ECE3084-L10 Laplace Transform

Thursday, March 2, 2017 4:05 AM

Recall:

Fourier Transform: $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$\underbrace{x_1(t) * x_2(t)}_{\text{convolution}} \iff \underbrace{X_1(j\omega)\, X_2(j\omega)}_{\text{multiplication}}$

Particularly useful for freq domain analysis:

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$Y(j\omega) = X(j\omega) H(j\omega)$

$x(t) = A\cos(\omega t + \phi)$

then $y(t) = |H(j\omega)|\, A \cos\left(\omega t + \phi + \angle H(j\omega)\right)$

The FT doesn't work for many signals, e.g.

$x(t) = u(t)$

$X(j\omega) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-j\omega t}\, dt \leftarrow$ This integral does not exist.

However, the FT does exist if the step fn decays: $x(t) = e^{-at} u(t)$

If we can introduce a factor $e^{-\sigma t}$ into the transform, then it could work for this input:

$$X(\sigma + j\omega) = \int_0^{\infty} e^{-\sigma t} e^{-j\omega t}\, dt \qquad \text{for } \sigma > 0$$

$$= \int_0^{\infty} e^{-(\sigma + j\omega)t}\, dt$$

$$= \frac{1}{-(\sigma + j\omega)}\, e^{-(\sigma + j\omega)t} \Big|_0^{\infty}$$

$$= \frac{1}{\sigma + j\omega}$$

Laplace Transform

Let $s = \sigma + j\omega$ $\quad s \in \mathbb{C}$ ← set of complex numbers

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Bilateral (or two-sided) Laplace Transform

We just replaced the $j\omega$ from the FT with a more general complex number, $s = \sigma + j\omega$

The FT projects signals onto complex sinusoids $e^{j\omega t}$, which are unable to represent some types of signals (e.g. signals that go to ∞ as $t \to \infty$).

The LT projects signals onto $e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t}$.

These are complex sinusoids that can exponentially grow or decay in magnitude depending on what the value of σ is.

This is a more general basis set that allows us to capture a richer set of signals.

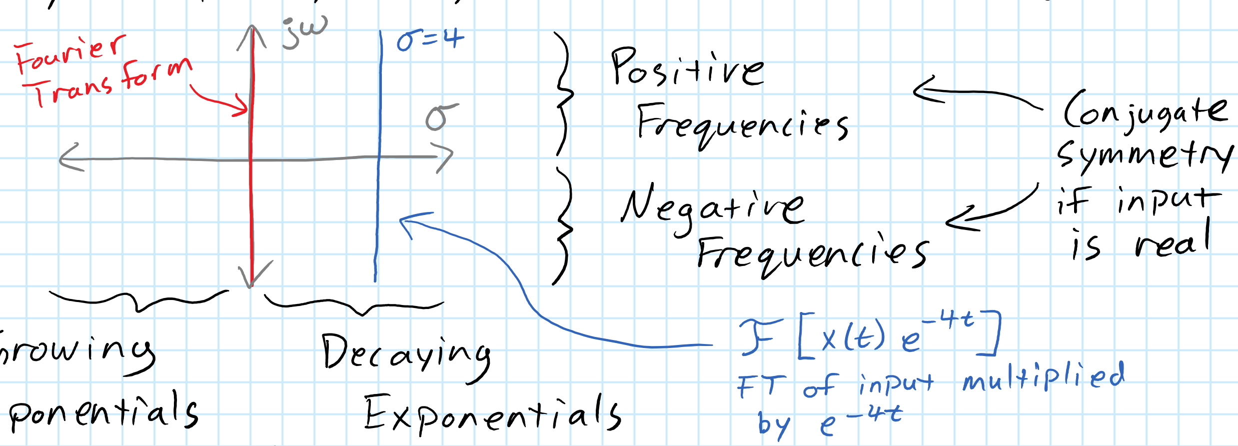
*Note that if we choose $\sigma = 0$ so that $s = j\omega$, then the LT reduces to the FT. In other words, the FT is a special case or subset of the LT.

For a particular signal $x(t)$, the <u>region of convergence</u> (ROC) is the set of all s for which the integral converges.

The ROC only constrains the real part of s (It's bounded by vertical lines in the s-plane).

One way to think about the LT:*

To find the values along a vertical line $s = \sigma$ in the s-plane, multiply the input by $e^{-\sigma t}$, then take the Fourier Transform.



*The LT is not really a two-stage process like this though.

