ECE3084-L15 LT and ZT

Monday, April 3, 2017 11:43 PM

Our model of sampling:

$$x(t) \longrightarrow \bigotimes \longrightarrow x_s(t) = \sum_{n=0}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

$$\uparrow$$

$$\sum_{n=0}^{\infty} \delta(t-nT_s)$$

Letting $x[n] = x(nT_s)$, $\quad x_s(t) = \sum_{n=0}^{\infty} x[n]\,\delta(t-nT_s)$

Take LT: $$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x[n]\,\delta(t-nT_s)\,e^{-st}\,dt$$

$$= \sum_{n=0}^{\infty} x[n] \int_0^{\infty} \delta(t-nT_s)\,e^{-st}\,dt$$

Apply sifting property

$$= \sum_{n=0}^{\infty} x[n]\,e^{-snT_s}$$

$$X_s(s) = \sum_{n=0}^{\infty} x[n]\,(e^{sT_s})^{-n}$$

Recall the Z-transform from 2026: $X(z) = \sum_{n=0}^{\infty} x[n]\,z^{-n}$

These are the same if $\boxed{z = e^{sT_s}}$ ← Mapping from the s-plane to the z-plane.

This mapping can be inverted: $s = \dfrac{\ln(z)}{T_s}$

ln of negative/complex numbers? → Most computational packages handle it fine, & return $-\infty$ for $\ln(0)$, which works fine for our purposes.

Sampling caused periodic replication in the Fourier domain. What about the s domain?

$$X_s(s + j\underbrace{k}_{\text{integer}}\omega_s) = X_s\left(s + j\tfrac{2\pi}{T_s}k\right)$$

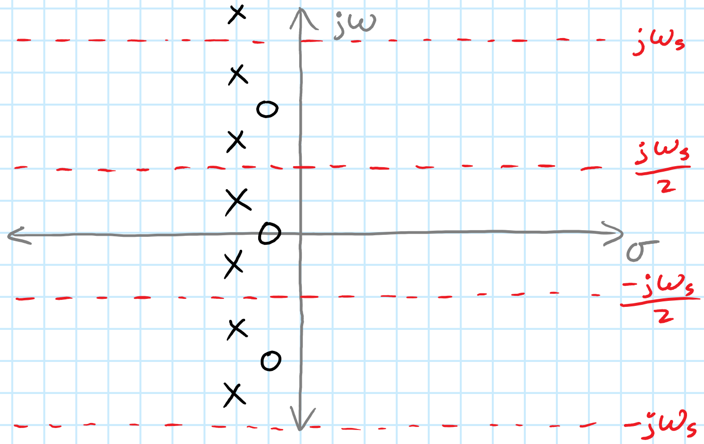
$$= \sum_{n=0}^{\infty} x[n]\,e^{-(s + j\frac{2\pi}{T_s}k)nT_s}$$

$$= \underbrace{\sum_{n=0}^{\infty} x[n]\,e^{-snT_s}}_{= X_s(s)}\ \underbrace{e^{-j2\pi kn}}_{=1 \text{ for integer } n \,\&\, k}$$

$X_s(s + jk\omega_s) = X_s(s)$ The LT is periodic vertically in the s-plane.

Can visualize as identical horizontal strips stacked vertically.

All poles & zeros get replicated (and can alias into neighboring strips if ω_s isn't high enough).

