ECE3084-L11 Partial Fraction Expansion

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The Laplace Transform works well with linear, ordinary differential equations with initial conditions.

one independant variable (not a PDE)

For example: $\dot{y} = -y + 1 \qquad y(0^-) = 0$

We don't care what happened before $t=0$, so we can treat it like it's multiplied by $u(t)$. I.E., the $\underline{1}$ becomes $u(t)$.

$$\xRightarrow{\mathcal{L}} \quad sY(s) - y(0^-) = -Y(s) + \frac{1}{s} \qquad \mathcal{L}[u(t)] = \frac{1}{s}$$

$$sY(s) = -Y(s) + \frac{1}{s}$$

Solve for $Y(s)$, then we can inverse transform to get the sol'n.

$$(s+1)Y(s) = \frac{1}{s} \qquad Y(s) = \frac{1}{s(s+1)}$$

Not in table...

<u>Inverse Laplace Transform</u>

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}\,ds$$

This is a line integral – integrate along a vertical line in the s-plane.
You can choose any constant c that lies in the ROC.
Could view it as doing an inverse FT, followed by dividing out whatever exponential the signal was scaled by to get back the original signal.
Doing this integral is very difficult – leave it to the mathematicians.

So how do we find the ILT? Need to make it look like things in the table. If we could split the denominator apart, it would work.

Partial Fraction Expansion

We have $Y(s) = \frac{1}{s(s+1)} = \underbrace{\frac{A}{s} + \frac{B}{s+1}}$

we want this so we can ILT it using the tables.

$$\frac{1}{s(s+1)} = \frac{A(s+1) + Bs}{s(s+1)}$$

$$1 = As + A + Bs$$

$$0s + 1 = (A+B)s + A$$

Equate the coefficients for each power of s:

s^1: $A + B = 0$

s^0: $A = 1$

Solve the system of equations to find A and B.

$$1 + B = 0 \Rightarrow B = -1$$

Plug back in to $Y(s)$: $Y(s) = \frac{1}{s} - \frac{1}{s+1}$

$u(t) \Leftrightarrow \frac{1}{s}$

$e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$

Can now use tables: $y(t) = u(t) - e^{-t}u(t)$

$$y(t) = (1 - e^{-t})\,u(t)$$

This is the solution to the original diff eq $\dot{y} = -y + 1$ with $y(0^-) = 0$. Can plug it in to make sure it works.

When solving diff eqs using the LT, we often get rational functions of s:

$$X(s) = \frac{B(s)}{A(s)} \quad \begin{matrix} \leftarrow \text{order } M \text{ polynomial} \\ \leftarrow \text{Order } N \text{ polynomial} \end{matrix}$$

$$= \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

* No common factors between $A(s)$ and $B(s)$

Let $p_1, p_2, \ldots, p_N$ be the roots of $A(s)$:

$$A(s) = a_N (s-p_1)(s-p_2)\cdots(s-p_N)$$

$$X(s) = \frac{B(s)}{a_N (s-p_1)(s-p_2)\cdots(s-p_N)}$$

The p_i, $i=1,2,\cdots,N$, are the poles of $X(s)$

To do PFE, the fraction must be strictly proper: $M < N$

If you try PFE when $M \geq N$, you'll get something like:

$$x^2 + 1 = (A+B)x + A$$

x^2: $1 = 0$??? Doesn't work!

x^1: $0 = A + B$

x^0: $1 = A$

For now, assume $M < N$.

Case 1: Distinct Roots

Example: $\dddot{x} + 3\ddot{x} + 9\dot{x} + 27x = 0$

Initial conditions: $x(0^-) = 0$ $\quad \dot{x}(0^-) = 0$ $\quad \ddot{x}(0^-) = 4$

LT $\Rightarrow$ $s\mathcal{L}[\ddot{x}] - \ddot{x}(0^-) + 3s^2X(s) + 9sX(s) + 27X(s) = 0$

$$s^3X(s) - 4 + 3s^2X(s) + 9sX(s) + 27X(s) = 0$$

$$X(s) = \frac{4}{s^3+3s^2+9s+27}$$

$$= \frac{4}{(s+3)(s^2+9)}$$

$$= \frac{4}{(s+3)(s+3j)(s-3j)}$$

synthetic division

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 9 & 27 \\ & & -3 & 0 & -27 \\ \hline & 1 & 0 & 9 & \boxed{0} \end{array}$$

