Fixed-Charge Transportation with Product Blending

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Abstract

Numerous planning models within the chemical, petroleum, and process industries involve coordinating the movement of raw materials in a distribution network so that they can be blended into final products. The uncapacitated fixed-charge transportation problem with blending (FCTPwB) studied in this paper captures a core structure encountered in many of these environments. We model the FCTPwB as a mixed-integer linear program and derive two classes of facets, both exponential in size, for the convex hull of solutions for the problem with a single consumer and show that they can be separated in polynomial time. Furthermore, we prove that in certain situations these classes of facets, along with the continuous relaxation of the original constraints, yield a description of the convex hull. Finally, we present a computational study that demonstrates that these classes of facets are effective in reducing the integrality gap and solution time for more general instances of the FCTPwB with arc capacities and multiple consumers.

1 Introduction and Problem Statement

In many operational and planning models within the chemical, petroleum, and process industries, a common issue involves blending raw materials with varying attributes and concentration levels into homogeneous intermediate or end products. Blending raw materials affords an organization the opportunity to realize sizable cost savings, while meeting demand for an array of final products and satisfying pre-determined specification requirements for each type of product [20]. The inherent flexibility of the blending process can be exploited to optimize the allocation and transportation of raw materials to production facilities. This motivates the study of what we call the *fixed-charge transportation problem with product blending* (FCTPwB). The feasible region of this problem arises as a substructure within many applications in the petrochemical industry, and potentially in other areas including supply chain management, agriculture, and the energy sector.

A general form of the standard fixed-charge transportation problem for a single product can be described as follows [17]. Consider a set of suppliers $S = \{1, \ldots, m\}$ and a set of consumers $C = \{1, \ldots, n\}$. Each supplier $i \in S$ has a minimum and maximum supply of a given product, denoted l_i and u_i, respectively. Similarly, each consumer $j \in C$ has a minimum and maximum demand for the product, denoted l_j and u_j, respectively. Product can be sent from suppliers to consumers on an underlying directed bipartite graph $G = (S \cup C, \mathcal{A})$, where $\mathcal{A}$ is the set of arcs. For each arc $(i, j) \in \mathcal{A}$, let c_{ij} denote the unit revenue for flow shipped from supplier i to consumer j and u_{ij} denote the capacity of flow on arc (i, j). What makes this problem more interesting than the classical transportation problem is the additional assumption that a fixed cost f_{ij} is incurred if arc (i, j) is opened. It is important to emphasize that fixed costs are incurred when arcs are opened as opposed to when suppliers are opened, as would happen in the facility location problem.

The FCTPwB incorporates an additional *proportionality requirement* on the quality of the product. Specifically, let $\tilde{p}_i$ denote the nominal quality (or purity) of product available from supplier $i \in S$ and $\tilde{p}_j^{\min}$ denote the minimum quality required at consumer $j \in C$. Then the additional constraint, which we refer to as a *linear blending constraint*, requires that the average quality of all product received by consumer j must be at least $\tilde{p}_j^{\min}$, where we assume that product received by a consumer can be *blended* together to meet this requirement. A similar constraint could be imposed based on a maximum quality requirement $\tilde{p}_j^{\max}$.

In this variant of the problem we assume that there is a single product as well as a single attribute associated with that product. The blending constraint applies to this single attribute. More generally, there could be multiple products/commodities each with multiple attributes, and consumers could demand different products with varying minimum and maximum quality requirements. In addition, the problem described above consists of a single period in which a product is distributed, but one could envision a multi-period problem in which the supply and demand inventories are affected by exogeneous factors, which is why we have chosen to describe the supply and demand as having to satisfy pre-determined inventory level requirements.

To cast this problem as a mixed-integer program, we introduce continuous decision variables x_{ij} to denote the amount of product sent from supplier i to consumer j and binary decision variables y_{ij} which take value 1 if arc (i, j) is opened and 0 otherwise. Let $p_{ij} := \tilde{p}_i - \tilde{p}_j^{\min}$ be the "purity difference" between supplier i

and consumer j, $\forall(i,j) \in \mathcal{A}$. This yields the arc-based formulation:

$$\begin{aligned}
\text{(FCTPwB)} \qquad & \max_{\mathbf{x},\mathbf{y}} && \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij} - \sum_{(i,j)\in\mathcal{A}} f_{ij}y_{ij} && \text{(1a)}\\
& \text{s.t.} && \sum_{i\in S} p_{ij}x_{ij} \geq 0, \qquad \forall j \in C && \text{(1b)}\\
& && l_i \leq \sum_{j\in C} x_{ij} \leq u_i, \qquad \forall i \in S && \text{(1c)}\\
& && l_j \leq \sum_{i\in S} x_{ij} \leq u_j, \qquad \forall j \in C && \text{(1d)}\\
& && 0 \leq x_{ij} \leq u_{ij}y_{ij}, \qquad \forall (i,j) \in \mathcal{A} && \text{(1e)}\\
& && y_{ij} \in \{0,1\}, \qquad \forall (i,j) \in \mathcal{A}\ . && \text{(1f)}
\end{aligned}$$

The objective of this formulation is to maximize profit, defined as the revenue from shipping product from suppliers to consumers minus the fixed cost incurred from opening the arcs on which goods are sent. Constraint (1b) models the linear blending constraint since it is a re-statement of the blending constraint

$$\frac{\sum_{i\in S}\tilde{p}_i x_i}{\sum_{i\in S} x_i} \geq \tilde{p}_j^{\min}\ ,$$

as it would appear in its natural form.

An interesting history of blending in the petroleum industry is given in [5] and [20]. These two works, along with [21], describe successful deployments of decision support systems in which blending is an integral component and underscore the importance of mathematical programming methodologies. In the chemical, petroleum, and wastewater treatment industries, several blending and pooling problems have undergone extensive study. The survey paper by Misener and Floudas [16] discusses five relevant classes of pooling problems.

When formulated as mathematical programs, most practical blending problems are modelled as mixed-integer nonlinear mathematical programming problems (MINLPs). However, because of the difficulty in solving these MINLPs, mixed-integer linear programming (MIP) formulations are commonly used to approximate MINLP formulations [11, 14]. In these MIP models, nonlinearities that arise from blending constraints are linearized (through reformulation) or approximated (sometimes iteratively) [14, 15].

The fixed-charge transportation problem (FCTP) *without* blending has been studied for years, with early work dating back to Balinski [3]. In the standard FCTP, each supplier $i \in S$ has a fixed supply $s_i = l_i = u_i$ and each consumer $j \in C$ has a fixed demand $d_j = l_j = u_j$. This problem is known to be $\mathcal{NP}$-hard. As a consequence, the FCTPwB is $\mathcal{NP}$-hard since if $p_{ij} > 0, \forall(i,j) \in \mathcal{A}$, then the blending constraints (1b) become redundant and the resulting problem is simply the FCTP. By and large, researchers have focused on developing heuristics and exact algorithms for solving the FCTP [1, 4, 6, 10, 12, 13, 19, 22]. More generally, the FCTP is a special case of the fixed-charge network flow problem for which substantial polyhedral theory and numerous algorithms have been developed. Notable inequalities derived from studying the single-node fixed-charge flow model include flow cover cuts [7, 18], flow path cuts [23], and flow pack cuts [2]. These cutting planes are now standard in many commercial MIP solvers. The relation between our facets and flow cover cuts is discussed in Section 2.4. We are not aware of any literature in which blending constraints are also considered.

Despite the abundance of research on blending and fixed-charge problems, there is a dearth of literature in which both themes are studied simultaneously from a polyhedral vantage point. In this paper, we strive to fill this void by investigating polyhedral aspects of the uncapacitated FCTPwB in which fixed charges and linear blending constraints are present. Our contributions are a polyhedral analysis of the FCTPwB, including two new families of facet-defining valid inequalities which fully exploit the presence of a linear blending requirement, and computational results that demonstrate the effectiveness of the inequalities. In Section 2, we introduce two exponentially-sized facet classes for the single-consumer uncapacitated FCTPwB polytope and provide intuition for their validity using arguments based on lifting facets of lower-dimensional sets. We also show that these facets can be separated with a low-order polynomial-time separation routine. In Section 3, we prove that in two special cases these facet classes, along with the continuous relaxation of the original formulation constraints, yield the convex hull of the feasible region. These results lend theoretical support to our claim that our two facet classes are strong. In Section 4, computational results are presented to illustrate the effectiveness of our facets at reducing the integrality gap and solution time on instances with multiple consumers and arc capacities. These results also provide empirical support that our separation procedure is extremely fast in practice. Some discussion of the relevance and applicability of these cuts to other models is provided in Section 5.

