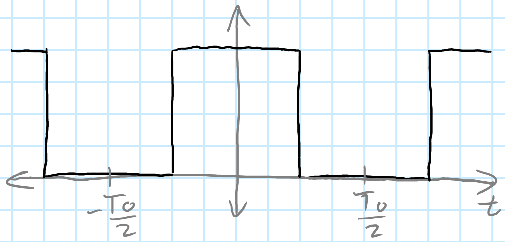
$$= -\frac{T_0}{j2\pi k}\left(\frac{T_0}{2}e^{-j\pi k} + \frac{T_0}{j2\pi k}e^{-j\pi k} + \frac{T_0}{2}e^{j\pi k} - \frac{T_0}{j2\pi k}e^{j\pi k}\right)$$

$$= \frac{jT_0^2}{2\pi k}\left(\underbrace{\cos(\pi k)}_{=(-1)^k} - \frac{1}{\pi k}\underbrace{\sin(\pi k)}_{=0}\right) = \frac{jT_0^2(-1)^k}{2\pi k}$$

The $T_0^2$ cancels w/ the $\frac{1}{T_0}$ outside the integral

Square Wave:

$$x(t) = \begin{cases} 1 & \frac{T_0}{4} \le t < \frac{T_0}{4} \\ 0 & \text{otherwise} \end{cases} \quad \text{For } -\frac{T_0}{2} \le t < \frac{T_0}{2}$$




& is periodic w/ period $T_0$

$$a_k = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} e^{-j\frac{2\pi}{T_0}kt}\,dt$$

$$= \frac{1}{T_0}\left(\frac{1}{-j\frac{2\pi}{T_0}k}\right) e^{-j\frac{2\pi}{T_0}kt}\Big|_{t=-T_0/4}^{t=T_0/4}$$

$$= \frac{1}{j2\pi k}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)$$

$$= \frac{1}{\pi k}\sin\left(\frac{\pi}{2}k\right)$$

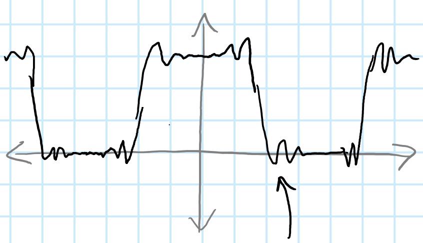
$$a_k = \begin{cases} 0 & \text{for even } k \text{ where } k \neq 0 \\ \frac{1}{\pi k} & \text{for } k = \ldots, -7, -3, 1, 5, 9, \ldots \\ -\frac{1}{\pi k} & \text{for } k = \ldots, -9, -5, -1, 3, 7, \ldots \end{cases}$$

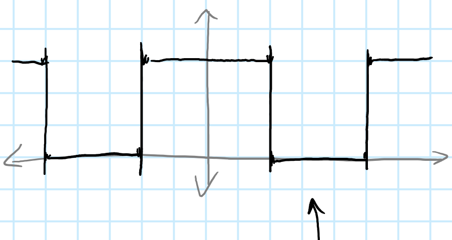
$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} 1\,dt = \frac{1}{T_0}\left(\frac{T_0}{2}\right) = \frac{1}{2}$$

← The "average" value / DC component

If you reconstruct it with a finite number of terms $-L \le k \le L$:

- This is like running it through an ideal lowpass filter ("Brick wall")





Gibb's phenomenon ("ringing"). Increasing $L$ shrinks it side-to-side, but not vertically. As $L$ increases, the average error approaches zero even though the Gibbs peaks never go away. This is an artifact of reconstructing a discontinuous function using continuous functions.