ECE3084-L09 Sampling and Periodicity

Tuesday, February 14, 2017 1:15 AM

From 2026: Create discrete time signal by sampling a continuous time signal:

$$x[n] = x(nT_s) \quad \text{where } T_s = \text{period between samples}$$

Sampling rate: $f_s = \frac{1}{T_s}$

Nyquist Rate = Twice the highest frequency component in $x(t)$.

Nyquist Sampling Theorem: To be able to reconstruct $x(t)$ from its samples $x[n]$, you must sample at a rate greater than the Nyquist Rate.

Normalized discrete-time frequency: $\hat{\omega} = \frac{\omega}{f_s}$ or $\hat{f} = \frac{f}{f_s}$

Units?

$$\hat{f} = \frac{f}{f_s} = \frac{\#\text{ oscillations per second}}{\#\text{ samples per second}} = \#\text{ oscillations per sample}$$

$$\hat{\omega} = \frac{\text{radians per second}}{\text{samples per second}} = \text{radians per sample}$$

Suppose we have a periodic square wave with fundamental frequency f_0. What is the Nyquist rate for this signal?

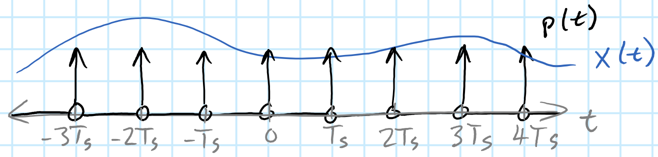
- Kind of a trick question. A square wave is not bandlimited. (has nonzero frequencies forever... they approach 0, but never hit it and then stay there.) Thus, it has no Nyquist rate.

In general, there is no Nyquist rate for a finite length signal, but we can get "close enough".

Sampling from a CT perspective:

Think of it as multiplying by a train of impulses ("comb"), $p(t)$: or "impulse train"

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$x_s(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)$$

Sampling property

Note: This is just a conceptual model of sampling, not what actual ADC hardware does.

What does this do in the frequency domain?

$p(t)$ is periodic, so can represent as a Fourier Series:

$$p(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j\omega_0 k t}$$

where we get the a_k coefficients with the FS analysis integral:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} p(t)\, e^{-j\omega_0 k t}\, dt$$

$T_0 = T_s \qquad \omega_0 = \omega_s$ — The period of $p(t)$ is equal to the sampling rate (The "comb")

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j\omega_s k t}\, dt$$

Only integrating over the period centered @ the origin, so only catch $\delta(t)$.

$$= \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, dt$$

Sampling / sifting property.

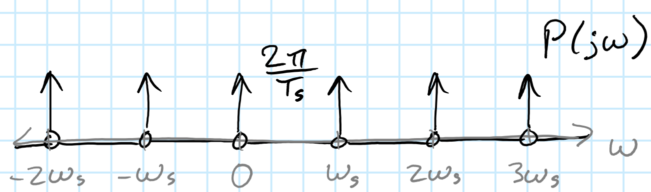
$$= \frac{1}{T_s}$$

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{j\omega_s k t} \qquad \text{(FS Representation)}$$

We can take the FT of each term in the sum: $e^{j\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi\,\delta(\omega - \omega_0)$

$$P(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k\omega_s)$$

Also could use "impulse train" entry from the table of FT pairs.



So, the FT of an impulse train is another impulse train.

Note: squishing the impulses closer together in one domain will spread them apart in the other (and vice versa).

Generally, the FT of a function does not look anything like the original function. Exceptions: sinc, Gaussian, impulse train.

Multiplied in time domain $\Rightarrow$ convolve in freq domain:

$$x_s(t) = x(t)\, p(t)$$

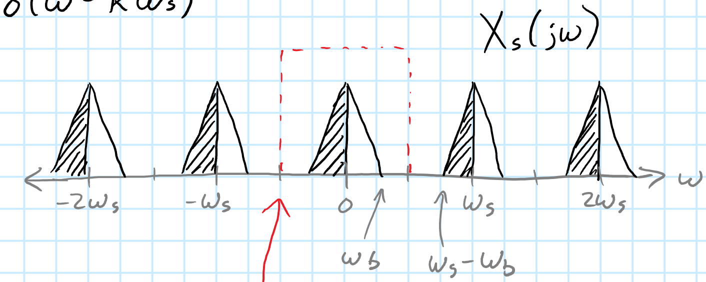
$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi} X(j\omega) * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - k\omega_s)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$\rightarrow$ shifted copies of $X(j\omega)$.