2 An Uncapacitated Single-Consumer Model

In this section, we study polyhedral aspects of an uncapacitated single-consumer model. We begin by collecting several assumptions that we will use throughout the remainder of the paper. We assume that each supplier can send product to a single consumer, that the consumer's (supplier's) lower bound on demand (supply) is 0, and that the consumer's (supplier's) upper bound on demand (supply) is 1, which is without loss of generality since we can scale parameters accordingly. Having unequal lower and upper bounds is not critical, but will permit us to work with a set that is full dimensional. We assume that arc capacities are arbitrarily large. Given that only one consumer is present, we drop the subscript for the consumer. We assume $p_1 > p_2 > \cdots > p_m$ and $p_i \neq 0, \forall i \in S$. This, again, is done for mathematical convenience. In fact, when we return to the multi-consumer case we will continue to assume that $p_{ij} \neq p_{kj}$ and $p_{ij} \neq 0$, $\forall i, k \in S, \forall j \in C$. Let $S^+ = \{1, \cdots, m_+\}$ be the set of *good* suppliers (i.e., suppliers whose purity difference p_i is positive) and analogously define $S^- = \{m_+ + 1, \cdots, m\}$ to be the set of *bad* suppliers. Let $S = S^+ \cup S^-$ be the set of all suppliers. We assume $m_+ = |S^+| \geq 1$ and $m_- = |S^-| \geq 1$.

The feasible region, denoted by X_{m_+,m_-}, of the single-consumer uncapacitated FCTPwB is the set of points $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times \{0,1\}^m$ satisfying

$$\text{(blending constraint)} \qquad \textstyle\sum_{i \in S^+} q_i x_i - \sum_{k \in S^-} r_k x_k \geq 0 \tag{2a}$$

$$\text{(demand constraint)} \qquad \textstyle\sum_{i \in S} x_i \leq 1 \tag{2b}$$

$$x_i \leq y_i, \forall\, i \in S, \tag{2c}$$

where $q_i = p_i, \forall\, i \in S^+$, $r_k = -p_k, \forall\, k \in S^-$. Note that $q_1 > \cdots > q_{m_+} > 0$ and $0 < r_{m_+ + 1} < \cdots < r_m$. We have introduced the parameters q_i and r_i for convenience so that all coefficients are positive. Our primary goal is to obtain a polyhedral description of the convex hull of X_{m_+,m_-}, denoted by $\text{conv}(X_{m_+,m_-})$.

2.1 Extreme Points

We now characterize the extreme points of $\text{conv}(X_{m_+,m_-})$. The intuition behind their structure is simple. The extreme points of the projection of $\text{conv}(X_{m_+,m_-})$ onto the continuous variables correspond to one of the three following cases: (i) the origin, (ii) one good supplier sending one unit of flow to satisfy demand while all other suppliers send nothing, or (iii) one good supplier and one bad supplier each sending product in such a way that both the blending and demand constraints are tight. When we return to the original space $\text{conv}(X_{m_+,m_-})$, we must also consider the y variables.

Proposition 1 *The extreme points of* $\text{conv}(X_{m_+,m_-})$ *are*

$$\Big(\mathbf{0}, \sum_{i\in T}\mathbf{e}_i\Big), \qquad \forall\ T \subseteq S \tag{3a}$$

$$\Big(\mathbf{e}_i, \mathbf{e}_i + \sum_{j\in T}\mathbf{e}_j\Big), \qquad \forall\ i \in S^+, \forall\ T \subseteq S \setminus \{i\} \tag{3b}$$

$$\Big(\frac{r_k}{q_i + r_k}\mathbf{e}_i + \frac{q_i}{q_i + r_k}\mathbf{e}_k, \mathbf{e}_i + \mathbf{e}_k + \sum_{j\in T}\mathbf{e}_j\Big), \qquad \forall\ i \in S^+, k \in S^-, \forall\ T \subseteq S \setminus \{i, k\}, \tag{3c}$$

where $\mathbf{e}_i \in \mathbb{R}^m$ *is the i-th unit vector. All nontrivial extreme points of* $\text{conv}(X_{m_+,m_-})$ *have exactly one positive value among the variables* $x_i, i \in S^+$*, and possibly one additional positive value among the variables* $x_k, k \in S^-$*.*

Proof It suffices to prove that the extreme points of $\{\mathbf{x} \in \mathbb{R}^m_+ : (2a);\ (2b)\}$, the continuous projection of $\text{conv}(X_{m_+,m_-})$, have the desired structure. This follows because the set only has two nontrivial constraints (2a) and (2b), and therefore when choosing which constraints to fix at equality at an extreme point, at most two variables (satisfying the specified conditions) will be positive. □

Corollary 1 *The set* $\text{conv}(X_{m_+,m_-})$ *is full-dimensional.*

Corollary 2 $x_i \geq 0$ *and* $y_i \leq 1$ *for all* $i \in S$ *are trivial facets of* $\text{conv}(X_{m_+,m_-})$*.*

Corollary 3 *The blending constraint* $\sum_{i\in S^+} q_i x_i - \sum_{k\in S^-} r_k x_k \geq 0$ *is a facet of* $\text{conv}(X_{m_+,m_-})$*. The inequalities* $\sum_{i\in S} x_i \leq 1$ *and* $x_i \leq y_i$ *for* $i \in S^+$ *are facets of* $\text{conv}(X_{m_+,m_-})$ *when* $m_+ \geq 2$*.*

Proof We can easily pick $2m+1$ affinely independent extreme points for Corollary 1 and $2m$ such points for Corollaries 2 and 3.

2.2 Facets of the Uncapacitated Single-Consumer FCTPwB Polytope

We now state and prove our main result.

Theorem 1 (Facet Class 1: Lifted Blending Facets) *The inequalities*

$$\sum_{i\in T} x_i + \sum_{k\in S^-} \min\Big\{1, \frac{r_k}{r_l}\Big\} x_k \leq \sum_{i\in S^+\setminus T}\Big(\frac{q_i}{r_l}\Big)x_i + \sum_{i\in T} y_i, \qquad \forall\ T \subseteq S^+, \forall\ l \in S^-\ , \tag{4}$$

are valid for $\text{conv}(X_{m_+,m_-})$*. They are facet-defining in all cases except when (a)* $T = \emptyset$ *and* $l < m$*, or (b)* $T = S^+$ *and* $l > m_+ + 1$*.*

Theorem 2 (Facet Class 2: Lifted Variable Upper Bound Facets) *Let* $S_j^+ = \{1, \ldots, j\}$ *for* $j \in S^+ \cup \{0\}$, *with* $S_0^+ = \emptyset$. *Let* $S_l^- = \{m_+ + 1, \ldots, l\}$ *for* $l \in S^- \cup \{m_+\}$, *with* $S_{m_+}^- = \emptyset$. *The inequalities*

$$\sum_{i \in T^+} \frac{r_l(q_i - q_j)}{q_i} x_i + \sum_{k \in T^- \cup \{l\}} (q_j + r_k) x_k \leq \sum_{k \in T^- \cup \{l\}} q_j y_k + \sum_{i \in S_{j-1}^+ \setminus T^+} (q_i - q_j) x_i + \sum_{i \in T^+} \frac{r_l(q_i - q_j)}{q_i} y_i, \tag{5}$$
$$\forall\, T^+ \subseteq S_{j-1}^+, \forall\, T^- \subseteq S_{l-1}^-, \forall\, j \in S^+, \forall\, l \in S^-$$

are valid for $\operatorname{conv}(X_{m_+,m_-})$. *If the conditions* $T^+ = S_{j-1}^+$ *and* $T^- \neq \emptyset$ *do not hold simultaneously, then the inequalities* (5) *are also facet-defining for* $\operatorname{conv}(X_{m_+,m_-})$.

Before proving these theorems, we give a brief explanation about their derivation as well as an illustrative example. Note that in Facet Class 1 when $l = m$ and $T = \emptyset$, the constraint becomes the original blending constraint (2a). Similarly, note that in Facet Class 2 when $j = 1$, $l \in S^-$, and $T^- = T^+ = \emptyset$, the constraint becomes a variable upper bound constraint $x_l \leq \frac{q_1}{q_1 + r_l} y_l$ on a bad supplier $l \in S^-$. Wherever possible, we will use subscripts i and j when indexing good suppliers and k and l when indexing bad suppliers.

We refer to these inequalities as *lifted* facets because they can be derived from lifting blending or variable upper bound inequalities from lower-dimensional sets. Specifically, for Facet Class 1, if we fix $T \subseteq S^+$ and $l \in S^-$, and set $x_i = y_i = 0, \forall i \in T$, and $x_k = y_k = 0, \forall k \in S^-, k \neq l$, we may lift the pairs of variables (x_t, y_t), which were fixed at 0, by considering the lifting function associated with the blending constraint $\sum_{j \in S^+ \setminus T} q_j x_j - r_l x_l \geq 0$, which is a facet on this restricted set. Similarly, for Facet Class 2, we fix a good supplier $j \in S^+$, a bad supplier $l \in S^-$, and set $x_i = y_i = x_k = y_k = 0, \forall i \in S_{j-1}^+, \forall k \in S_{l-1}^-$. We may then lift the pairs of variables (x_t, y_t), which were fixed at 0, by considering the lifting function associated with the variable upper bound constraint $x_l \leq \frac{q_j}{q_j + r_l} y_l$, which is a facet on this restricted set. Moreover, it can be shown that this lifting function is superadditive, hence, we obtain the computationally attractive property known as sequence independent lifting [8].

