ECE3084-L12 Transfer Functions and Stability

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We can use the LT & PFE to solve differential equations for LTI systems, but it may be painful (lots of work). Only want to do it if necessary.

We don't necessarily need the solution to analyze the input-ouput behavior or even to control a system.

Many LTI systems have this form:

$$\frac{d^n}{dt^n} y + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_1 \dot{y} + a_0 y = b x$$

x= input / forcing fn

Laplace Transfom:

$$s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_1 s Y(s) + a_0 Y(s) - P_{init}(s) = b X(s)$$

Initial conditions; order n-1 polynomial of s.

We assume $x(0^-) = 0$

$$(s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s) = P_{init}(s) + (b_m s^m + \cdots + b_1 s + b_0) X(s)$$

$$Y(s) = \underbrace{\frac{P_{init}(s)}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}}_{\substack{\text{Due to initial conditions.} \\ \text{"Zero-input response"}}} + \underbrace{\frac{b}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}}_{\substack{\text{Due to input or forcing function} \\ \text{"Zero-state response"}}} X(s)$$

In general, can analyze them independantly & sum for full response:

$$y(t) = y_{natural}(t) + y_{forced}(t)$$

If no initial conditions (no stored energy in the system), then we get the zero state response:

$$Y(s) = \underbrace{\frac{b}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}}_{H(s)} X(s)$$

$H(s) \leftarrow$ The transfer function

↳ "Transfers" inputs to outputs.

$$Y(s) = H(s) X(s)$$

Zero/pole/gain form

$$H(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

s terms in numerator can arise if there are derivatives of x on the RHS of the diff eq.

Using PFE:

$$Y(s) = H(s)X(s) = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{c_1}{s-p_1} + \cdots + \frac{c_n}{s-p_n}$$

(May also have repeated poles) the p_i are called "poles"

Each term of the PFE corresponds to a "mode" of the system:

$$\frac{c_i}{s-p_i} \overset{\mathcal{L}^{-1}}{\Longrightarrow} c_i \underbrace{e^{p_i t} u(t)}_{\text{This is a mode of the system}}$$

A type of term that will show up in the output

If the root is repeated, we get

$$\frac{c_{i1}}{s-p_i} + \cdots + \frac{c_{ik}}{(s-p_i)^k} \overset{\mathcal{L}^{-1}}{\Longrightarrow} p(t)\, e^{p_i t} u(t)$$

a polynomial of t

How does each mode in the output behave as $t \to \infty$?

The exponential term will always dominate any polynomial term from repeated roots for large t. So we only need to know how it behaves:

If p_i is real:

- $p_i > 0$: exponential blows up. $y \to \pm\infty$
- $p_i = 0$: the mode is 1 (i.e. a constant)
- $p_i < 0$: decaying exponential. Disappears as $t \to \infty$

If $p_i = \sigma + j\omega$ (complex pole, $\omega \neq 0$, come in conjugate pairs for real coefs in the diff eq).

- $\sigma > 0$: oscillation with exponentially increasing magnitude
- $\sigma = 0$: pure/constant oscillation.
- $\sigma < 0$: decaying oscillation, disappears as $t \to \infty$

The zero-input response and the transfer function from the zero state response have the same denominator, and thus the same poles, modes, and asymptotic behavior. Thus, we can ignore the zero-input response (initial conditions) when analyzing stability.

So we'll just concern ourselves with:

$$Y(s) = H(s)X(s)$$

Stability

For $Y(s) = H(s)X(s)$, poles can come from both $H(s)$ and $X(s)$. But we want to analyze the system, not a specific input. So we just look at $H(s)$.

Def: An input-output system is BIBO (bounded-input bounded-output) if a bounded input results in a bounded output. "Bounded" $\exists M$: $|f(t)| < M \ \forall t$

Inputs will "excite" (put energy into) the different modes of the system. If all of these modes decay, then nothing will blow up when you have a bounded input.

Theorem: Let the poles of $H(s)$ be given by $p_1, p_2, \ldots, p_n$. The system is BIBO if and only if $\mathrm{Re}(p_i) < 0, \ \forall i$.

What about $\mathrm{Re}(p_i) = 0$? Those modes don't blow up or decay.

Example: $H(s) = \frac{1}{s}$ (pole @ origin)

If $x(t) = Ae^{-\alpha t}u(t)$ for $\alpha > 0$ (decaying exponential)

$X(s) = \frac{A}{s+\alpha}$

$$Y(s) = H(s)X(s) = \frac{A}{s(s+\alpha)} = \frac{c_1}{s} + \frac{c_2}{s+\alpha}$$

$$c_1 = \left.\frac{A}{s+\alpha}\right|_{s=0} = \frac{A}{\alpha} \qquad c_2 = \left.\frac{A}{s}\right|_{s=-\alpha} = -\frac{A}{\alpha}$$

$$Y(s) = \frac{A}{\alpha}\left(\frac{1}{s} - \frac{1}{s+\alpha}\right)$$

$$y(t) = \frac{A}{\alpha}\left(1 - e^{-\alpha t}\right)u(t)$$ This is bounded.