The bilateral LT is rarely used by practicing engineers and complicates dealing with ROCs. For causal systems with inputs for $t \geq 0$ and some set of initial conditions at $t = 0^-$, we use:

Unilateral Laplace Transform ("Laplace Transform"):

$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

Includes the point $t=0$ (ie if there's a δ there)

Unless we specifically say "bilateral," this is what we are referring to. The system outputs $y(t)$ we derive with this are only valid for $t \geq 0$.

Examples:

1. $x(t) = u(t)$

$$X(s) = \int_0^{\infty} u(t) e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^{\infty}$$

$$= -\frac{1}{s} e^{-s(\infty)} + \frac{1}{s}$$

goes to 0 if $Re(s) > 0$

$$X(s) = \frac{1}{s}$$

$$\boxed{\mathcal{L}[u(t)] = \frac{1}{s} \qquad Re(s) > 0}$$

ROC is the RHP (right half plane)

2. $x(t) = e^{-at} u(t)$

$$X(s) = \int_0^{\infty} e^{-at} e^{-st} dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^{\infty}$$

$$= -\frac{1}{s+a} e^{-(s+a)\infty} + \frac{1}{s+a}$$

Goes to zero if $Re(s) > -a$

$$\boxed{\mathcal{L}[e^{-at} u(t)] = \frac{1}{s+a} \qquad Re(s) > -a}$$

I think most of the entries on the table with an e^{-at} have a ROC of $Re(s) > -a$.

3. $x(t) = t e^{-at}$

$$X(s) = \int_0^\infty t e^{-at} e^{-st}\, dt$$

$$= \int_0^\infty t e^{-(a+s)t}\, dt$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

(Can derive this by integrating the product rule of differentiation.)

$u = t \qquad dv = e^{-(a+s)t}$

$du = 1 \qquad v = \frac{-1}{a+s} e^{-(a+s)t}$

$$X(s) = \left(\frac{-1}{a+s} t e^{-(a+s)t} - \int \frac{-1}{a+s} e^{-(a+s)t}\, dt \right|_0^\infty \qquad \mathrm{Re}(s) > a$$

$$= \left(\frac{-1}{a+s} t e^{-(a+s)t} - \frac{1}{(a+s)^2} e^{-(a+s)t} \right|_0^\infty$$

$$= (0 - 0) - \left(0 + \frac{-1}{(a+s)^2} \right)$$

$$\mathcal{L}\left[t e^{-at} u(t)\right] = \frac{1}{(a+s)^2} \qquad \mathrm{Re}(s) > a$$

The $u(t)$ shows up in every entry of the table (except where redundant) to emphasize that this is the unilateral LT.

In general, better to use tables & properties rather than integrating.

Key Properties of the LT (see table for others)

- Linearity: $\mathcal{L}[a x_1 + b x_2] = a\mathcal{L}[x_1] + b\mathcal{L}[x_2]$
  $= a X_1(s) + b X_2(s)$

  Follows easily from linearity of integrals

- Time Delay: $\mathcal{L}[x(t-T)] = e^{-sT} X(s)$

$$\mathcal{L}\left[x(t-T)\right] = \int_0^\infty x(t-T) e^{-st}\, dt$$

Change of vars:

$\tau = t - T \qquad t = \tau + T$

$dt = d\tau$

$$= \int_{\tau+T=0}^{\tau+T=\infty} x(\tau) e^{-s(\tau+T)}\, dt$$

$$= e^{-sT} \int_{-T}^\infty x(\tau) e^{-s\tau}\, d\tau$$

Assume $x(t) = 0$ for $t \in [-T, 0)$

$$= e^{-sT} \int_0^\infty x(\tau) e^{-s\tau}\, d\tau$$

$$= e^{-sT} X(s)$$

- Differentiation: $\mathcal{L}[x'(t)] = sX(s) - x(0^-)$

small x!

$$\mathcal{L}[x'(t)] = \int_0^\infty x'(t)\, e^{-st}\, dt$$

Integration by parts: $\int u\,dv = uv - \int v\,du$

$u = e^{-st}$ $\qquad dv = x'(t)\,dt$

$du = -se^{-st}dt$ $\qquad v = x(t)$

$$= \left(e^{-st} x(t) - \int x(t)(-se^{-st})\,dt \right|_0^\infty$$

$$= \underbrace{x(t)e^{-st}\Big|_0^\infty}_{0 - x(0)} + \underbrace{s\int_0^\infty x(t)e^{-st}\,dt}_{X(s)}$$

$$= sX(s) - x(0^-)$$

0^- because that's how we defined the lower bound of the LT. We just get lazy & don't always write it.