Example. There are two good suppliers, $S^+ = \{1, 2\}$, two bad suppliers, $S^- = \{3, 4\}$, and $\mathbf{p} = (11, 7, -3, -5)$. The lifted blending facets are

	T	l	
$3x_1 - 7x_2 + 3x_3 + 3x_4 \leq 3y_1$	$\{1\}$	3	(LB 3a)
$-11x_1 + 3x_2 + 3x_3 + 3x_4 \leq 3y_2$	$\{2\}$	3	(LB 3b)
$x_1 + x_2 + x_3 + x_4 \leq y_1 + y_2$	$\{1,2\}$	3	(LB 3c)
$-11x_1 - 7x_2 + 3x_3 + 5x_4 \leq 0$	$\emptyset$	4	(LB 4a)
$5x_1 - 7x_2 + 3x_3 + 5x_4 \leq 5y_1$	$\{1\}$	4	(LB 4b)
$-11x_1 + 5x_2 + 3x_3 + 5x_4 \leq 5y_2$	$\{2\}$	4	(LB 4c)

As described above, these facets are obtained by "turning off" all good suppliers in T and all bad suppliers besides l, and then lifting back in the pairs (x_t, y_t) of variables that were "turned off" starting from the lower-dimensional blending constraint $\sum_{j \in S^+ \setminus T} q_j x_j - r_l x_l \geq 0$. Note that facet (LB 4a) is the original blending constraint. Facet (LB 3c) states that at least one good supplier must be "turned on" if any product is sent from a supplier.

The lifted variable upper bound facets are

$$
\begin{array}{rrrrrrrrrrrcccc}
 & & & & & & & & & & & T^{+} & T^{-} & j & l \\
 & & 14x_3 & & & \leq & & & 11y_3 & & & \emptyset & \emptyset & 1 & 3 & \text{(LVUB 13)} \\
 & & & & 16x_4 & \leq & & & & & 11y_4 & \emptyset & \emptyset & 1 & 4 & \text{(LVUB 14)} \\
-4x_1 & + & 10x_3 & & & \leq & & & 7y_3 & & & \emptyset & \emptyset & 2 & 3 & \text{(LVUB 23a)} \\
12x_1 & + & 110x_3 & & & \leq & 12y_1 & + & 77y_3 & & & \{1\} & \emptyset & 2 & 3 & \text{(LVUB 23b)} \\
-4x_1 & & & + & 12x_4 & \leq & & & & & 7y_4 & \emptyset & \emptyset & 2 & 4 & \text{(LVUB 24a)} \\
-4x_1 & + & 10x_3 & + & 12x_4 & \leq & & & 7y_3 & + & 7y_4 & \emptyset & \{3\} & 2 & 4 & \text{(LVUB 24b)} \\
20x_1 & & & + & 132x_4 & \leq & 20y_1 & & & + & 77y_4 & \{1\} & \emptyset & 2 & 4 & \text{(LVUB 24c)}
\end{array}
$$

Note that when $j = 1$, the variable upper bound inequality $(q_1 + r_l)x_l \leq q_1 r_l$, for $l \in S^-$, is already facet-defining. When $j = 2$ and $l = 3$, i.e., when supplier 1 alone is "turned off" at the outset, there are two ways to lift in the pair (x_1, y_1) to obtain a facet as shown in (LVUB 23a) and (LVUB 23b). When $j = 2$ and $l = 4$, i.e., when suppliers 1 and 3 are "turned off" at the outset, there are three ways to lift in the pairs (x_1, y_1) and (x_3, y_3) to obtain a facet as shown in (LVUB 24a) – (LVUB 24c).

In addition to the bound inequalities, inequalities (LB) and (LVUB), the following three facets are needed to describe the convex hull of X_{m_+,m_-} for this example:

$$
\begin{array}{rrrrrrrrrrrrrr}
-110x_1 & + & 30x_2 & + & 30x_3 & + & 162x_4 & \leq & & + & 30y_2 & & & + & 77y_4 \\
30x_1 & - & 42x_2 & + & 128x_3 & + & 30x_4 & \leq & 30y_1 & & & + & 77y_3 & & \\
60x_1 & & & + & 550x_3 & + & 396x_4 & \leq & 60y_1 & & & + & 385y_3 & + & 231y_4 \; .
\end{array}
$$

Proof of Theorem 1: Let $(\mathbf{x}^*, \mathbf{y}^*) \in X_{m_+,m_-}$, $T \subseteq S^+$, and $l \in S^-$. If $y_i^* = 0, \forall\, i \in T$, then inequality (4) reduces to a weakened version (because of the min operator) of the blending constraint (2a) under the restriction $x_i = y_i = 0, \forall\, i \in T$. Otherwise, we have

$$\sum_{i \in T} x_i^* + \sum_{k \in S^-} \min\left\{1, \frac{r_k}{r_l}\right\} x_k^* \leq \sum_{i \in T} x_i^* + \sum_{k \in S^-} x_k^* \leq 1 \leq \sum_{i \in T} y_i^* \leq \sum_{i \in S^+ \setminus T} \left(\frac{q_i}{r_l}\right) x_i^* + \sum_{i \in T} y_i^* \; .$$

In all but the two exceptional cases, to prove that inequality (4) is facet-defining for a given choice of $T \subseteq S^+$ and $l \in S^-$, let $u \in S^+ \setminus T$ and $v \in T$. One can verify that the following $2m - 1$ points, along with the origin, are affinely independent:

$$\Big(\mathbf{0}, \mathbf{e}_i\Big), \qquad \forall\, i \in S^+ \setminus T \tag{6a}$$

$$\Big(\mathbf{e}_i, \mathbf{e}_i\Big), \qquad \forall\, i \in T \tag{6b}$$

$$\left(\frac{r_l}{q_i + r_l}\mathbf{e}_i + \frac{q_i}{q_i + r_l}\mathbf{e}_l, \mathbf{e}_i + \mathbf{e}_l\right), \qquad \forall\, i \in S^+ \tag{6c}$$

$$\Big(\mathbf{0}, \mathbf{e}_k\Big), \qquad \forall\, k \in S^- \tag{7a}$$

$$\left(\frac{r_k}{q_u + r_k}\mathbf{e}_u + \frac{q_u}{q_u + r_k}\mathbf{e}_k, \mathbf{e}_u + \mathbf{e}_k\right), \qquad \forall\, k \in S^-, k < l \tag{7b}$$

$$\left(\frac{r_k}{q_v + r_k}\mathbf{e}_v + \frac{q_v}{q_v + r_k}\mathbf{e}_k, \mathbf{e}_v + \mathbf{e}_k\right), \qquad \forall\, k \in S^-, k > l \tag{7c}$$

Note that (6a)–(6c) contribute $2m_+$ points and (7a)–(7c) contribute $2m_- - 1$ points. ☐

Proof of Theorem 2: Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an extreme point of $\text{conv}(X_{m_+,m_-})$. Let $j \in S^+, l \in S^-, T^+ \subseteq S^+_{j-1}$, and $T^- \subseteq S^-_{l-1}$. If $x^*_i = 1$ for some $i \in S^+$ or if $x^*_k > 0$ for some $k \in S^- \setminus (T^- \cup \{l\})$, then validity is immediate. So suppose $(\mathbf{x}^*, \mathbf{y}^*)$ takes the form (3c) for some $i \in S^+$ and some $k \in T^- \cup \{l\}$.

Case 1: If $i \geq j$ $(q_j \geq q_i)$, then $(q_j + r_k)x^*_k = (q_j + r_k)\left(\frac{q_i}{q_i+r_k}\right) \leq q_j = q_j y^*_k$.

Case 2: If $i \in S^+_{j-1} \setminus T^+$, then, since $x^*_i + x^*_k = 1$ and $r_k x^*_k - q_i x^*_i = 0$ is readily seen, we obtain

$$(q_j - q_i)x^*_i + (q_j + r_k)x^*_k = q_j(x^*_i + x^*_k) + r_k x^*_k - q_i x^*_i = q_j = q_j y^*_k \ .$$

Case 3: If $i \in T^+$, then $\left(\frac{r_l(q_i-q_j)}{q_i}\right)x^*_i + (q_j + r_k)x^*_k = \left(\frac{r_l(q_i-q_j)}{q_i}\right)\left(\frac{r_k}{q_i+r_k}\right) + (q_j + r_k)\left(\frac{q_i}{q_i+r_k}\right) \leq q_j + \frac{r_l(q_i-q_j)}{q_i} = q_j y^*_k + \left(\frac{r_l(q_i-q_j)}{q_i}\right) y^*_i$, with equality holding only when $k = l$.