But what if $x(t) = u(t)$? $X(s) = \frac{1}{s}$

$Y(s) = \frac{1}{s^2}$ $\qquad y(t) = t\,u(t)$ $\qquad$ NOT bounded.

$\therefore$ Not BIBO (must remain bounded for ALL bounded inputs).

This is called resonance – when you excite it at the same frequency that one of the modes naturally oscillates at. With no decay, the oscillation will grow larger & larger unbounded.

In this case, the frequency was zero (just a DC term).

Example: $H(s) = \frac{1}{X^2+4}$

If $x(t) = u(t)$:

$$Y(s) = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A = \frac{1}{s^2+4}\bigg|_{s=0} = \frac{1}{4}$$

$$1 = A(s^2+4) + Bs^2 + Cs$$
$$1 = (A+B)s^2 + Cs + A4$$
$$C = 0$$
$$A + B = 0 \qquad B = -A = -\frac{1}{4}$$

$$Y(s) = \left(\frac{1}{4}\right)\frac{1}{s} - \left(\frac{1}{4}\right)\frac{s}{s^2+4}$$

$$y(t) = \frac{1}{4}u(t) - \frac{1}{4}\cos(2t)u(t)$$ Bounded.

What if $x(t) = \sin(2t)u(t)$ $\quad X(s) = \frac{2}{s^2+4}$

$$Y(s) = \frac{2}{(s^2+4)^2}$$ ⟵ Ugly PFE... just look at form of sol'n.

$$= \frac{2}{(s-2j)^2(s+2j)^2} = \underbrace{\frac{A}{s-2j} + \frac{A^*}{s+2j}}_{\substack{\text{terms like} \\ \cos(2t+\phi) \\ \text{Bounded}}} + \underbrace{\frac{B}{(s-2j)^2} + \frac{B^*}{(s+2j)^2}}_{\substack{\text{terms like} \\ t\cos(2t+\phi) \\ \text{Unbounded}}}$$

So we cant have poles on the $j\omega$ axis & still be BIBO. The poles must be in the left hand side of the complex plane (s-plane).

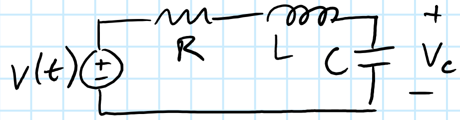
For BIBO systems — No longer need the disclaimer "as long as the limit exists" for the final value theorem (assuming bounded input).

Note: If a polynomial has real coefficients, then its roots will exhibit conjugate symmetry (i.e. complex roots will only occur in conjugate pairs). Because of this, the poles & zeros will occur in conjugate pairs if the diff eq has real coefficients.

– Makes sense. You need to add two counter-rotating complex sinusoids to cancel out the imaginary parts and get a real sinusoid.

Example: Series RLC circuit:

$$V_c(s) = \underbrace{\frac{1}{LCs^2 + RCs + 1}}_{H(s)} V(s)$$



Poles are at $s = \dfrac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}$

We know $R > 0$, $C > 0$, $L > 0$

If $R^2C^2 - 4LC > 0$ (overdamped)

We know $|RC| > |\sqrt{R^2C^2 - 4LC}|$ because the $-4LC$ can only make the right hand side smaller. $\therefore$ the $-RC$ makes the real part negative.

If $R^2C^2 - 4LC < 0$

then the poles are $s = \dfrac{-RC \pm j\sqrt{4LC - R^2C^2}}{2LC}$

which has a negative real part.

(Underdamped) Conjugate poles show up as sinusoid term in sol'n — oscillation.

If you remove the Resistor (set $R=0$), then the poles are purely imaginary:

$$s = \frac{\pm j\sqrt{4LC}}{2LC} = \pm \frac{j}{\sqrt{LC}}$$

so the resonant freq is $\dfrac{1}{\sqrt{LC}}$

Do you know what this circuit does?

2^{nd} order lowpass filter.

Inductor looks like open circuit @ high freq.

Cap looks like short circuit @ high freq.

All the high freq voltage gets dropped across the inductor.

At low freq:

Inductor $\rightarrow$ short circuit

Cap $\rightarrow$ Open circuit

All low freq voltage gets dropped across the cap. Gain = 1