$s^2 + 0s + 9 = s^2+9$

$s=-3$ is a root

$$\frac{4}{(s+3)(s+3j)(s-3j)} = \frac{A}{s+3} + \frac{B}{s+3j} + \frac{C}{s-3j}$$

$$= \frac{A(s+3j)(s-3j) + B(s+3)(s-3j) + C(s+3)(s+3j)}{(s+3)(s+3j)(s-3j)}$$

$$4 = As^2 + 9A + Bs^2 + (3-3j)Bs - 9jB + Cs^2 + (3+3j)Cs + 9jC$$

$$4 = (A+B+C)s^2 + ((3-3j)B + (3+3j)C)s + 9A - 9jB + 9jC$$

s^2: $\quad A + B + C = 0$

s^1: $\quad (3-3j)B + (3+3j)C = 0$

s^0: $\quad 9A - 9jB + 9jC = 4$

Solve $\Rightarrow$ $A = \frac{2}{9}$ $\quad B = \frac{-1+j}{9}$ $\quad C = \frac{-1-j}{9}$

I just used cSolve() on a TI-89

$$X(s) = \left(\frac{2}{9}\right)\left(\frac{1}{s+3}\right) + \left(\frac{-1+j}{9}\right)\left(\frac{1}{s+3j}\right) + \left(\frac{-1-j}{9}\right)\left(\frac{1}{s-3j}\right)$$

$e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$

$$x(t) = \frac{2}{9}e^{-3t} + \left(\frac{-1+j}{9}\right)e^{-3jt} + \left(\frac{-1-j}{9}\right)e^{3jt}$$

The complex conjugate terms look like a cosine. Convert complex scalars to polar form $Ae^{j\theta}$

$$= \frac{2}{9}e^{-3t} + \frac{\sqrt{2}}{9}e^{j\frac{3\pi}{4}}e^{-3jt} + \frac{\sqrt{2}}{9}e^{-j\frac{3\pi}{4}}e^{3jt}$$

$$= \frac{2}{9}e^{-3t} + \frac{\sqrt{2}}{9}\left(e^{-j(3t-\frac{3\pi}{4})} + e^{j(3t-\frac{3\pi}{4})}\right)$$

$$\boxed{x(t) = \frac{2}{9}e^{-3t} + \frac{2\sqrt{2}}{9}\cos\left(3t - \frac{3\pi}{4}\right) \quad \text{For } t \geq 0}$$

You can verify this by taking its 3 derivatives & plugging them in to the original diff eq. Also plug 0 into each derivative to check initial conditions.

Case 2: Repeated Roots

$$X(s) = \frac{1}{(s-p)^n} = \frac{A_0 + A_1 s + A_2 s^2 + \cdots + A_{n-1} s^{n-1}}{(s-p)^n}$$

More work to take the ILT

OR

$$= \frac{B_1}{s-p} + \frac{B_2}{(s-p)^2} + \cdots + \frac{B_n}{(s-p)^n}$$

More work to calculate B_i's

How do we take the ILT of that?

$$\frac{1}{s^k} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{(k-1)!} t^{k-1} u(t)$$

Polynomial entry on table (modified a bit)

$$\frac{1}{(s-\alpha)^k} \Rightarrow \frac{1}{(k-1)!} t^{k-1} e^{\alpha t} u(t)$$

Modulation Prop

$$\frac{s}{(s-\alpha)^k} \Rightarrow \frac{d}{dt}\left(\frac{1}{(k-1)!} t^{k-1} e^{\alpha t}\right) u(t)$$

Derivative Prop

$$\frac{s^2}{(s-\alpha)^k} \Rightarrow \frac{d^2}{dt^2}\left(\frac{1}{(k-1)!} t^{k-1} e^{\alpha t}\right) u(t)$$

$$\vdots$$

Example:

$$X(s) = \frac{1}{(s+2)^2(s+1)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+1}$$

$$= \frac{A(s+2)(s+1) + B(s+1) + C(s+2)^2}{(s+2)^2(s+1)}$$

$$1 = As^2 + 3As + 2A + Bs + B + Cs^2 + 4Cs + 4C$$

$$1 = (A+C)s^2 + (3A+B+4C)s + (2A+B+4C)$$

$A + C = 0$

$3A + B + 4C = 0$

$2A + B + 4C = 1$

Subtract these two equations: $A = -1$

$-1 + C = 0$ $\qquad$ $-2 + B + 4 = 1$

$C = 1$ $\qquad$ $B = -1$

$$X(s) = \frac{-1}{s+2} - \frac{1}{(s+2)^2} + \frac{1}{s+1}$$

$$t^k e^{-at} u(t) \Leftrightarrow \frac{k!}{(s+a)^{k+1}}$$

$$x(t) = -e^{-2t} - te^{-2t} + e^{-t}$$

$$\boxed{x(t) = e^{-t} - (t+1)e^{-2t} \qquad t \geq 0}$$

The book does this same example using the other style of PFE

Residue Method for PFE (Ch 11 in the book)

Distinct Roots:

$$Y(s) = \frac{3}{s^2+7s+10}$$

$$= \frac{3}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5}$$

To find A, multiply both sides by $(s+2)$

$$\frac{3}{(s+5)} = A + \frac{B(s+2)}{s+5}$$

Then plug in $s=-2$: $\frac{3}{3} = A + 0 \qquad A = 1$

For B, multiply by $(s+5)$ and plug in $s=-5$

$$\frac{3}{s+2} = \frac{A(s+5)}{s+2} + B \quad \Rightarrow \quad \frac{3}{-3} = 0 + B \qquad B = -1$$

Can view it as "covering up" that term & plugging in the s that zeros it out.