Can apply repeatedly:

$$\mathcal{L}[x''(t)] = s\mathcal{L}[x'(t)] - x(0^-)$$
$$= s^2X(s) - sx'(0^-) - x(0^-)$$

<u>Notice</u>: The derivatives disappeared and became polynomials in the s-domain. Thus, <u>differential equations</u> in the time domain become <u>algebraic equations</u> in the s-domain! This is why the LT is so useful for solving diff eqs.

- Integration:

Let $v(t) = \begin{cases} 0 & t<0 \\ \int_0^t x(\tau)\,d\tau & t \ge 0 \end{cases}$

So $x(t) = v'(t)$

$$\mathcal{L}[v'(t)] = sV(s) - v(0)$$

$= 0$ by definition

$$\mathcal{L}[x(t)] = sV(s)$$
$$X(s) = sV(s)$$
$$V(s) = \frac{1}{s}X(s)$$

$$\mathcal{L}\left[\int_0^t x(\tau)\,d\tau\right] = \frac{1}{s}X(s)$$

- Final Value Theorem: $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

If the limit exists, and $x(t)$ & $x'(t)$ have LTs.

Proof: $$\lim_{s\to 0} sX(s) = \lim_{s\to 0}\left(sX(s) - x(0) + x(0)\right)$$
$$= \lim_{s\to 0}\left(\mathcal{L}[x'(t)]\right) + x(0)$$
$$= \lim_{s\to 0}\int_0^\infty x'(t)e^{-st}\,dt + x(0)$$
$$= \int_0^\infty x'(t)\left(\lim_{s\to 0} e^{-st}\right)dt + x(0)$$
$$= \int_0^\infty x'(t)\,dt + x(0)$$
$$= \left(x(\infty) - x(0)\right) + x(0)$$
$$= x(\infty)$$

- Initial Value Theorem: $\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$

Proof: Similar, just change the limit to $s\to\infty$ instead of $s\to 0$.

Examples

- $x(t) = \cos(\omega t)\,u(t)$

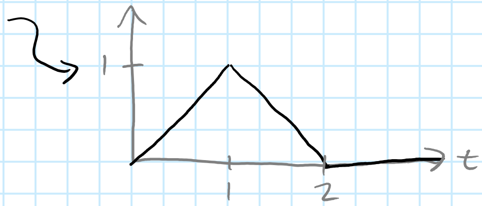
$$= \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t) \qquad x(t)e^{at} \overset{\mathcal{L}}{\Longleftrightarrow} X(s-a)$$

$$X(s) = \tfrac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \qquad u(t) \Longleftrightarrow \frac{1}{s}$$
$$= \frac{1}{2}\left(\frac{s+j\omega + s - j\omega}{s^2+\omega^2}\right) = \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right)$$
$$= \frac{s}{s^2+\omega^2}$$

$$\int_0^t x(\tau)\,d\tau \Longleftrightarrow \frac{1}{s}X(s)$$

- $x(t) = t\,u(t) = \int_0^t u(t)\,dt$ $\qquad u(t) \Longleftrightarrow \frac{1}{s}$

$$X(s) = \frac{1}{s}\mathcal{L}[u(t)] = \frac{1}{s^2}$$

- $x(t) =$ 

$$x(t) = t\,u(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2)$$

$$X(s) = \frac{1}{s^2} - 2e^{-s}\left(\frac{1}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right)$$

- $x(t) = \delta(t)$

$$X(s) = \int_{0^-}^{\infty} \delta(t)\,e^{-st}\,dt = \left(e^{-st}\right|_{t=0} = 1$$

$$X(s) = \underline{1}$$

- $x(t) = \sin(t - \frac{\pi}{3})\,u(t)$ Note: $\sin(A-B) = \sin A \cos B - \cos A \sin B$

$$= \left(\sin(t)\cos\left(\frac{\pi}{3}\right) - \cos(t)\sin\left(\frac{\pi}{3}\right)\right)u(t)$$

$$= \left(\frac{1}{2}\sin(t) - \frac{\sqrt{3}}{2}\cos(t)\right)u(t) \qquad \omega = 1$$

$$X(s) = \frac{1}{2}\left(\frac{1}{s^2+1}\right) - \frac{\sqrt{3}}{2}\left(\frac{s}{s^2+1}\right) = \frac{1}{2}\left(\frac{-\sqrt{3}\,s + 1}{s^2+1}\right)$$

Why can't we use the time delay property?

→ the step fn was not delayed.