ECE3084-L13 Frequency Response of 2nd Order Systems

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First order system:

$$\dot{y} - py = kx \quad \xrightarrow{\mathcal{L}} \quad sY(s) - y(0^-) - pY(s) = k\,X(s)$$

Ignore init cond for steady state freq resp

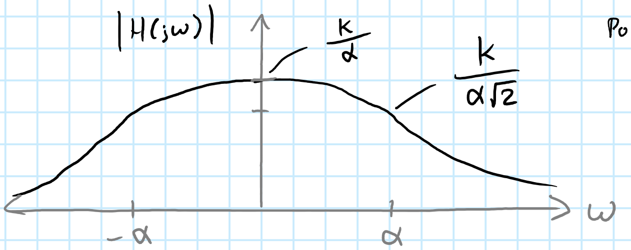
$$(s-p)\,Y(s) = k\,X(s)$$

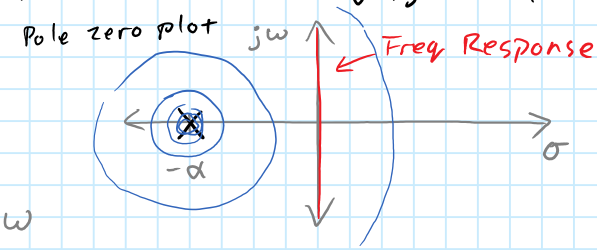
$$H(s) = \frac{Y(s)}{X(s)} = \frac{k}{s-p}$$

To be BIBO stable, must have $p < 0$. (We're also assuming p is real — you'll typically have real coefficients in the diff eq in real life.)

Let $p = -\alpha$. $\quad H(s) = \frac{k}{s+\alpha}$

Plug in $s = j\omega$ to get freq response: $\quad H(j\omega) = \frac{k}{\alpha + j\omega} \quad |H(j\omega)| = \frac{k}{\sqrt{\alpha^2+\omega^2}}$





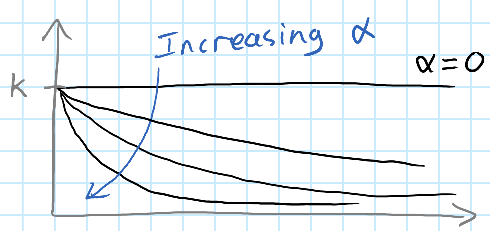
Cutoff freq

($\frac{1}{2}$ of max output power)

What contour lines might look like.

As the pole gets closer to the origin (α decreasing), the low pass filter gets narrower and sharper

Impulse response: $\quad h(t) = ke^{-\alpha t}u(t)$



Higher $\alpha \Rightarrow$ faster decay.

$$\alpha = \frac{1}{\tau}$$

Time constant

Poles must be in LHP (i.e. it's BIBO stable) to calculate freq resp.

If they're not, then the $j\omega$ axis is not in the ROC and the freq resp does not exist.

Second order systems:

2 common notations for the denominator of $H(s) = \frac{N(s)}{D(s)}$

$$D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$D(s) = s^2 + \frac{\omega_n}{Q} s + \omega_n^2$$

ζ = Damping Ratio

ω_n = Undamped natural frequency

Q = Quality Factor

Related by $Q = \frac{1}{2\zeta}$ or $\zeta = \frac{1}{2Q}$

Control systems engineers — Prefer damping ratio (ζ), worry about it being too low (causing oscillation).

Filter designers — Prefer quality factor (Q), usually desire high Q (gives a sharper cutoff / taller peak)

Use quadratic formula to find poles:

$$s_p = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) = \omega_n\left(-\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}\right)$$

Three cases:

