ECE3084-L14 Step Response

Wednesday, March 22, 2017 10:44 PM

Impulse responses characterize LTI systems mathematically, but:

- We can't usually generate a realistic impulse.
- Even if we could, it would likely cause nonlinear effects

(Remember: our equations are <u>models.</u> Usually only valid within a certain regime, and even then are still approximations / make assumptions. The question is, are they "good enough"?)

What we usually measure instead is the <u>step response.</u>

$\mathcal{L}[\delta(t)] = 1$ (for all s) $\rightarrow$ Excites / puts energy into every possible mode of the system.

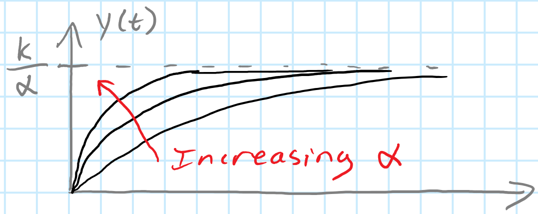
$\mathcal{L}[u(t)] = \frac{1}{s}$ $\rightarrow$ Still excites all the modes, just not evenly like the impulse response did.

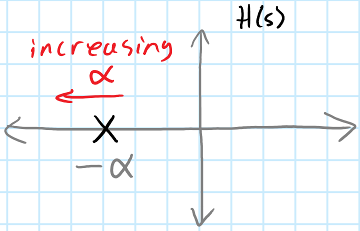
1st Order Systems

- $H(s) = \frac{k}{s+\alpha}$ (low pass filter) $X(s) = \frac{1}{s} = \mathcal{L}[u(t)]$

$Y(s) = \frac{k}{s(s+\alpha)} = \frac{A}{s} + \frac{B}{s+\alpha}$ $A = \frac{k}{\alpha}$ $B = -\frac{k}{\alpha}$

$y(t) = \frac{k}{\alpha}(1 - e^{-\alpha t})\,u(t)$



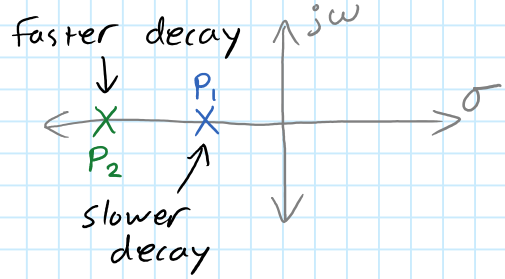


- $H(s) = \frac{s}{s+\alpha}$ (High pass filter)

$Y(s) = \frac{s}{s(s+\alpha)} = \frac{1}{s+\alpha}$ $y(t) = e^{-\alpha t}u(t)$

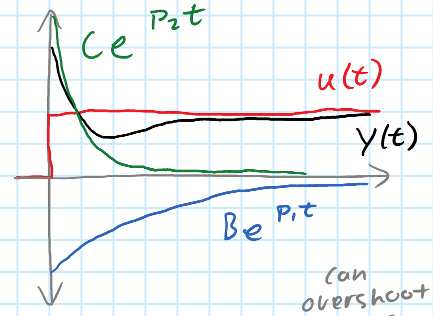
- Perfectly fast rise time
- Ignores the step input in steady state.
- Kind of like an edge detector.

2^{nd} Order Low Pass Filter



$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Case 1: Overdamped — 2 real poles on the negative real axis. $\zeta > 1$

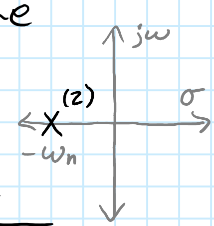


$$Y(s) = H(s)X(s) = H(s)\left(\frac{1}{s}\right) = \frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2}$$

$$A = H(s)\Big|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1 \quad \text{passive filter}$$

$$y(t) = \underline{u(t)} + \underline{Be^{p_1 t}u(t)} + \underline{Ce^{p_2 t}u(t)}$$

The slower term (or the term with the pole closer to the imaginary axis) tends to dominate the behavior of the response. (The faster term dies out more quickly.)