Repeated Roots:

$$Y(s) = \frac{2}{(s+2)(s+1)^3} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

Above method works for A & D:

$$A = \frac{2}{(s+1)^3}\bigg|_{s=-2} = \frac{2}{(-1)^3} = -2$$

$$D = \underbrace{\frac{2}{s+2}}_{\text{LHS}}\bigg|_{s=-1} = \frac{2}{1} = 2$$

Multiply both sides by $(s+1)^3$ for this one.

(after multiplying both sides by $(s+1)^3$)

To find the k^{th} lower order term: $\left(\frac{1}{k!}\frac{d^k}{ds^k}\left(\text{LHS}\right)\right)\bigg|_{s=\text{ the root}}$

$$C = \left(\frac{1}{1!}\frac{d}{ds}\left(\frac{2}{s+2}\right)\right)\bigg|_{s=-1} = \left(\frac{(s+2)(0)-2(1)}{(s+2)^2}\right)\bigg|_{s=-1}$$

$$= \frac{-2}{(s+2)^2}\bigg|_{s=-1} = \frac{-2}{1} = -2$$

$$B = \left(\frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{2}{s+2}\right)\right)\bigg|_{s=-1} = \left(\frac{1}{2}\frac{d}{ds}\left(\frac{-2}{(s+2)^2}\right)\right)\bigg|_{s=-1}$$

$$= \left(\frac{1}{2}\frac{(s+2)^2(0)+2(2)(s+2)}{((s+2)^2)^2}\right)\bigg|_{s=-1} = \frac{2s+4}{(s+2)^4}\bigg|_{s=-1} = \frac{2}{1} = 2$$

PFE of Improper Fractions: $M \geq N$ (Also in Ch 11)

$$Y(s) = \frac{B(s)}{A(s)} \begin{array}{l} \leftarrow \text{Order } M \\ \leftarrow \text{order } N \end{array}$$

$$= Q(s) + \frac{R(s)}{A(s)}$$

Strictly Proper

Find $Q(s)$ and $R(s)$ using polynomial long division.

Example:

$$Y(s) = \frac{2s^2 + 8s + 68}{s^2 + 4s + 29}$$

$$\begin{array}{r|l} & \quad\quad\quad\quad 2 \quad \leftarrow Q(s) \\ \hline s^2 + 4s + 29 & 2s^2 + 8s + 68 \\ & -2s^2 + -8s + -58 \\ \hline & 0 \quad\quad 0 \quad\quad 10 \quad \leftarrow R(s) \end{array}$$

$$Y(s) = 2 + \frac{10}{s^2 + 4s + 29} = 2 + \frac{10}{(s+2-5j)(s+2+5j)}$$

$$\text{poles at } s = \frac{-4 \pm \sqrt{16 - 4(1)(29)}}{2(1)} = \frac{-4 \pm \sqrt{-100}}{2} = -2 \pm 5j$$

Roots can be complex!

$$\frac{10}{(s+2-5j)(s+2+5j)} = \frac{A}{s+2-5j} + \frac{B}{s+2+5j}$$

$$A = \left(\frac{10}{s+2+5j}\right)\Bigg|_{s=-2+5j} = \frac{10}{-2+5j+2+5j} = \frac{10}{10j} = -j$$

$$B = \left(\frac{10}{s+2-5j}\right)\Bigg|_{s=-2-5j} = \frac{10}{-2-5j+2-5j} = \frac{10}{-10j} = j$$

$$Y(s) = 2 + \frac{-j}{s+2-5j} + \frac{j}{s+2+5j}$$

$1 \Longleftrightarrow \delta(t)$

$e^{-at}u(t) \Longleftrightarrow \frac{1}{s+a}$

$$y(t) = 2\delta(t) - je^{-(2-5j)t} + je^{-(2+5j)t}$$

$$= 2\delta(t) - e^{-2t}\left(je^{j5t} - je^{-j5t}\right)$$

$$\boxed{y(t) = 2\delta(t) - 2e^{-2t}\sin(5t) \quad \text{for } t \geq 0}$$