In all but the exceptional cases, to prove that inequality (5) is facet-defining for a given choice of $j \in S^+, l \in S^-, T^+ \subseteq S^-_{j-1}$, and $T^- \subseteq S^-_{l-1}$, let $u \in S^+ \setminus S^+_{j-1}$ and $v \in S^+_{j-1} \setminus T^+$. One can verify that the following $2m - 1$ points, along with the origin, are affinely independent:

$$\Big(\mathbf{0}, \mathbf{e}_i\Big), \qquad \forall\ i \in (S^+_{j-1} \setminus T^+) \cup (S^+ \setminus S^+_{j-1}) \tag{8a}$$

$$\Big(\mathbf{e}_i, \mathbf{e}_i\Big), \qquad \forall\ i \in T^+ \cup (S^+ \setminus S^+_{j-1}) \tag{8b}$$

$$\left(\frac{r_l}{q_i + r_l}\mathbf{e}_i + \frac{q_i}{q_i + r_l}\mathbf{e}_l, \mathbf{e}_i + \mathbf{e}_l\right), \qquad \forall\ i \in S^+_{j-1} \tag{8c}$$

$$\Big(\mathbf{0}, \mathbf{e}_k\Big), \qquad \forall\ k \in (S^-_{l-1} \setminus T^-) \cup (S^- \setminus S^-_l) \tag{9a}$$

$$\left(\frac{r_k}{q_j + r_k}\mathbf{e}_j + \frac{q_j}{q_j + r_k}\mathbf{e}_k, \mathbf{e}_j + \mathbf{e}_k\right), \qquad \forall\ k \in T^- \cup \{l\} \cup (S^- \setminus S^-_l) \tag{9b}$$

$$\left(\frac{r_k}{q_u + r_k}\mathbf{e}_u + \frac{q_u}{q_u + r_k}\mathbf{e}_k, \mathbf{e}_u + \mathbf{e}_k\right), \qquad \forall\ k \in S^-_{l-1} \setminus T^- \tag{9c}$$

$$\left(\frac{r_k}{q_v + r_k}\mathbf{e}_v + \frac{q_v}{q_v + r_k}\mathbf{e}_k, \mathbf{e}_v + \mathbf{e}_k\right), \qquad \forall\ k \in T^- \tag{9d}$$

Note that (8a)–(8c) contribute $2m_+$ points and (9a)–(9d) contribute $2m_- - 1$ points. ☐

2.3 Separation

The next proposition shows that separation of the lifted blending constraints (4) and the lifted variable upper bound constraints (5) can be done in polynomial time, i.e., the former can be done in $O(m^2)$ time while the latter can be done in $O(m^3)$ time.

Proposition 2 *Let $(\mathbf{x}^*, \mathbf{y}^*)$ be an optimal solution to the LP relaxation.*

1. *Fix $l \in S^-$. If*

$$\zeta(l) := \sum_{k \in S^-} \min\left\{1, \frac{r_k}{r_l}\right\} x^*_k + \sum_{i \in S^+} \left(\left(1 + \left(\frac{q_i}{r_l}\right)\right) x^*_i - y^*_i\right)^+ - \left(\frac{q_i}{r_l}\right) x^*_i \tag{10}$$

is positive, where $(x)^+ := \max\{0, x\}$, then the most violated lifted blending inequality (4) for this $l \in S^-$ is given by the subset $T := \{i \in S^+ : \left(\left(1 + \left(\frac{q_i}{r_l}\right)\right) x^*_i - y^*_i\right) > 0\}$. If $\zeta(l) \leq 0, \forall l \in S^-$, then there is no violated lifted blending inequality (4).

2. Fix $j \in S^+$ and $l \in S^-$. If

$$\psi(j,l) := -\sum_{i=1}^{j-1}\frac{(q_i-q_j)}{r_l}x_i^* + \sum_{i=1}^{j-1}\left(\frac{(q_i-q_j)}{q_i}(x_i^*-y_i^*)+\frac{(q_i-q_j)}{r_l}x_i^*\right)^+ + \sum_{k=m_+ +1}^{l}\left(\frac{(q_j+r_k)}{r_l}x_k^* - \frac{q_j}{r_l}y_k^*\right)^+ \tag{11}$$

is positive, then the most violated lifted variable upper bound inequality (5) *for this $j \in S^+$ and $l \in S^-$ is given by the subsets $T^+ := \{i \in \{1,\ldots,j-1\} : \left(\frac{(q_i-q_j)}{q_i}(x_i^*-y_i^*)+\frac{(q_i-q_j)}{r_l}x_i^*\right) > 0\}$ and $T^- := \{k \in \{m_+ +1,\ldots,l\} : \left(\frac{(q_j+r_k)}{r_l}x_k^* - \frac{q_j}{r_l}y_k^*\right) > 0\}$. If $\psi(j,l) \le 0, \forall j \in S^+, \forall l \in S^-$, then there is no violated lifted variable upper bound inequality* (5).

Proof. 1. For each bad supplier $l \in S^-$, one can find the most violated blending inequality (4), or determine that no such violated inequality exists, by checking if

$$\zeta(l) = \kappa + \max_{T\subseteq S^+}\sum_{i\in T}(x_i^*-y_i^*) - \sum_{i\in S^+\setminus T}\left(\frac{q_i}{r_l}\right)x_i^*$$

is positive, where $\kappa = \sum_{k\in S^-}\min\left\{1,\frac{r_k}{r_l}\right\}x_k^*$ is a constant independent of the subset T. Notice that the maximization is trivial: if $x_i^* - y_i^* > -\left(\frac{q_i}{r_l}\right)x_i^*$, set $i \in T$; otherwise, $i \in S^+ \setminus T$. Consequently, if $\zeta(l)$, as defined in (10), is positive, set $T = \{i \in S^+ : \left(\left(1+\left(\frac{q_i}{r_l}\right)\right)x_i^* - y_i^*\right) > 0\}$. Since $\zeta(l)$ can be computed by summing over all good suppliers $j \in S^+$, of which there are at most m, and this operation must be done for each bad supplier $l \in S^-$, of which there are also at most m, we can determine the most violated lifted blending cuts (4) in $O(m^2)$ time.

2. For each good supplier $j \in S^+$ and each bad supplier $l \in S^-$, one can find the most violated variable upper bound inequality (5), or determine that no such violated inequality exists, by checking if

$$\psi(j,l) = \max_{T^+\subseteq S^+_{j-1},\,T^-\subseteq S^-_{l-1}} \sum_{i\in T^+}\frac{(q_i-q_j)}{q_i}(x_i^*-y_i^*) + \sum_{k\in T^-\cup\{l\}}\left(\frac{(q_j+r_k)}{r_l}x_k^* - \frac{q_j}{r_l}y_k^*\right) - \sum_{i\in S^+_{j-1}\setminus T^+}\frac{(q_i-q_j)}{r_l}x_i^*$$

is positive. As above, this maximization problem is trivial: if $\left(\frac{r_l(q_i-q_j)}{q_i}\right)(x_i^*-y_i^*) > -(q_i-q_j)x_i^*$ for $i \in S^+_{j-1}$, set $i \in T^+$; otherwise, set $i \in S^+_{j-1} \setminus T^+$. Similarly, if $(q_j+r_k)x_k^* - q_jy_k^* > 0$ for $k \in S^-_{l-1}$, set $k \in T^-$; otherwise, set $k \in S^-_{l-1} \setminus T^-$. Hence, if $\psi(j,l)$, as defined in (11), is positive, set T^+ and T^- accordingly. In the worst case, it requires $O(m^3)$ time to find the most violated lifted variable upper bound facets over all (j,l) pairs. This follows because looping over all (j,l) pairs, for $j \in S^+$ and $l \in S^-$, requires $O(m^2)$ time, and for a given (j,l) pair, the above summation requires $O(m)$ time. ☐

2.4 Relation to Single-Node Flow Covers

We close this section by comparing the constraint set $X_{m+,m-}$ with that of the single-node flow model since the latter has been studied extensively in the literature [7, 18]. The constraint set for a single-node flow model is given by

$$F := \left\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^m_+ \times \{0,1\}^m : \sum_{j\in N^+}x_j - \sum_{j\in N^-}x_j \le b, x_j \le a_jy_j, \forall\, j \in N\right\},$$

where the set N of arcs has been partitioned into incoming arcs N^- and outgoing arcs N^+, each arc j has a fixed capacity $a_j \in \mathbb{R}_+$ if opened, and $b \in \mathbb{R}$ is the exogeneous supply/demand at this node. There are two ways to relate the set $X_{m+,m-}$ to F.