Case 2: Critically damped. $\zeta = 1$, $p_1 = p_2 = -\omega_n$

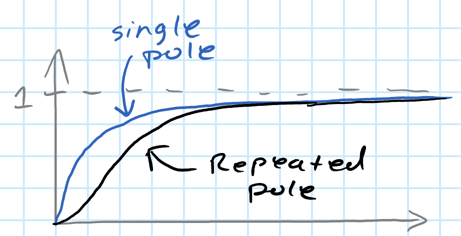
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}X(s) = \frac{\omega_n^2}{(s+\omega_n)^2}\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$= \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2} \qquad A = \frac{\omega_n^2}{(s+\omega_n)^2}\Big|_{s=0} = 1$$

$$C = \frac{\omega_n^2}{s}\Big|_{s=-\omega_n} = -\omega_n \qquad B = \frac{1}{1!}\frac{d}{ds}\frac{\omega_n^2}{s}\Big|_{s=-\omega_n} = (-1)\frac{\omega_n^2}{s^2} = -1$$

$$y(t) = u(t) - e^{-\omega_n t}u(t) - \omega_n t e^{-\omega_n t}u(t)$$

$$= u(t) - (1+\omega_n t)e^{-\omega_n t}u(t)$$



Compare to 1^{st} order:

$$y(t) = u(t) - e^{-\alpha t}u(t)$$

Slower response. The growing $(1+\omega_n t)$ term counteracts the decay of the $e^{-\omega_n t}$ term initially, slowing down the response.

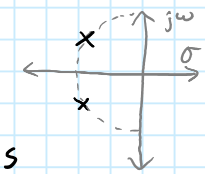
Can also view it as the high freqs rolling off more quickly (−40 dB/decade instead of −20 dB/decade) — it can't respond as quickly in the time domain with less of the higher frequencies.

Case 3: Underdamped. $0 < \zeta < 1$, complex poles.

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

↳ complete the square ↗

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the <u>damped frequency</u>



$$Y(s) = \frac{\omega_n^2}{s((s+\zeta\omega_n)^2 + \omega_d)} = \frac{A}{s} + \frac{Bs + C}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$A = \frac{\omega_n^2}{\omega_n^2} = 1$$

clear fractions: $\omega_n^2 = A((s+\zeta\omega_n)^2 + \omega_d^2) + Bs^2 + Cs$

s^2: $0 = A + B \quad \Rightarrow \quad B = -A = -1$

s^1: $0 = A(2\zeta\omega_n) + C \quad C = -2\zeta\omega_n$

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} = \frac{1}{s} - \frac{(s+\zeta\omega_n) + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

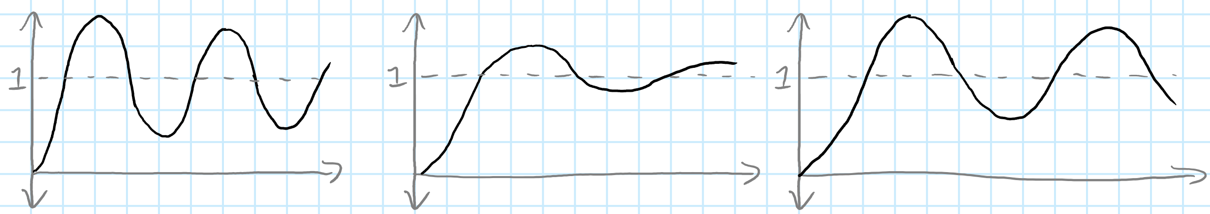
$$y(t) = u(t) - e^{-\zeta\omega_n t}\underbrace{\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)}_{\text{same freq. Can combine into single cos (see book)}}u(t)$$

$\sin(\omega_d t) = \cos(\omega_d t - \frac{\pi}{2})$

$$y(t) = \left(1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)\right)u(t) \qquad \phi = \tan^{-1}\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{\omega_n}{\omega_d} = \frac{\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{1}{\sqrt{1-\zeta^2}}$$

Transient response is a decaying sinusoid



Small ζ | Bigger ζ | Smaller ω_n (lower freq)

Fewer oscillations / lower freq
Less overshoot

↑ (ω_d smaller relative to ω_n)

Observations:

1. Step response has "ripples" (oscillates) if underdamped at all ($\zeta < 1$), but a resonant peak in the freq domain only exists for $\zeta < \frac{1}{\sqrt{2}}$ (meaning output ripples can have bigger magnitude than input).
2. The resonant freq $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ is always less than $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the freq of oscillation in the in the time domain. Both approach ω_n for small ζ.

ω_r = location of peak in freq response.

