ECE3084-L19 Parseval's Theorem

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Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Note: The change of variables $\omega = 2\pi f$ would make the $1/2\pi$ go away.

Energy in time domain = Energy in frequency domain

<u>If</u>: $x(t)$ and $X(j\omega)$ are "square integrable" (both integrals must converge to finite numbers).

→ This theorem does not apply to things like δ fns & sinusoids.

View δ fn as the limit of an area 1 rect as its width goes to zero:

$$\int_0^1 1^2 \, dt = 1 \qquad \int_0^{1/2} 2^2 \, dt = \frac{4}{2} = 2 \qquad \int_0^{1/h} h^2 \, dt = \frac{h^2}{h} = h \qquad \int_{0^-}^{0^+} \delta^2(t) \, dt \overset{?}{=} \infty$$

Version for periodic functions, where a_k are the FS coefs of $x(t)$:

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

<u>If</u> both sides are finite.

Generalized form:

$$\int_{-\infty}^{\infty} f(t) g^*(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) G^*(j\omega) \, d\omega$$

Periodic (a_k & b_k are FS coefs of $f(t)$ & $g(t)$):

$$\frac{1}{T_0} \int_{T_0} f(t) g^*(t) \, dt = \sum_{k=-\infty}^{\infty} a_k b_k^*$$

Useful because sometimes integrals we want to take match this form, but are much easier to compute in one domain compared to the other (i.e. integrating over a sinc vs. integrating over a rect).

Derivation: Start with multiplication property:

$$\mathcal{F}[f(t)g(t)] = \frac{1}{2\pi} F(j\omega) * G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\lambda)\, G(j(\omega-\lambda))\, d\lambda$$

$$\mathcal{F}[f(t)g^*(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\lambda)\, G^*(j(\lambda-\omega))\, d\lambda$$

conjugation property

$$\int_{-\infty}^{\infty} f(t)g^*(t)e^{-j\omega t}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\lambda)\, G^*(j(\lambda-\omega))\, d\lambda$$

Definition of FT

Evaluate at $\omega = 0$

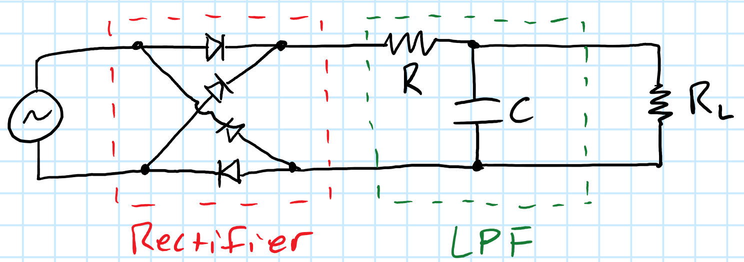
$$\int_{-\infty}^{\infty} f(t)g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\lambda)\, G^*(j\lambda)\, d\lambda$$

Change λ to ω to get Parseval's.

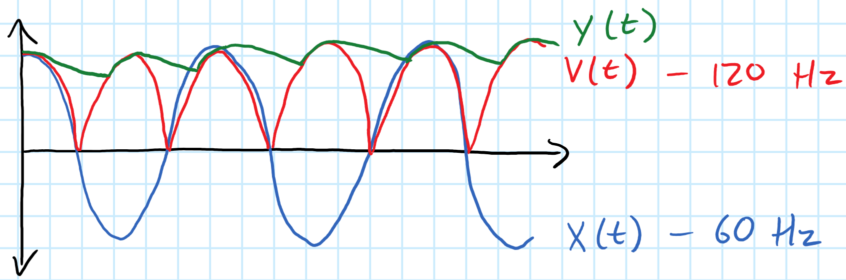
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, G^*(j\omega)\, d\omega$$

Example: Voltage Rectifier (AC to DC conversion / power supply)

$$x(t) \to \boxed{\text{Rectifier}} \xrightarrow{v(t)} \boxed{\text{LPF}} \to y(t)$$



Could have an inductor instead of a resistor in the LPF, but this is less common. These inductors can be big, heavy, expensive, and not very ideal (in behavior).



How much ripple is there in the output $y(t)$?

Quantify as $\dfrac{RMS_{harmonics}}{\text{DC value}}$ $\qquad \dfrac{\text{RMS amplitude of AC component}}{\text{RMS amplitude of DC}}$

Before the LPF:

FS coefficients of rectified cosine: $a_k = \frac{2}{\pi(1-4k^2)}$ Given by Professor Lanterman

$f_0 = 120$ Hz $\quad \omega_0 = 240\pi$

$T_0 = \frac{1}{120}$

$$RMS^2 = \frac{1}{T_0}\int_{T_0} \Big| \underbrace{\sum_{k\neq 0} a_k e^{-jk240\pi t}}_{\text{FS synthesis sum for rectified cosine minus the DC term } (k=0).} \Big|^2 dt$$

$a_0 = \frac{2}{\pi} = 0.6366 \leftarrow$ DC value

$a_1 = a_{-1} = \frac{2}{-3\pi} = -0.2122$

$a_2 = a_{-2} = \frac{2}{-15\pi} = -0.0424$

$a_3 = a_{-3} = \frac{2}{-35\pi} = -0.0182$

And keep getting smaller...

AC Harmonics

Approx with 6 terms:

$k = \pm 1, \pm 2, \pm 3$

By Parseval's:

$$RMS^2 = \sum_{k\neq 0} |a_k|^2 \approx 2\left((-0.2122)^2 + (-0.0424)^2 + (-0.0182)^2\right)$$

$$RMS^2 \approx 0.0944$$

$$RMS \approx \sqrt{0.0944} = 0.3072$$ (0.3001 from the fundamental)

$$\% \text{ distortion} = \frac{.3072}{.6366} \times 100\% = 48\%$$ (that's a lot)

After the LPF: Voltage divider: $H(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \frac{\left(\frac{j\omega}{R}\right)}{\left(\frac{j\omega}{R}\right)} = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$

$$|H(j\omega)| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j240\pi k t} = \sum_{k=-\infty}^{\infty} a_k H(j240\pi k)\, e^{j240\pi k t}$$

Let $C = 1\,\mu F$ and $R = 10\,k\Omega$

Evaluate $|H(j\omega)|$ for $k = \pm 0, 1, 2, 3$ & calc $|b_k|$

$|H(j0)| = 1$ $\quad |b_0| = (.6366)(1) = .6366$

$|H(j240\pi)| = 0.1315$ $\quad |b_1| = (.2122)(.1315) = .0279$

$|H(j480\pi)| = 0.0662$ $\quad |b_2| = (.0424)(.0662) = .0028$

$|H(j720\pi)| = 0.0442$ $\quad |b_3| = (.0182)(.0442) = .0008$

RMS ≈ 0.0412 (0.0395 from fundamental)

$$\%\ \text{distortion} = \frac{0.0412}{0.6366} \times 100\,\% = 6.5\,\%$$

much smaller

May not be good enough for some applications (i.e. audio, where 120 Hz is very audible). Could design a better filter.