- Interpretation 1: After setting $a_j = 1, \forall j \in S$, $b = 1$, and $N^- = \emptyset$, one can treat the demand constraint $\sum_{i \in S} x_i \leq 1$ as the constraint $\sum_{j \in N^+} x_j - \sum_{j \in N^-} x_j \leq b$ in F and intersect F with a single homogeneous linear inequality $\sum_{i \in S^+} q_i x_i - \sum_{k \in S^-} r_k x_k \geq 0$ to obtain the set $X_{m+,m-}$ as it was originally defined in (2).
- Interpretation 2: After setting $a_j = |p_j|, \forall j \in S$, and $b = 0$, and introducing an auxiliary decision variable $z_j = |p_j| x_j, \forall j \in S$, one can rewrite $\sum_{j \in S} p_j x_j \geq 0$ as $\sum_{j \in S^-} z_j - \sum_{j \in S^+} z_j \leq b$. Thus, S^- and S^+ play the role of N^+ and N^-, respectively, in F. In addition, one must intersect these constraints with the demand constraint, which becomes $\sum_{i \in S} \frac{z_j}{|p_j|} \leq 1$, to obtain the set

$$Z := \left\{ (\mathbf{z}, \mathbf{y}) \in \mathbb{R}_+^m \times \{0,1\}^m : \sum_{j \in S^-} z_j - \sum_{j \in S^+} z_j \leq 0, \sum_{j \in S} \frac{z_j}{|p_j|} \leq 1, z_j \leq |p_j| y_j, \forall\, j \in S \right\} .$$

Since $X_{m+,m-}$ and Z are subsets of F, valid cuts generated by well known procedures for the single-node flow covers, e.g., lifted flow cover inequalities, are valid for $X_{m+,m-}$ and Z. However, it is easy to verify that our two facet classes *cannot* be obtained as flow cover inequalities from $X_{m+,m-}$ or Z when the additional side constraint is omitted.

3 Special Cases: One Good or One Bad Supplier

In this section, we consider two special cases of the FCTPwB in which S^+ or S^- is a singleton. In both cases, we show that the continuous relaxation of X_{m_+,m_-} along with Facet Classes 1 and 2 yield the convex hull of X_{m_+,m_-}. These results lend theoretical support to our claim that inclusion of our two facet classes lead to strong formulations of the FCTPwB. Note that, as shown in the example from Section 2.2, when $|S^+| > 1$ and $|S^-| > 1$, the continuous relaxation of the original formulation constraints and the two facet classes are not enough to describe $\text{conv}(X_{m_+,m_-})$.

3.1 One Good Supplier and Many Bad Suppliers

First consider the simplified single-consumer model in which there is a single good supplier and one or more bad suppliers, i.e., $S^+ = \{1\}$ and $S^- = \{2, \ldots, m\}$. In this case, the lifted blending and variable upper bound facets for $X_{1,m-1}$ become:

$$\sum_{i \in S} x_i \leq y_1 \tag{12a}$$

$$x_k \leq \frac{q_1}{q_1 + r_k} y_k, \forall\, k \in S^-. \tag{12b}$$

Constraint (12a) states that if any product is sent, then the arc originating from the lone good supplier must be "on" (otherwise, the blending constraint cannot be met). Similarly, the maximum amount of product that can be sent from a bad supplier $k \in S^-$ is bounded above by the ratio $\frac{q_1}{q_1+r_k}$.

Theorem 3 [A Polyhedral Description of $\mathrm{conv}(X_{1,m-1})$] *Let* $P := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times [0,1]^m : (2a),\ (12a),\ (12b)\}$. *Then* $P = \mathrm{conv}(X_{1,m-1})$.

Proof Let $(\mathbf{x}^*, \mathbf{y}^*) \in P$ with some fractional $y_i^* \in (0,1)$. We show that $(\mathbf{x}^*, \mathbf{y}^*)$ cannot be an extreme point of P (see, e.g., Approach 2 on p.145 of [24]). Without loss of generality, we assume that the p_i's have been normalized so that $q_1 = 1$. The proof is split into four cases:

Case 1: Suppose $i \in S^-$ and $x_i^* < \frac{y_i^*}{r_i+1}$. Then for some $\varepsilon > 0$ we have $(\mathbf{x}^*, \mathbf{y}^* \pm \varepsilon \mathbf{e}_i) \in P$. Therefore $(\mathbf{x}^*, \mathbf{y}^*)$ is not extreme.

Case 2: Suppose $\sum_{k\in S} x_k^* = \alpha < 1$. Then the points

$$x_k^1 = \frac{x_k^*}{\alpha}, \quad y_k^1 = \min\left\{1, \frac{y_k^*}{\alpha}\right\}, \forall\ k \in S, \qquad \text{and} \qquad x_k^2 = 0, \quad y_k^2 = \max\left\{0, \frac{y_k^* - \alpha}{1-\alpha}\right\}, \forall\ k \in S,$$

satisfy $(\mathbf{x}^1, \mathbf{y}^1), (\mathbf{x}^2, \mathbf{y}^2) \in P$ and yield $(\mathbf{x}^*, \mathbf{y}^*) = \alpha(\mathbf{x}^1, \mathbf{y}^1) + (1-\alpha)(\mathbf{x}^2, \mathbf{y}^2)$. Thus, $i \neq 1$ and we must have $\sum_{k\in S} x_k^* = 1$ at any nontrivial extreme point of P.

Case 3: Suppose $\sum_{k\in S} x_k^* = 1$ (which implies $y_1^* = 1$), $x_i^* = \frac{y_i^*}{r_i+1}$ and $x_1^* - \sum_{k\in S^-} r_k x_k^* > 0$. The point $(\mathbf{x}^1, \mathbf{y}^1)$ with

$$x_1^1 = \frac{r_i + \sum_{k\neq 1,i}(r_k - r_i)x_k^*}{r_i+1}, \quad x_i^1 = \frac{1 - \sum_{k\neq 1,i}(r_k+1)x_k^*}{r_i+1},$$
$$y_1^1 = 1, \quad y_i^1 = 1 - \sum_{k\neq 1,i}(r_k+1)x_k^*, \quad (x_k^1, y_k^1) = (x_k^*, y_k^*), \forall\ k \neq 1, i$$

and $(\mathbf{x}^2, \mathbf{y}^2)$ with

$$x_1^2 = 1 - \sum_{k\neq 1,i} x_k^*, \quad x_i^2 = 0, \quad y_1^2 = 1, \quad y_i^2 = 0, \quad (x_k^2, y_k^2) = (x_k^*, y_k^*), \forall\ k \neq 1, i$$

belong to P and there is some $\lambda \in (0,1)$ with $(\mathbf{x}^*, \mathbf{y}^*) = \lambda(\mathbf{x}^1, \mathbf{y}^1) + (1-\lambda)(\mathbf{x}^2, \mathbf{y}^2)$.

Case 4: Suppose $\sum_{k\in S} x_k^* = 1$, $x_i^* = \frac{y_i^*}{r_i+1}$ and $x_1^* - \sum_{k\in S^-} r_k x_k^* = 0$. Then $y_i^* + \sum_{k\in S^-\setminus\{i\}}(r_k+1)x_k^* = 1$, which implies that $0 \leq x_l^* < \frac{1}{r_l+1}, \forall l \in S^- \setminus \{i\}$, and that there exists some $k \in S^- \setminus \{i\}$ such that $x_k^* > 0$. Since $0 < x_k^* < \frac{1}{r_k+1}$, $y_k^* = 1$ (otherwise, we are in Case 1). Define the direction vector $\mathbf{d} \in \mathbb{R}^m$ as

$$d_1 = \left(\frac{r_i+1}{r_k+1} - 1\right), \quad d_i = 1, \quad d_k = -\frac{r_i+1}{r_k+1}, \quad d_j = 0, \forall j \notin \{1, i, k\},$$

and note that $\sum_{j\in S} d_j = 0$ and $d_1 - \sum_{l\in S^-} r_l d_l = 0$. For $\varepsilon > 0$, define $y_i^1 = (r_i+1)(x_i^* + \varepsilon)$, $y_i^2 = (r_i+1)(x_i^* - \varepsilon)$, and let $\mathbf{x}^1 = \mathbf{x}^* + \varepsilon\mathbf{d}$, $\mathbf{x}^2 = \mathbf{x}^* - \varepsilon\mathbf{d}$, $y_j^1 = y_j^2 = y_j^*, \forall\ j \neq i$. Then if ε is small enough, $(\mathbf{x}^1, \mathbf{y}^1), (\mathbf{x}^2, \mathbf{y}^2) \in P$, and $(\mathbf{x}^*, \mathbf{y}^*)$ is their midpoint, so it cannot be extreme. ☐

3.2 Many Good Suppliers and One Bad Supplier

A polyhedral description of $\mathrm{conv}(X_{m-1,1})$ is more complex than $\mathrm{conv}(X_{1,m-1})$, in which there were only a polynomial number of facets. When $S^+ = \{1, \ldots, m-1\}$ and $S^- = \{m\}$, the lifted blending and variable upper bound facets for $X_{m-1,1}$ become:

$$\sum_{i \in T} x_i + x_m \leq \sum_{i \in S^+ \setminus T} \left(\frac{q_i}{r_m}\right) x_i + \sum_{i \in T} y_i, \qquad \forall\, T \subseteq S^+ \tag{13a}$$

$$\sum_{i \in T} \frac{r_m(q_i - q_j)}{q_i} x_i + (q_j + r_m) x_m \leq q_j y_m + \sum_{i \in S^+_{j-1} \setminus T} (q_i - q_j) x_i + \sum_{i \in T} \frac{r_m(q_i - q_j)}{q_i} y_i, \tag{13b}$$
$$\forall\, T \subseteq S^+_{j-1}, \forall\, j \in S^+$$

Theorem 4 [A Polyhedral Description of conv($X_{m-1,1}$)] *Let* $P := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m_+ \times [0,1]^m : x_i \leq y_i, \forall i \in S^+, (2b), (13a), (13b)\}$. *Then* $P = \text{conv}(X_{m-1,1})$.

