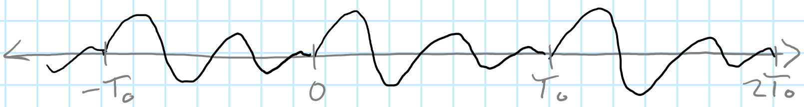
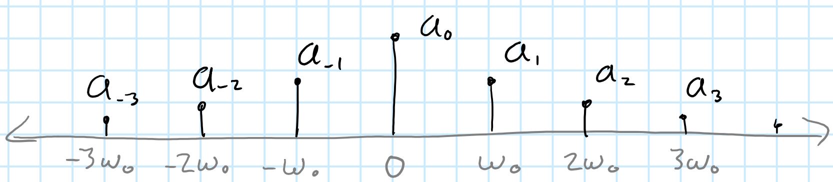
ECE3084-L07-FourierTransform

Wednesday, February 1, 2017 12:50 PM

From Fourier Series to Fourier Transform



FS: Assumes signal is periodic, gives spectral lines:



What if we push $T_0 \to \infty$?

Spectral lines get closer together. spacing $= \omega_0 = \frac{2\pi}{T_0}$

Can hand-wavily derive the FT equations by taking this limit (see the book). FS synthesis sum becomes an integral.

A signal with "infinite" period is no longer periodic.

Fourier Transform Pair:

$$\boxed{\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \end{aligned}}$$

Analysis

Synthesis

Note that the analysis integral is identical to the frequency response:

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$ Freq response of $h(t)$

FT of two signals convolved together:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \; e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{t=\infty} h(t-\tau)\, e^{-j\omega t}\, dt\, d\tau$$

Change of variables: $\lambda = t - \tau$
$t = \lambda + \tau$
$dt = d\lambda$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{\lambda+\tau=-\infty}^{\lambda+\tau=\infty} h(\lambda)\, e^{-j\omega(\lambda+\tau)}\, d\lambda\, d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau}_{X(j\omega)} \; \underbrace{\int_{-\infty}^{\infty} h(\lambda)\, e^{-j\omega\lambda}\, d\lambda}_{H(j\omega)}$$

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

So, convolution in time $\longleftrightarrow$ multiplication in ω

one of the most celebrated properties of Fourier Transforms

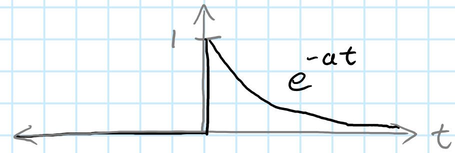
Cascaded Systems:

$$\begin{matrix} x(t) \\ X(j\omega) \end{matrix} \longrightarrow \underset{H_1(j\omega)}{\boxed{h_1(t)}} \longrightarrow \underset{H_2(j\omega)}{\boxed{h_2(t)}} \longrightarrow \begin{matrix} y(t) \\ Y(j\omega) = X(j\omega)H_1(j\omega)H_2(j\omega) \end{matrix}$$

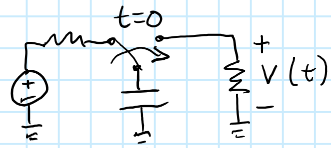
Equivalent System:

$$\begin{matrix} x(t) \\ X(j\omega) \end{matrix} \longrightarrow \boxed{\begin{matrix} h_1(t) * h_2(t) \\ H_1(j\omega)\, H_2(j\omega) \end{matrix}} \longrightarrow \begin{matrix} y(t) \\ Y(j\omega) \end{matrix}$$

FT Example: Decaying Exponential



$x(t) = e^{-at}u(t)$ with $a > 0$



$a = \frac{1}{\tau} = \frac{1}{RC}$

$$X(j\omega) = \int_0^\infty e^{-at}e^{-j\omega t}\,dt$$

$$= \int_0^\infty e^{-(a+j\omega)t}\,dt$$

$$= \frac{e^{-(a+j\omega)}}{-(a+j\omega)}\Bigg|_0^\infty = 0 + \frac{1}{-(a+j\omega)}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

What does this look like in freq domain?