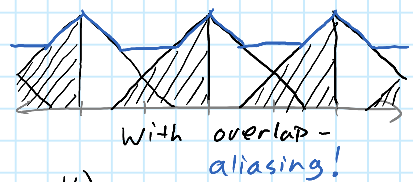
"periodic replication"



For no overlap, $\omega_s - \omega_b > \omega_b$

$\Rightarrow \quad \omega_s > 2\omega_b$

$\omega_b < \frac{1}{2}\omega_s$ Nyquist criterion



Can get $x(t)$ back with a perfect LPF (brick wall):

$\omega_{co} = \omega_s/2$

$$H(j\omega) = \begin{cases} T_s & \text{for } |\omega| \le \omega_s/2 \\ 0 & \text{otherwise} \end{cases}$$

In the time domain:

$\omega_s = \frac{2\pi}{T_s}$

$$h(t) = T_s \frac{\sin\left(\frac{\omega_s}{2} t\right)}{\pi t} = \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\frac{\pi}{T_s} t}$$

Reconstructed signal:

$$x_r(t) = x_s(t) * h(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \right) * \frac{\sin\left(\frac{\pi}{T_s} t\right)}{\frac{\pi}{T_s} t}$$

convolve w/ delta fns

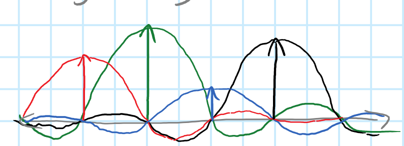
$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)}$$

Sum of scaled & shifted sinc functions

$= x(t) \quad \Rightarrow$ Perfect if $\omega_b < \frac{\omega_s}{2}$, but non-causal

This is the sinc interpolation formula. Tells us how to reconstruct a bandlimited signal from its samples.

Note: There's always aliasing because no finite length signal is band limited.



"Good enough" depends on the application.

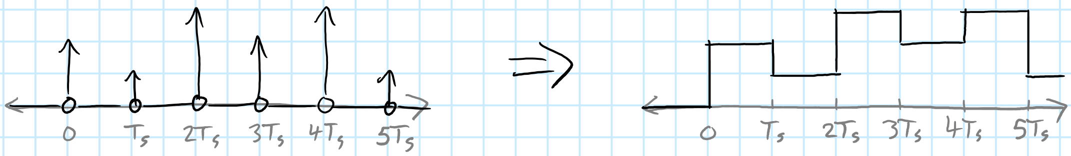
– Aliasing effects might fall below the "noise floor."

At integer multiples $x_r(mT_s)$, only one sinc contributes to that output (the one centered there). It's at a zero for all the other sincs.

What we've just done is of theoretical interest → Proves the Nyquist Sampling theorem.

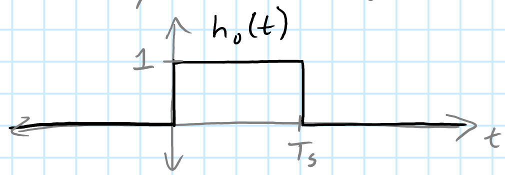
Practical Reconstruction: "sample and hold" / zero order hold (This is the typical approach)

Hold the output constant until the next sample arrives:



What would the impulse response of a system that does this look like?

$$h_0(t) = \begin{cases} 1 & \text{for } 0 \le t < T_s \\ 0 & \text{otherwise} \end{cases}$$



Subscript zero because it's a "zero order hold"

This is causal. Also, because of the sampled nature of the input, it is easy to implement. (e.g. with a capacitor & an electronic switch. A filter like this that would operate on a generic continuous time signal would be hard to construct.)

$$x_{r_0}(t) = h_0(t) * x_s(t)$$ Reconstructed signal using a zero order hold

$$= h_0(t) * \left(\sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \right)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, h_0(t - nT_s)$$

In the frequency domain: $X_{r_0}(j\omega) = H_0(j\omega)\, X_s(j\omega)$

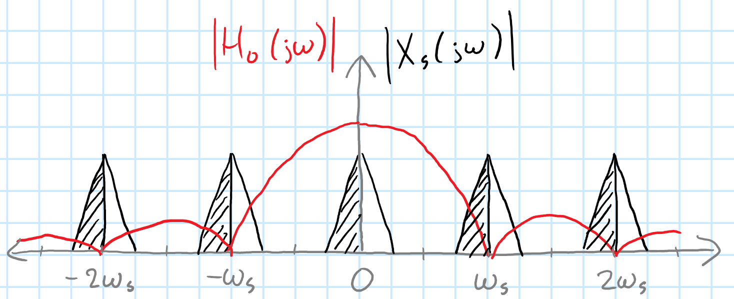
$$H_0(j\omega) = \frac{\sin\left(\frac{\omega T_s}{2}\right)}{\omega/2}\, e^{-j\omega \frac{T_s}{2}}$$

Square pulse of width T_s time delayed by $\frac{T_s}{2}$

Zeros happen at $\frac{\omega T_s}{2} = n\pi$

$$\omega = \frac{n 2\pi}{T_s} = n\omega_s$$ where $\omega_s = \frac{2\pi}{T_s}$

Zeros land @ centers of aliased copies of the sampled signal



Still need a low pass filter to get rid of the aliased parts that bleed through around the zeros.

The signal gets shaped/distorted by the shape of the center lobe.