Sketch of Proof. We show that for any cost vector $(\mathbf{c}, \mathbf{f}) \in \mathbb{R}^{m \times m}, (\mathbf{c}, \mathbf{f}) \neq (\mathbf{0}, \mathbf{0})$, the set $M(\mathbf{c}, \mathbf{f})$ of optimal solutions to the problem $\max\{\mathbf{c}^T\mathbf{x} - \mathbf{f}^T\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in X_{m-1,1}\}$ coincides with at least one of the hyperplanes associated with an inequality defining P (see, e.g., Approach 6 on p.146 of [24]). The proof, which is outlined in Figure 1, proceeds by partitioning the space of cost vectors and by gradually eliminating cost vectors from consideration. Initially, cost vectors that lead to optimal solutions that lie on one of the trivial or formulation facets are considered. Finally, cost vectors that lead to optimal solutions in which we are indifferent between sending product (a) exclusively from a single good supplier and (b) jointly from a good supplier and the bad supplier are considered. This last case requires the most care, but also sheds light on when our two facet classes are necessary. A complete proof is provided in the appendix. □

4 Computational Results

In this section, computational results are presented to illustrate the effectiveness of our two facet classes. In our first experiment, we investigate the reduction in the root node integrality gap due to our blending facets on uncapacitated single-consumer FCTPwB instances. Since our facets do not give the convex hull of X_{m_+,m_-} when $m_+ > 1$ and $m_- > 1$, this experiment provides empirical evidence concerning the strength of our facets with respect to the set X_{m_+,m_-}. In our second experiment, we solve capacitated multi-consumer FCTPwB instances to provable optimality and show that integrating our cuts in a branch-and-cut algorithm yields significant reductions in the overall solution time and the number of nodes explored in the search tree.

All experiments have the following characteristics: All computations were carried out on a Linux machine with kernel 2.6.18 running on a 64-bit x86 processor equipped with two Intel Xeon E5520 chips, which run at 2.27 GHz, and 32GB of RAM. The LP and MIP solvers of Gurobi 3.0 were used [9]. For every set of parameters, 100 instances were randomly generated. All cuts are generated via the separation routine described in Proposition 2. Specifically, for each good and each bad supplier, the most violated blending cuts are generated and are only added if the violation is at least $\epsilon := 0.0001$. Note that when multiple consumers are present, the number and set of good and bad suppliers differ for each consumer. Separation is performed for each consumer.

4.1 Uncapacitated Single-Consumer FCTPwB

In our first experiment, we present results for instances of the uncapacitated single-consumer FCTPwB. In light of Theorems 3 and 4, all instances have at least two good and bad suppliers so that the convex hull is not already known. Since our facets, along with the original formulation constraints, do not yield the convex hull of X_{m_+,m_-}, our main curiosity in this experiment is to obtain empirical evidence concerning how effective our cuts are at tightening the LP relaxation. Specifically, we aim to answer the following question: What is the reduction in the integrality gap due to our two facet classes and how many of these cuts are necessary to achieve this gap reduction? The integrality gap is defined as $(z^* - z^{LP})/z^*$, where z^* is the true optimal objective function value (computed in advance) and z^{LP} is the objective function value of the LP relaxation.

To answer this question, we could compare the integrality gap of the LP relaxation with that of a cutting plane algorithm in which only blending cuts are separated. However, in addition to this comparison, we may also want to know the value of our blending cuts when they are embedded in a MIP solver in which standard MIP cuts are used. To this end, we compare the integrality gap at the root node for four different options: the LP relaxation (denoted by 'LP' in the tables), Gurobi on its own, i.e., without blending cuts, ('GRB'), a user-implemented cutting plane algorithm ('User') in which only our blending cuts are added to the model until the LP relaxation ceases to improve by at least ϵ or no violated cuts are found, and Gurobi with both standard MIP cuts enabled and blending cuts added through a callback ('GRB+User'). We also experimented with turning off all default Gurobi cuts and having Gurobi use only our cuts through a callback. However, this option was almost always worse than default Gurobi and was always worse than our cutting plane implementation. Note that in this first experiment MIP preprocessing ('presolve') is turned *off* to understand how our blending cuts improve the quality of the original formulation.

A particular instance is generated as follows. First, we select the number of good and bad suppliers m_+ and m_-, respectively. Fixed costs are set such that $f_i = m - i + 1, \forall i \in S$. Unit cost are set such that $c_i = m + 1, \forall i \in S^+$, and $c_k = m + 1 + \Delta\text{bad}, \forall k \in S^-$, where $\Delta\text{bad} \in \mathbb{Z}_+$ is a parameter representing an increase in revenue (i.e., an incentive) for using bad suppliers. It is important to note that without an appreciable incentive for using bad suppliers, the optimal solution is trivial: send everything from a single good supplier. In this case, our blending cuts will not help. Nominal purity levels are generated as $\tilde{p}_i \sim \text{Normal}(0,1), \forall i \in S$. To have exactly m_+ good and m_- bad suppliers, respectively, we sort the $\tilde{p}_i$'s in decreasing order, re-index so that $\tilde{p}_1 > \cdots > \tilde{p}_m$ and set $\tilde{p}^{\min} = (\tilde{p}_{m_+} + \tilde{p}_{m_++1})/2$. Finally, we set $p_i = \tilde{p}_i - \tilde{p}^{\min}, \forall i \in S$.

The results are shown in Tables 1 and 2. The heading '# Good' refers to the number of good suppliers. The next four columns indicate the average integrality gap (%) at the root node of the branch-and-bound tree for the four different options discussed above. To reiterate, this gap is exact since it is relative to the true optimal MIP solution. The remaining columns show cut-specific information. 'Cuts (User)' and 'Cuts (GRB+User)' refer to cut information associated with the 'User' and 'GRB+User' option, respectively. 'LB' and 'LVUB' denote the average number of lifted blending cuts (4) and lifted variable upper bound cuts (5) that were generated through separation, respectively. 'Rounds' refers to the average number of separation rounds, i.e., the average number of times an attempt to separate the current optimal solution to the LP relaxation with a blending cut.

The results in Tables 1 and 2 suggest that our blending cuts are effective at reducing the integrality gap of the model. In fact, the smallest gap is often achieved when only blending cuts are added. These results provide compelling empirical evidence that the subset of facets of X_{m_+,m_-} identified in Theorems 1 and 2 work well by themselves. We also see that when the number of suppliers is larger and when the incentive for using bad suppliers (Δbad) increases, our cuts are more valuable, i.e., the difference between the integrality gap of 'GRB' and 'User' and between 'GRB' and 'GRB+User' becomes more pronounced.

Given that blending cuts alone are so effective, one might assume that coupling blending cuts with standard MIP cuts added by Gurobi would further reduce the integrality gap. The results indicate that this is not the case when we simply add blending cuts as user cuts through a callback in Gurobi. It appears that with default settings Gurobi prefers not to generate cuts as aggressively as our implemented cutting plane method. Two possible explanations for this behavior are: (i) if the absolute value of the ratio (violation of cut)/(norm of cut) does not exceed Gurobi's default tolerance, the cut may be rejected, and (ii) if two cuts are close to parallel, one of them may be rejected (Z. Gu, personal communication, August 13, 2010). At any rate, these results also serve as a useful reminder: Care has to be taken when setting up computational experiments and with interpreting computational results. If we had just used a callback implementation, we would have drawn completely different conclusions about the value of our blending cuts!

Data		Root Gap (%)				Cuts (User)			Cuts (GRB+User)		
Δbad	# Good	LP	GRB	User	GRB+User	LB	LVUB	Rounds	LB	LVUB	Rounds
5	5	1.19	**0.00**	**0.00**	**0.00**	0	6	1	0	4	1
5	10	1.45	**0.00**	0.01	**0.00**	3	49	3	1	17	1
5	15	3.63	**0.08**	0.32	0.19	38	208	15	7	226	7
15	5	16.72	0.42	**0.12**	1.27	33	86	8	51	56	5
15	10	29.67	13.93	**5.51**	14.44	308	652	43	102	801	11
15	15	17.47	2.66	**2.20**	7.21	315	300	69	58	712	12
25	5	24.52	2.28	**0.50**	4.97	110	196	14	119	438	9
25	10	22.59	9.92	**1.09**	8.57	301	439	39	108	874	11
25	15	16.71	2.70	**1.08**	7.13	281	155	59	56	716	12
50	5	9.29	0.66	**0.01**	0.87	73	51	6	79	301	5
50	10	13.51	4.19	**0.15**	2.66	221	154	26	96	742	11
50	15	13.04	4.82	**0.32**	4.63	227	79	47	58	715	13
100	5	4.20	0.22	**0.00**	0.23	49	24	3	56	215	3
100	10	7.49	0.68	**0.04**	0.35	142	68	15	69	536	8
100	15	8.97	2.84	**0.13**	2.08	175	63	36	55	642	12

Table 1: Root information for the Uncapacitated Single-Consumer FCTPwB with 20 Suppliers

4.2 Capacitated Multi-Consumer FCTPwB

In our next experiment, we show the strength of our two cut classes for capacitated multi-consumer FCTPwB instances described by Formulation (1). In this capacitated setting, our inequalities remain valid, but may no longer be facet-defining. The set-up for this experiment resembles what was done above, except in addition to investigating the root relaxation, we also observe that our cuts are effective at solving these instances to provable optimality. In some cases, embedding blending cuts within Gurobi reduces solution time by two orders of magnitude.