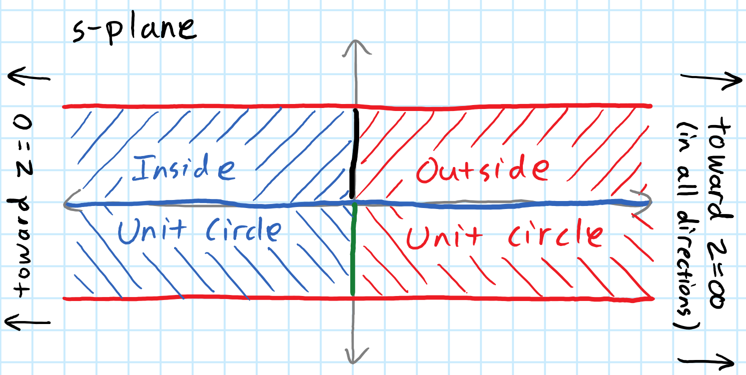


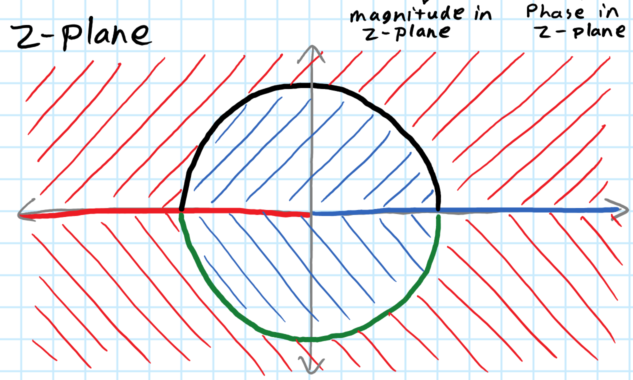
The mapping $z = e^{sT_s}$ uniquely maps a single strip to the entire z-plane.

The inverse mapping $s = \frac{\ln(z)}{T_s}$ gives the appropriate point in the central strip between $-\frac{j\omega_s}{2}$ and $\frac{j\omega_s}{2}$ (it's not unique).

What does the mapping look like?

$$z = e^{sT_s} = \underbrace{e^{Re[sT_s]}}_{\text{magnitude in z-plane}} \underbrace{e^{j\,Im[sT_s]}}_{\text{Phase in z-plane}}$$





Frequency response: plug $s = j\omega$ in to Laplace Transform

$$z = e^{sT_s}\Big|_{s=j\omega} = e^{j\omega T_s} = e^{j\frac{\omega}{F_s}} = e^{j\hat{\omega}}$$ ← This corresponds to the unit circle in the z-plane.

where $\hat{\omega} = \frac{\omega}{F_s}$ is the normalized frequency.

Plug $s = j\omega$ into the LT to get the CTFT: $H(j\omega)$ lies along $j\omega$ axis.

Plug $z = e^{j\hat{\omega}}$ into the ZT to get the DTFT: $H(e^{j\hat{\omega}})$ lies along the unit circle as you go around it.

The Z-transform is a special case of the Laplace transform with a convenient, custom notation.

Note: Stable poles for Z-transform are inside the unit circle in the z-plane. Corresponds to left half of s-plane.

What if we want to implement a continuous time filter digitally?

Could create an approximation:

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{ADC}\\(T_s)\end{matrix}} \xrightarrow{x[n]} \boxed{H(e^{j\hat{\omega}})} \xrightarrow{y[n]} \boxed{\begin{matrix}\text{DAC}\\(T_s)\end{matrix}} \longrightarrow y(t)$$

Matched z-transform method (AKA Pole-zero mapping/matching):

Use $z = e^{sT_s}$ to map all poles & zeros from the s-domain to the z-domain, giving $H_{pzm}(z)$

The effective frequency response of the full cascade (including ADC/DAC) is:

$$H_{eff}(j\omega) = \begin{cases} H_{pzm}\left(e^{j\frac{\omega}{F_s}}\right) & \text{for } |\omega| < \frac{\omega_s}{2} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Frequencies above $\omega_s/2$ result in aliasing, & cause non-linear effects in the full cascade. The idea of a frequency response is no longer valid without linearity.

$H_{pzm}\left(e^{j\frac{\omega}{F_s}}\right) \approx H(j\omega)$ (They're not identical)

Gets very close for large ω_s ($\omega_s \gg \omega_c$)

See example of converting a 2nd order Butterworth filter in the book.

There are other methods of approximating s-domain systems in the z-domain (e.g. "bilateral transformation"). Beyond scope of this class.