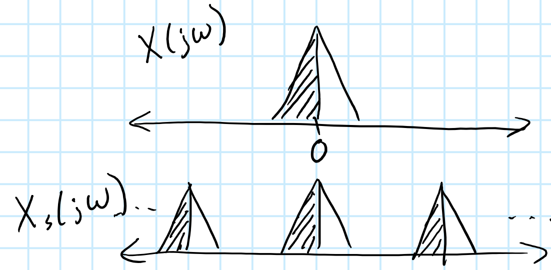
can have a more gradual cutoff slope

Increase ω_s to reduce aliasing, minimize center lobe distortion, & relax LPF specs.

→ Broadens the main lobe with respect to $X_s(j\omega)$, more space between aliased copies.

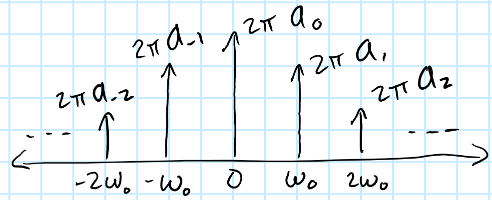
Reconstructed signal is delayed by $T_s/2$. Often unimportant, but can be important in control applications. Tradeoff between performance (small delay) and cost (high ω_s).

Duality of sampling & periodicity



Notice that $X(j\omega)$ is not periodic, but when we sampled $x(t)$ to get $x_s(t)$, $X_s(j\omega)$ became periodic.

Multiplying by an impulse train ("sampling") in one domain is a convolution with an impulse train in the other domain (which replicates the signal and makes it periodic).



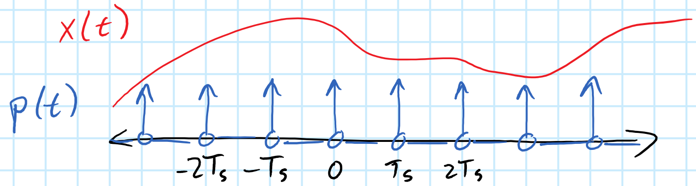
This matches what we've seen with FS previously. If you take the FT of a periodic signal, you get an impulse train in the freq domain, scaled by 2π times the a_k coefficients.

Overview with demos

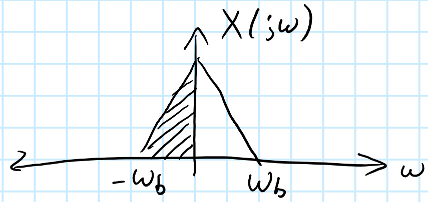
Thursday, March 2, 2017 2:25 AM

Our model of sampling:

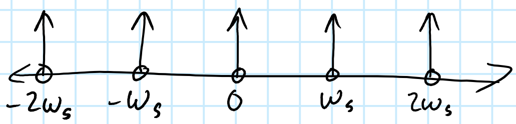
$x_s(t) = x(t)\,p(t)$



$X(j\omega)$
→ Assume bandlimited with max freq ω_b



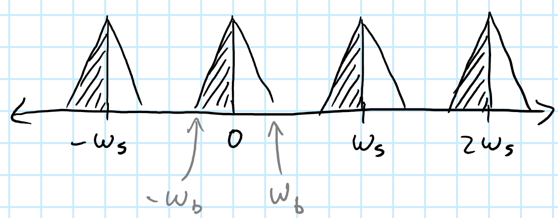
$P(j\omega)$



$\omega_s = 2\pi f_s = \frac{2\pi}{T_s}$

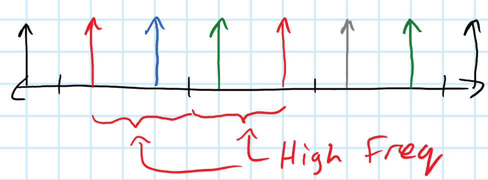
$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

Can reconstruct the original signal $x(t)$ from $x_s(t)$ if there's no overlap



Need $\omega_s > 2\,\omega_b$ to avoid overlap / aliasing

Aliasing: After sampling, frequencies higher than $\frac{1}{2}\omega_s$ look like lower frequencies (they're indistinguishable).



samplingVisualization.m

samplingASinc.m

Extra

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Relation between CTFT & DTFT:

$x(t)$ = CT signal

$x[n] = x(nT_s)$ = DT signal

$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(n - nT_s)$ = CT sampled signal

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Could compute $X_s(j\omega)$ directly:

$$X_s(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s)\,e^{-j\omega t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t - nT_s)\,e^{-j\omega t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n T_s}$$

Note that $\hat{\omega} = \frac{\omega}{F_s} = \omega T_s$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\hat{\omega} n} \quad \longleftarrow \text{DTFT formula}$$

Can derive inverse DTFT by low-pass filtering the inverse CTFT $X_s(j\omega)$ to recover the central $x(t)$.

Derivations in the book.

Can view Fourier Series as samples of the CTFT.