A particular instance is generated as follows. There are $m = 20$ suppliers and the number of consumers varies depending on data set used. Table 3 specifies the number of consumers as well as the number of good suppliers for each consumer. For example, in Data Set 1, the first consumer has 15 good suppliers; the last consumer has 6. As above, nominal purity levels are generated as Normal(0,1) random variables and purity differences are computed so that the appropriate number of good suppliers aligns with what is stated in

Data		Root Gap (%)				Cuts (User)			Cuts (GRB+User)		
Δbad	# Good	LP	GRB	User	GRB+User	LB	LVUB	Rounds	LB	LVUB	Rounds
5	10	0.25	**0.00**	**0.00**	**0.00**	0	15	1	0	8	0
5	20	0.22	**0.00**	**0.00**	**0.00**	0	78	2	0	20	0
5	30	0.39	**0.00**	**0.00**	**0.00**	1	248	3	0	464	7
15	10	3.99	0.15	**0.05**	0.31	36	247	9	20	41	2
15	20	3.53	0.28	**0.18**	0.53	62	1214	16	29	246	3
15	30	12.74	7.83	**5.98**	8.68	997	1065	105	100	2471	10
25	10	19.04	4.23	**0.30**	5.76	111	744	17	135	138	5
25	20	24.65	12.73	**8.96**	12.46	1300	1939	74	283	4422	14
25	30	14.13	8.71	**3.91**	8.87	1295	1556	136	161	4346	16
50	10	23.89	14.94	**1.75**	12.10	1068	1610	51	304	2415	10
50	20	16.13	11.46	**2.77**	7.83	1741	2059	99	261	4583	13
50	30	12.69	7.53	**2.31**	6.86	1111	772	114	136	3738	14
100	10	9.60	4.91	**0.05**	5.10	597	487	24	219	1940	7
100	20	9.89	6.06	**0.77**	4.03	1439	999	78	255	4320	14
100	30	10.19	6.32	**0.69**	5.56	1050	357	106	134	3610	13

Table 2: Root information for the Uncapacitated Single-Consumer FCTPwB with 40 Suppliers

Data Set	# Consumers	# Good Suppliers per Consumer
1	10	15,14,13,12,11,10,9,8,7,6
2	17	18,17,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2

Table 3: Data Sets

Table 3. For each arc $(i, j) \in \mathcal{A}$, we set $f_{ij} = m - i - (j/m)$; $c_{ij} = m + 1$ if $p_{ij} > 0$ and $c_{ij} = m + 1 + i + \Delta\text{bad}$ if $p_{ij} < 0$. We set $l_i = l_j = 0$, $u_i = n$, and $u_j = 1, \forall i \in S, j \in C$. Finally, we distinguish between *weakly* and *highly* capacitated instances in which arc capacities u_{ij} are randomly generated as Uniform(0.80,0.95) and Uniform(0.25,0.50), respectively. In the tables, weakly and highly capacitated instances are denoted with a 'W' and an 'H,' respectively.

The results are shown in Tables 4–7. Tables 4 and 5 present information related to the root node of the search tree while Tables 6 and 7 focus on information related to solving the instances to provable optimality. 'Cap' refers to the capacity of the instance. Note that MIP preprocessing ('presolve') is turned *on*, just as a user would do. Tables 4 and 5 report the same information reported in the first set of experiments. In Tables 6 and 7, under the '# Cuts' heading, 'LB' and 'LVUB' refer to the number of lifted blending and lifted variable upper bound cuts that were ever generated. '# Nodes' refers to the number of nodes that were explored in the search tree.

After solving the capacitated multi-consumer FCTPwB model to provable optimality and averaging the results, the following observations are apparent. No blending (user) cuts were ever generated after the root node. This does not necessarily mean that there are no violated blending cuts at nodes other than the root node. However, with default parameter settings Gurobi chooses never to execute our cut callback beyond the root node and therefore never attempts to generate blending cuts at nodes other than the root node. It should also be noted that default Gurobi cuts were almost never generated beyond the root node. In every case, fewer nodes in the branch-and-cut tree were explored when blending cuts were generated alongside default Gurobi cuts. This reduction in the number of nodes explored often led to an order of magnitude improvement in the overall solution time.

In contrast to what was observed in our first experiment, Gurobi often performed many more rounds of separation at the root node than our implemented cutting plane method in this second experiment. One possible explanation for this is that when arc capacities are introduced, our inequalities are no longer facet defining and are unable to reduce the integrality gap as much per iteration as in our first experiment. Meanwhile, with the introduction of arc capacities and multiple consumers, Gurobi is able to generate more of its own inequalities (30-40% of which are Gomory mixed-integer cuts and 25-35% of which are flow cover cuts). Note that arc capacities lead to multiple single-node flow cover sets and, therefore, greater potential for flow cover inequalities to be separated. This leads to more opportunities for us to generate more (weaker) inequalities, which in turn leads to more opportunities for Gurobi to generate more inequalities, and so forth. Thus we end up with many more separation rounds and slow convergence.

In preliminary experimentation, we also learned that when the parameter Δbad was large, it was important to place an upper bound on the number of each type of blending cut that can be generated or on the number of separation rounds. Without such a constraint, an excessive number of blending cuts could be generated at the root node, bogging down the computations at subsequent iterations, ultimately resulting in longer solution times than default Gurobi. To avoid this, we imposed an upper bound of 5000 rounds of separation for all of the instances solved in this second experiment. As a final comment, in general, weakly capacitated instances are much easier to solve. Since our cuts were developed for an uncapacitated model, it seems natural that they should perform better on weakly capacitated instances.

Data		Root Gap (%)				# Cuts (User)			# Cuts (GRB+User)		
Cap	Δbad	LP	GRB	User	GRB+User	LB	LVUB	Rounds	LB	LVUB	Rounds
W	5	40.55	31.65	15.24	**9.40**	1219	4380	50	31342	178102	2128
W	15	35.37	27.30	14.11	**9.62**	856	3062	27	2860	10049	143
W	25	29.69	23.80	12.76	**8.10**	339	1714	3	1522	6783	27
W	50	22.60	16.47	10.99	**5.93**	327	1773	3	1696	7787	46
W	100	16.93	12.01	9.79	**4.75**	316	1712	3	2897	9681	174
H	5	43.60	20.89	23.71	**13.65**	91	57	12	574	466	466
H	15	36.13	23.19	20.67	**12.31**	212	48	25	1169	302	417
H	25	30.15	21.08	18.35	**11.23**	311	64	34	1796	376	1159
H	50	24.35	20.37	14.25	**11.86**	431	82	47	742	159	106
H	100	15.59	11.65	9.11	**6.81**	415	74	38	675	128	65

Table 4: Root information for Data Set 1

5 Future Research

We would like to extend our two facet classes in two ways. First, it would be interesting to determine similar cuts for the *capacitated* FCTPwB. We attempted to do this for the case of a single good supplier

Data		Root Gap (%)				# Cuts (User)			# Cuts (GRB+User)		
Cap	Δbad	LP	GRB	User	GRB+User	LB	LVUB	Rounds	LB	LVUB	Rounds
W	5	34.39	20.23	13.03	**7.97**	1473	4924	40	65557	281830	3665
W	15	30.49	18.59	12.38	**7.78**	629	2608	6	1717	7037	19
W	25	27.32	17.76	11.82	**7.76**	449	2035	3	1833	8021	20
W	50	21.36	13.50	10.37	**5.77**	447	2170	3	1760	8063	20
W	100	16.56	9.50	8.86	**3.71**	395	1879	3	2084	7346	51
H	5	24.54	7.37	13.99	**4.79**	240	69	30	1354	594	798
H	15	27.94	10.03	12.79	**5.30**	330	87	39	7061	1717	3646
H	25	26.22	11.71	12.80	**5.84**	450	98	53	6795	4927	2829
H	50	22.37	11.75	11.85	**6.29**	679	118	76	26381	3914	4568
H	100	15.06	7.96	9.29	**6.71**	684	118	68	12042	2878	3158