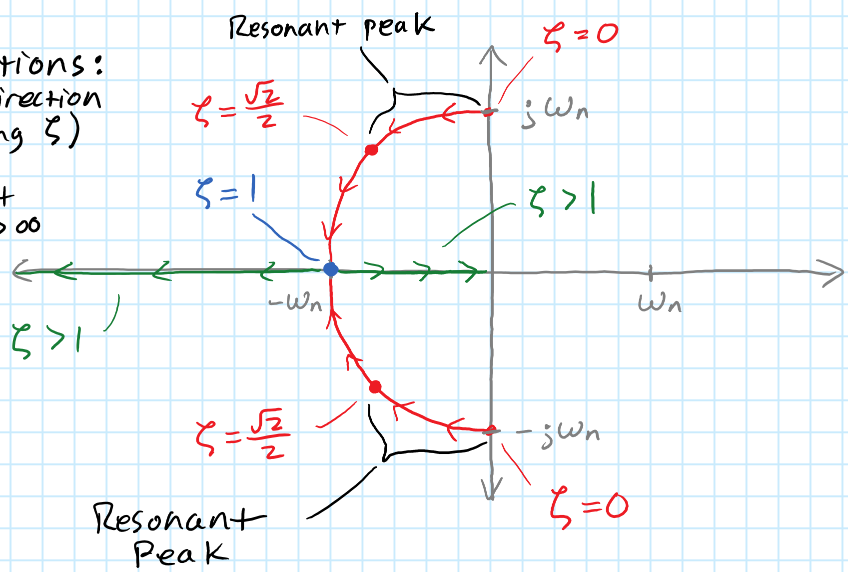
Case 1: $\zeta < 1$ or $Q > 1/2$ — The square root is imaginary and the poles form a complex conjugate pair. <u>Underdamped.</u>

Case 2: $\zeta = 1$ or $Q = 1/2$ — Both poles lie at $s_p = \omega_n$ (repeated pole). <u>Critically damped.</u>

Case 3: $\zeta > 1$ or $Q < 1/2$ — The square root is real & both poles lie on the negative real axis (b/c $\zeta > \sqrt{\zeta^2 - 1}$). <u>Overdamped.</u>

Pole locations:
(Arrows in direction of increasing ζ)

Hold ω_n constant
Vary ζ from $0 \to \infty$



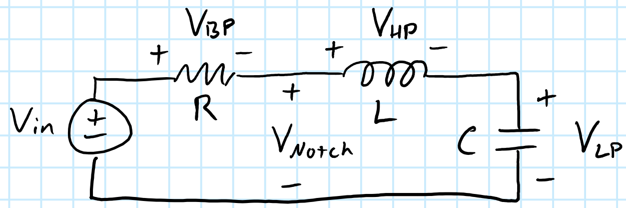
Canonical filter transfer functions:

Low pass: unity gain @ $\omega = 0$ $\quad H_{2LP}(s) = \dfrac{\omega_n^2}{D(s)}$

These are just voltage dividers across different components.

Band pass: unity gain @ $\omega = \omega_n$ $\quad H_{2BP}(s) = \dfrac{2\zeta\omega_n s}{D(s)} = \dfrac{(\omega_n/Q)\, s}{D(s)}$

High pass: unity gain @ $\omega = \infty$ $\quad H_{2HP}(s) = \dfrac{s^2}{D(s)}$



Low Pass Filter (2nd order)

$$H_{2LP}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{s^2 + \frac{\omega_n}{Q}s + \omega_n^2}$$

Can see that $H_{2LP}(j0) = 1$ and $H_{2LP}(j\infty) = 0$

Is there a peak in between? Minimize $|D(s)|$

$$\underset{s=j\omega}{\arg\min}\, |D(s)| = \underset{s=j\omega}{\arg\min}\, |s^2 + 2\zeta\omega_n s + \omega_n^2|$$

$$= \underset{\omega}{\arg\min}\, |-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2|$$

$$= \underset{\omega}{\arg\min}\, \sqrt{\underbrace{(\omega_n^2-\omega^2)^2}_{\text{real part}} + \underbrace{(2\zeta\omega_n\omega)^2}_{\text{imaginary part}}}$$

$$= \underset{\omega}{\arg\min}\left((\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right)$$

$$\frac{d}{d\omega}\left((\omega_n^2-\omega^2)^2 + (2\zeta\omega_n\omega)^2\right) = 0$$

$$2(\omega_n^2-\omega^2)(-2\omega) + 2(2\zeta\omega_n\omega)(2\zeta\omega_n) = 0$$

$$4\omega\left((\omega^2-\omega_n^2) + 2\zeta^2\omega_n^2\right) = 0$$

$$\omega^2 - \omega_n^2 = -2\zeta^2\omega_n^2$$

$$\omega^2 = \omega_n^2 - 2\zeta^2\omega_n^2 = \omega_n^2(1-2\zeta^2)$$

$$\boxed{\omega_{LP,r} = \omega_n\sqrt{1-2\zeta^2} \quad \text{Resonant Frequency}}$$

This doesn't make sense if the square root is imaginary.