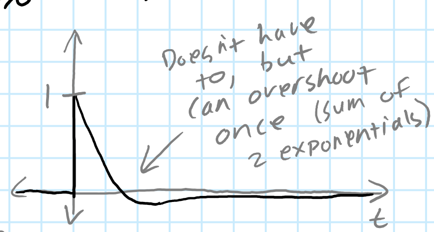
2nd Order Highpass Filter

$$H_{2HP}(s) = \frac{s^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y_{2HP}(s) = \frac{H_{2HP}(s)}{s} = \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

IVT: $y(0^+) = \lim_{s\to\infty} sY(s) = 1$ FVT: $y(\infty) = \lim_{s\to 0} sY(s) = 0$

Case 1: Overdamped, two real poles:

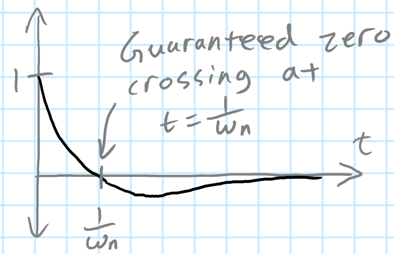


$$Y_{2HP}(s) = \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$

$$y_{2HP}(t) = Ae^{p_1 t} + Be^{p_2 t}$$

(no step fn this time, like for the LPF)

Case 2: Critically Damped, repeated poles, $\zeta = 1$:

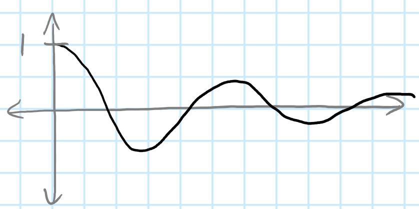


$$Y_{2HP}(s) = \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A}{s+\omega_n} + \frac{B}{(s+\omega_n)^2}$$

$$B = s\Big|_{s=-\omega_n} = -\omega_n \qquad s = A(s+\omega_n) - \omega_n$$
$$A = 1$$

$$y_{2HP}(t) = e^{-\omega_n t}u(t) - \omega_n t e^{-\omega_n t}u(t)$$

Case 3: Underdamped



$$Y_{2HP} = \frac{(s+\zeta\omega_n) - \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$y_{2HP}(t) = e^{-\zeta\omega_n t}\underbrace{\left(\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)}_{\text{Same term as for LPF}}u(t)$$

$$= \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\,u(t) \qquad \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

2^{nd} Order Bandpass Filter

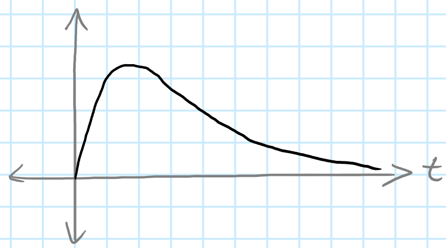
$$H_{2BP}(s) = \frac{2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad Y_{2BP}(s) = \frac{1}{s} H_{2BP}(s) = \frac{2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

IVT: $$y_{2BP}(0) = \lim_{s\to\infty} s\,Y_{2BP}(s) = \lim_{s\to\infty} s\,\frac{H_{2BP}(s)}{s} = 0$$

FVT: $$y_{2BP}(\infty) = \lim_{s\to 0} s\,Y_{2BP}(s) = H_{2BP}(0) = \frac{0}{\omega_n^2} = 0$$

Case 1: Overdamped, real poles

$$Y_{2BP}(s) = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$



$$y_{2BP}(t) = Ae^{p_1 t}u(t) + Be^{p_2 t}u(t)$$

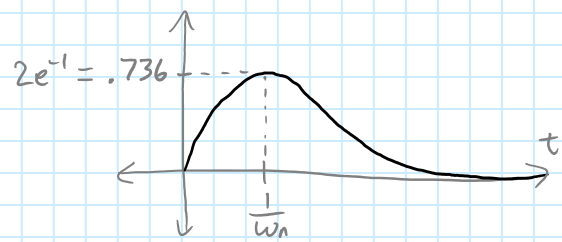
For $y_{2BP}(0) = 0$, must have $A = -B$

"Fast" pole controls rise time & has the negative coefficient.

"Slow" pole controls the decay & has the positive coefficient.

Case 2: Critically damped, $\zeta = 1$

$$Y_{2BP}(s) = \frac{2\omega_n}{(s+\omega_n)^2}$$



$$y_{2BP}(t) = 2\omega_n t e^{-\omega_n t} u(t)$$

Can find peak by taking derivative & setting equal to zero.

Case 3: Underdamped

$$Y_{2BP}(s) = \frac{2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$y_{2BP}(t) = \frac{2\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t)\,u(t)$$

$$= \frac{2\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)\,u(t)$$

plug in $\omega_d = \omega_n\sqrt{1-\zeta^2}$ Decaying sine.