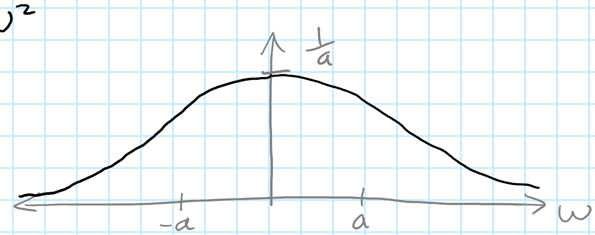
Magnitude Spectrum: $|X(j\omega)| = \sqrt{X(j\omega)\,X^*(j\omega)}$

$$|X(j\omega)| = \sqrt{\left(\frac{1}{a+j\omega}\right)\left(\frac{1}{a-j\omega}\right)} = \sqrt{\frac{1}{a^2+\omega^2}}$$

at $\omega = 0$ $\quad |X(j\omega)| = \frac{1}{a}$

at $\omega = a$ $\quad |X(j\omega)| = \sqrt{\frac{1}{2a^2}} = \frac{1}{a\sqrt{2}}$

as $\omega \to \infty$ $\quad |X(j\omega)| \to 0$



If $x(t)$ was the impulse response of a system:

It'd be a "single pole" low pass filter with a half-power cutoff frequency of $\omega_c = a$

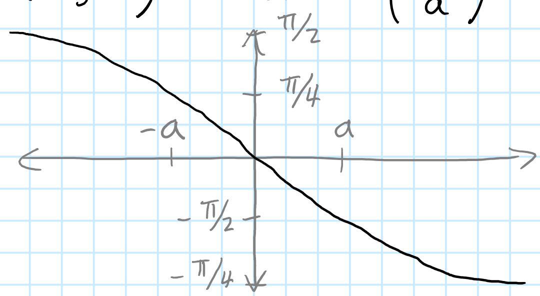
Phase spectrum:

$$\angle\{X(j\omega)\} = \angle\left\{\frac{1}{a+j\omega}\right\} = -\angle\{a+j\omega\} = -\arctan\left(\frac{\omega}{a}\right)$$

at $\omega = 0$ $\quad \angle\{X(j\omega)\} = 0$

at $\omega = \pm a$ $\quad \angle\{X(j\omega)\} = \mp\frac{\pi}{4}$

as $\omega \to \pm\infty$ $\quad \angle\{X(j\omega)\} \to \mp\frac{\pi}{2}$



With numbers: $h(t) = e^{-0.1t}u(t)$ $\quad x(t) = 3\cos(0.2t + 0.6)$

$a = 0.1$ $\quad \omega = 0.2$

$$H(j0.2) = \frac{1}{0.1 + j0.2}$$

$$|H(j0.2)| = \sqrt{\frac{1}{0.1^2 + 0.2^2}} = 4.47 \qquad \angle\{H(j0.2)\} = -\angle\{0.1 + j0.2\} = -1.11$$

$$y(t) = (4.47)(3)\cos(0.2t + 0.6 - 1.11) = 13.41\cos(0.2t - 0.51)$$

FT of a δ function

$x(t) = \delta(t - t_0)$

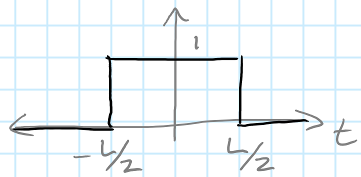
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt \qquad \text{use sampling/sifting property}$$
$$= e^{-j\omega t_0}$$

$$\delta(t - t_0) \longleftrightarrow e^{-j\omega t_0}$$ delay in time $\rightarrow$ linear phase shift in frequency

when $t_0 = 0$: $\quad \delta(t) \longleftrightarrow 1$

Sum of all cosines yields an impulse. They all line up at $t=0$, but interfere & cancel eachother out everywhere else.

FT of a pulse of length L



$x(t) = u(t + \frac{L}{2}) - u(t - \frac{L}{2})$

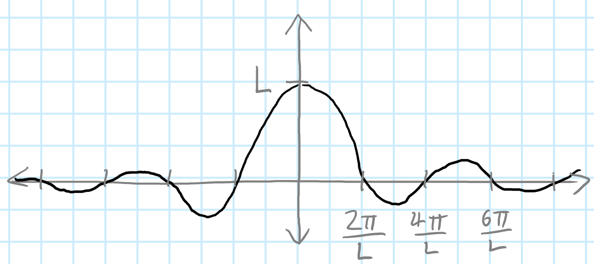
$$X(j\omega) = \int_{-L/2}^{L/2} e^{-j\omega t}\, dt$$
$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-L/2}^{L/2}$$
$$= \frac{-1}{j\omega}\left(e^{-j\omega \frac{L}{2}} - e^{j\omega \frac{L}{2}}\right)$$
$$= \frac{2}{\omega} \sin\left(\omega \frac{L}{2}\right)$$

What about $\omega = 0$? Use L'Hopitals, or plug in before integrating!

$$X(0) = \int_{-L/2}^{L/2} 1\, dt = L$$

This is called a sinc function

Has the form $\dfrac{\sin(\text{something})}{\text{something}}$



$x(t)$ bounded in time $\longrightarrow$ $X(j\omega)$ unbounded in frequency.