For it to be real, we need $1-2\zeta^2 \geq 0 \Rightarrow \zeta \leq \frac{1}{\sqrt{2}}$ (or $\frac{\sqrt{2}}{2}$)

A resonant peak only exists if $\zeta < \frac{1}{\sqrt{2}}$

Resonant freq $\rightarrow 0$ as $\zeta \rightarrow \frac{1}{\sqrt{2}}$

Can find magnitude of peak by plugging $\omega_{LP,r}$ into $|H(j\omega)|$

$$|H_{2LP}(j\omega_{LP,r})|^2 = \frac{\omega_n^4}{(\omega_n^2-\omega_{LP,r}^2)^2+(2\zeta\omega_n\omega_{LP,r})^2} \qquad \omega_{LP,r}=\omega_n\sqrt{1-2\zeta^2}$$

$$= \frac{\omega_n^4}{(\omega_n^2-\omega_n^2(1-2\zeta^2))^2+(2\zeta\omega_n^2\sqrt{1-2\zeta^2})^2}$$

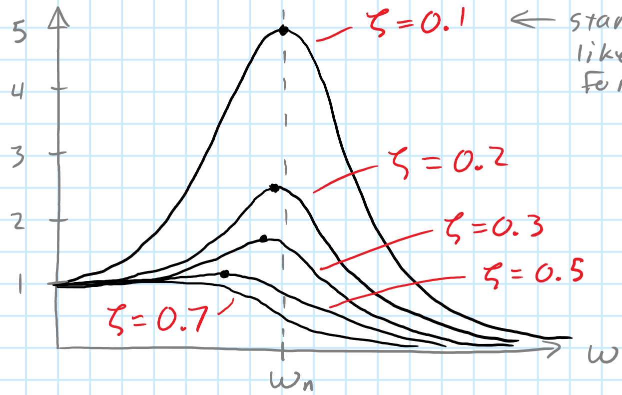
$$= \frac{\omega_n^4}{\omega_n^4(1-(1-2\zeta^2))^2+4\zeta^2\omega_n^4(1-2\zeta^2)}$$

$$= \frac{1}{4\zeta^4+4\zeta^2-8\zeta^4}$$

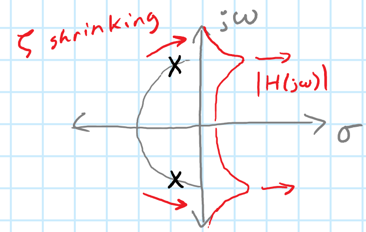
$$= \frac{1}{4\zeta^2(1-\zeta^2)}$$

$$|H_{2LP}(j\omega_{LP,r})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Only a function of ζ!



← starts looking more like a band pass filter for small ζ.



High pass filter (2nd order)

$$H_{2HP}(s) = \frac{s^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

It's clear that $H_{2HP}(j0)=0$ and $H_{2HP}(j\infty)=1$

Similar process to find resonant peak & its magnitude:

$$\omega_{HP,r} = \omega_n \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Peak only exists if $\zeta < \frac{1}{\sqrt{2}}$

Resonant freq $\to \infty$ as $\zeta \to \frac{1}{\sqrt{2}}$

$$|H_{2HP}(j\omega_{HP,r})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Band pass filter (2nd order)

$$H_{2BP}(s) = \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$|H_{2BP}(j\omega)|^2 = \frac{4\zeta^2\omega_n^2\omega^2}{(\omega_n^2-\omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}$$ divide everything by $\omega_n^2\omega^2$

$$= \frac{4\zeta^2}{\underbrace{\left(\frac{\omega_n}{\omega} - \frac{\omega}{\omega_n}\right)^2}+4\zeta^2}$$

Peak occurs when this term is zero, which happens at $\omega = \omega_n$. This gives unity gain at this frequency.

Low & high freqs with same gain (ω_L & ω_U):

$\omega_n = \sqrt{\omega_L \omega_U}$

Can show by equating $\left(\frac{\omega_n}{\omega_L} - \frac{\omega_L}{\omega_n}\right) = -\left(\frac{\omega_n}{\omega_U} - \frac{\omega_U}{\omega_n}\right)$

ω_n is the center frequency in a logarithmic sense:

$$\ln \omega_n = \frac{\ln \omega_L + \ln \omega_U}{2}$$

Bandwidth: $BW = \omega_{U\,1/2} - \omega_{L\,1/2}$ $\Leftarrow$ $\frac{1}{2}$ power frequencies

$$Q = \frac{\omega_n}{BW}$$

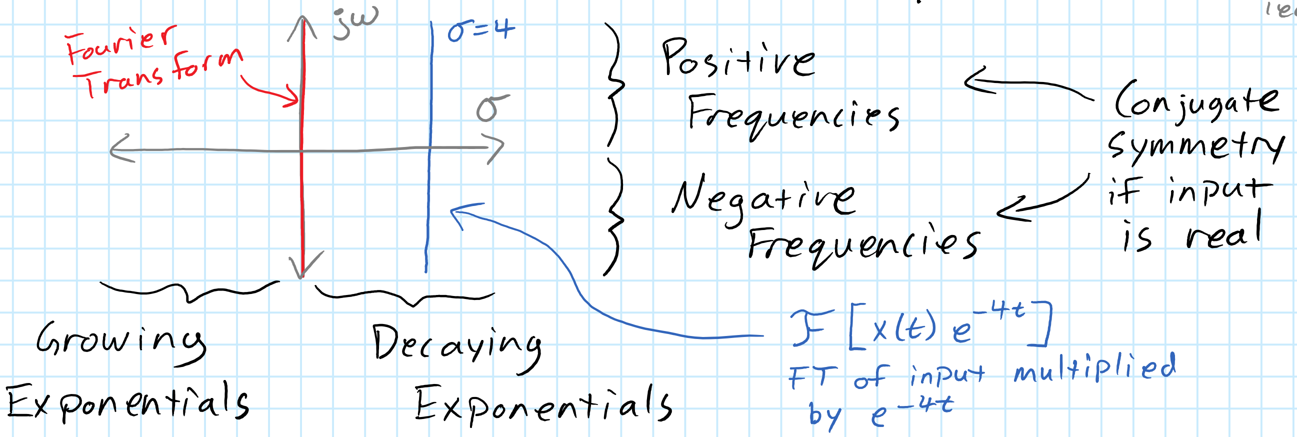
We perceive audio frequencies logarithmically, so controlling Q automatically adapts the BW so that it perceptually remains the same as ω_n is swept.

Butterworth Filter – Choose poles to get the steepest possible cutoff without creating a peak. Happens when $\zeta = \frac{1}{\sqrt{2}}$ (poles at $\pm 45°$ off the negative real axis in the s-plane).

EXTRA material for better intuition (conjugate symmetry & PFE).

Should probly move this to the prev lecture

Remember this? (Specifically the conjugate symmetry part?)



Poles & zeros will have conjugate symmetry as well (For diff eqs with real coefs)

Conjugate symmetric poles inverse transform into real sinusoids (i.e. scaled/shifted cosine) multiplied by an exponential (which is just a constant if on the $j\omega$ axis).

Use conjugate symmetry to make PFE easier:

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{A}{s-j2} + \frac{A^*}{s+j2} + \frac{B}{s-j3} + \frac{B^*}{s+j3}$$

Only need 2 equatios / applications of residue method

$$A = \left.\frac{1}{(s+j2)(s^2+9)}\right|_{s=j2} = \frac{1}{j4(-4+9)} = \frac{1}{j20} \qquad A^* = -\frac{1}{j20}$$

$$B = \left.\frac{1}{(s^2+4)(s+j3)}\right|_{s=j3} = \frac{1}{(-9+4)j6} = \frac{1}{-j30} \qquad B^* = \frac{1}{j30}$$

Then manipulate exponential's in time domain to turn into sin/cos.

$$\frac{1}{j20}e^{j2t}u(t) - \frac{1}{j20}e^{-j2t}u(t) - \frac{1}{j30}e^{j3t} + \frac{1}{j30}e^{-j3t}$$

$$\left(\frac{1}{10}\sin(2t) - \frac{1}{15}\sin(3t)\right)u(t)$$

Could also expand as $\displaystyle\frac{1}{(s^2+4)(s^2+9)} = \frac{A}{s^2+4} + \frac{B}{s^2+9}$

Not sure if this would work for other sys, like $\frac{1}{(s^2+4)(s+3)}$ but could still set up sys of eqns.

$$A = \left.\frac{1}{s^2+9}\right|_{s^2=-4} = \frac{1}{-4+9} = \frac{1}{5} \qquad B = \left.\frac{1}{s^2+4}\right|_{s^2=-9} = -\frac{1}{5}$$

$$\frac{1}{5}\left(\frac{\frac{1}{2}(2)}{s^2+2^2}\right) - \frac{1}{5}\left(\frac{\frac{1}{3}(3)}{s^2+3^2}\right) \xrightarrow{\mathcal{L}^{-1}} \left(\frac{1}{10}\sin(2t) - \frac{1}{15}\sin(3t)\right)u(t)$$

If the poles also had a real part, you'd get an exponential times the sin/cos.