Opposite case: bounded in frequency: $X(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c)$

$$x(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1\, e^{j\omega t}\, d\omega = \frac{\sin(\omega_c t)}{\pi t}$$ Unbounded in freq

Often use tables of FT pairs & properties to avoid having to integrate.

Properties of Fourier Transforms:

Delay: $x(t-t_d) \longleftrightarrow e^{-j\omega t_d} X(j\omega)$

Linearity: $a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(j\omega) + a_2 X_2(j\omega)$

Conjugation: $x^*(t) \longleftrightarrow X^*(-j\omega)$

Time Reversal: $x(-t) \longleftrightarrow X(-j\omega)$

Scaling: $x(at) \longleftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$

Note inverse relationship: stretch in time → compress in Freq

Convolution: $x(t) * h(t) \longleftrightarrow X(j\omega) H(j\omega)$

Multiplication: $x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$

Modulation: $x(t) \cos(\omega_0 t) \longleftrightarrow \frac{1}{2}\left(X(j(\omega-\omega_0)) + X(j(\omega+\omega_0))\right)$

Modulation: $x(t) e^{j\omega_0 t} \longleftrightarrow X(j(\omega-\omega_0))$

Differentiation: $\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$

Examples

① $x(t) = \delta(t) + \delta(t-2)$ delay property

$X(j\omega) = 1 + e^{-j2\omega} = e^{-j\omega}(e^{j\omega} + e^{-j\omega})$

$= 2e^{-j\omega}\cos(\omega)$

② $x(t) = e^{-t}u(t-2)$

$= e^{-(t-2)}u(t-2)e^{-2}$ delay property plus previous result for a decaying exponential

$X(j\omega) = e^{-2}\frac{1}{2+j\omega}e^{-j\omega 2}$

③ $x(t) = u(t) - u(t-4)$

$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = \frac{2\sin(2\omega)}{\omega}\underbrace{e^{-j2\omega}}_{\text{delay prop}}$

④ $x_1(t) = \cos(t)$

$x_2(t) = \sin(3t)$

Find $y(t) = x_1(t) * x_2(t)$ Use convolution property

$Y(j\omega) = X_1(j\omega)X_2(j\omega)$

$= \pi(\delta(\omega-1) + \delta(\omega+1))\left(\frac{\pi}{j}\right)(\delta(\omega-3) - \delta(\omega+3))$

$= 0$

⑤ $x(t) = \delta(t) - \delta(t-2)$

1st approach: shifted impulses $\delta(t-t_0) \Longleftrightarrow e^{-j\omega t_0}$

$X(j\omega) = e^{-j\omega 0} - e^{-j\omega 2} = 1 - e^{-j2\omega} = e^{-j\omega}(e^{j\omega} - e^{-j\omega})$

$= 2je^{-j\omega}\sin(j\omega)$

2nd approach: derivative property.

$u(t+T/2) - u(t-T/2) \Longleftrightarrow \frac{\sin(\omega T/2)}{\omega/2}$

$x(t-t_d) \Longleftrightarrow e^{-j\omega t_d}X(j\omega)$

Let $\tilde{x}(t) = u(t) - u(t-2)$

Then $x(t) = \frac{d}{dt}\tilde{x}(t) = \delta(t) - \delta(t-2)$

$\tilde{X}(j\omega) = \underbrace{\frac{\sin(\omega)}{\omega/2}}_{\text{Square pulse FT}}\underbrace{e^{-j\omega}}_{\text{Delay property}}$

$\frac{d\tilde{x}(t)}{dt} = x(t) \Longleftrightarrow j\omega\tilde{X}(j\omega) = X(j\omega)$

$X(j\omega) = j\omega\frac{\sin(\omega)}{\omega/2}e^{-j\omega}$

$= 2je^{-j\omega}\sin(\omega)$ Same as before