Table 5: Root information for Data Set 2

Data		Time (sec)		# Cuts		# Nodes	
Cap	Δbad	GRB	GRB+User	LB	LVUB	GRB	GRB+User
W	5	271.06	**7.42**	31342	178102	2018646	**3204**
W	15	217.10	**0.91**	2860	10049	1538488	**159**
W	25	59.97	**0.47**	1522	6783	443672	**17**
W	50	19.40	**0.55**	1696	7787	114445	**37**
W	100	59.42	**0.80**	2897	9681	300811	**167**
H	5	**0.40**	0.61	574	466	1433	**603**
H	15	2.41	**0.65**	1169	302	18523	**469**
H	25	28.48	**1.55**	1796	376	249101	**1485**
H	50	43.49	**0.26**	742	159	317443	**95**
H	100	29.95	**0.21**	675	128	218113	**53**

Table 6: Full solve information for Data Set 1

and many bad suppliers. However, even for this simple set, the form of the cuts became complicated. Second, it would be interesting to construct facet classes when the right-hand-side b, which in our model is set to 0, of the blending constraint $\sum_{i \in S} p_i x_i \geq b$ takes nonzero values. Obtaining facets for this set, i.e., $X := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^m \times \{0,1\}^m : \sum_{i \in S} p_i x_i \geq b, \sum_{i \in S} x_i \leq 1, x_i \leq y_i, \forall i \in S\}$, could have greater appeal to the MIP community as they could be used to solve general MIP instances in which this structure appears. Our initial efforts into the question suggest that when $b > 0$ Facet Class 2 inequalities remain valid and facet-defining. However, we also found that "new" facets surface. We believe that lifting arguments will help to resolve this issue.

Although not presented here, we have also tested our blending inequalities when there are multiple blending constraints present. Specifically, suppose that the single blending constraint $\sum_{i \in S} p_i x_i \geq 0$ is replaced by $\sum_{i \in S} p_i^a x_i \geq 0, \forall a \in A$, where A is a set of attributes and p_i^a is the purity difference for supplier i with respect to attribute $a \in A$. We have found that applying our cuts for each attribute independently

Data		Time (sec)		# Cuts		# Nodes	
Cap	Δbad	GRB	GRB+User	LB	LVUB	GRB	GRB+User
W	5	931.71	**12.45**	65557	281830	4176033	**5627**
W	15	221.64	**0.61**	1717	7037	962686	**8**
W	25	123.12	**0.56**	1833	8021	465171	**8**
W	50	63.83	**0.49**	1760	8063	230501	**8**
W	100	11.18	**0.60**	2084	7346	42606	**38**
H	5	2.11	**1.57**	1354	594	10382	**1089**
H	15	**5.90**	6.58	7061	1717	27501	**5500**
H	25	169.14	**46.41**	6795	4927	843040	**148599**
H	50	265.60	**159.21**	26381	3914	1028405	**444680**
H	100	273.77	**129.75**	12042	2878	1076130	**372344**

Table 7: Full solve information for Data Set 2

can reduce the root integrality gap by 80% on instances similar to those considered in Section 4.1. It would be interesting to explore how our cuts perform on multi-period models as well as multi-period models with multiple attributes.

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References

[1] J. S. Aguado. Fixed Charge Transportation Problems: a new heuristic approach based on Lagrangean relaxation and the solving of core problems. *Annals of Operations Research*, 172(1):45–69, 2009.

[2] A. Atamtürk. Flow pack facets of the single node fixed-charge flow polytope. *Operations Research Letters*, 29:107–114, 2001.

[3] M. L. Balinski. Fixed-cost transportation problems. *Naval Research Logistics Quarterly*, 8:41–54, 1961.

[4] G. Bell, B. Lamar, and C. Wallace. Capacity improvement, penalties, and the fixed charge transportation problem. *Naval Research Logistics*, 46(4):341–355, 1999.

[5] C. DeWitt, L. Lasdon, A. Waren, D. Brenner, and S. Melhem. OMEGA - An Improved Gasoline Blending System for Texaco. *Interfaces*, 19(1):85–101, 1989.

[6] F. Glover, M. Amini, and G. Kochenberger. Parametric ghost image processes for fixed-charge problems: A study of transportation networks. *Journal of Heuristics*, 11, 2005.

[7] Z. Gu, G. Nemhauser, and M. Savelsbergh. Lifted flow cover inequalities for mixed 0-1 integer programs. *Mathematical Programming*, 85(3):439–467, 1999.

[8] Z. Gu, G. Nemhauser, and M. Savelsbergh. Sequence independent lifting in mixed integer programming. *Journal of Combinatorial Optimization*, 4(1):109–129, 2000.

[9] Gurobi Optimization Inc. Gurobi Optimizer Reference Manual, Version 3.0, 2010.

[10] W. M. Hirsch and G. B. Dantzig. The fixed charge problem. *Naval Research Logistics Quarterly*, 15:413–424, 1968.

[11] R. Karuppiah, K. C. Furman, and I. E. Grossmann. Global optimization for scheduling refinery crude oil operations. *Computers & Chemical Engineering*, 32(11):2745–2766, 2008.

[12] A. Klose. Algorithms for solving the single-sink fixed-charge transportation problem. *Computers & Operations Research*, 35(6):2079–2092, 2008.

[13] B. Lamar and C. Wallace. Revised-modified penalties for fixed charge transportation problems. *Management Science*, 43(10):1431–1436, 1997.

[14] H. Lee, J. Pinto, I. Grossmann, and S. Park. Mixed-integer linear programming model for refinery short-term scheduling of crude oil unloading with inventory management. *Industrial & Engineering Chemistry Research*, 35(5):1630–1641, 1996.

[15] C. Mendez, I. Grossmann, I. Harjunkoski, and P. Kabore. A simultaneous optimization approach for off-line blending and scheduling of oil-refinery operations. *Computers & Chemical Engineering*, 30(4):614–634, 2006.

[16] R. Misener and C. A. Floudas. Advances for the pooling problem: modeling, global optimization, and computational studies survey. *Applied and Computational Mathematics*, 8(1):3–22, 2009.

[17] G. Nemhauser and L. Wolsey. *Integer and Combinatorial Optimization*. Wiley, New York, 1988.

[18] M. Padberg, T. Van Roy, and L. Wolsey. Valid Linear Inequalities for Fixed Charge Problems. *Operations Research*, 33(4):842–861, 1985.

[19] U. Palekar, M. Karwan, and S. Zionts. A branch-and-bound method for the fixed charge transportation problem. *Management Science*, 36(9):1092–1105, 1990.

[20] B. Rigby, L. Lasdon, and A. Waren. The Evolution of Texaco Blending Systems - From OMEGA To STARBLEND. *Interfaces*, 25(5):64–83, 1995.

[21] F. Romo, A. Tomasgard, L. Hellemo, M. Fodstad, B. H. Eidesen, and B. Pedersen. Optimizing the Norwegian Natural Gas Production and Transport. *Interfaces*, 39(1):46–56, 2009.

[22] M. Sun, J. Aronson, P. McKeown, and D. Drinka. A tabu search heuristic procedure for the fixed charge transportation problem. *European Journal of Operational Research*, 106(2-3):441–456, 1998.

[23] T. Van Roy and L. Wolsey. Valid Inequalities and Separation for Uncapacitated Fixed Charge Networks. *Operations Research Letters*, 4(4):105112, 1985.

[24] L. Wolsey. *Integer Programming*. Wiley, 1998.

Appendix

In this appendix, we prove Theorem 4. The next two propositions are used in the proof of Theorem 4. Let $\alpha_i = \frac{r_m}{q_i+r_m}$ and $(1-\alpha_i) = \frac{q_i}{q_i+r_m}, \forall\ i \in S^+$.

Proposition 3 *The extreme points of* $\mathrm{conv}(X_{m-1,1})$ *that lie in a lifted blending facet* (13a) *defined by the subset* $T \subseteq S^+$ *are:*

$$\left(\mathbf{0}, \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ U \subseteq S \setminus T \tag{14a}$$

$$\left(\mathbf{e}_i, \mathbf{e}_i + \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ i \in T, \forall\ U \subseteq S^+ \setminus T \tag{14b}$$

$$\left(\alpha_i \mathbf{e}_i + (1-\alpha_i)\mathbf{e}_m, \mathbf{e}_i + \mathbf{e}_m + \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ i \in S^+, \forall\ U \subseteq S^+ \setminus T, \tag{14c}$$

Proof By inspection. Substitute each extreme point of $\mathrm{conv}(X_{m-1,1})$ into the lifted blending facet defined by the subset $T \subseteq S^+$ and verify that the facet is only satisfied at equality by the above extreme points. $\square$

Proposition 4 *The extreme points of* $\mathrm{conv}(X_{m-1,1})$ *that lie in a lifted variable upper bound facet* (13b) *defined by* $j \in S^+$ *and the subset* $T \subseteq S_{j-1}$ *are:*

$$\left(\mathbf{0}, \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ U \subseteq S^+ \setminus T \tag{15a}$$

$$\left(\mathbf{e}_i, \mathbf{e}_i + \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ i \in (S^+ \setminus S_{j-1}) \cup T, \forall\ U \subseteq S^+ \setminus (T \cup \{i\}) \tag{15b}$$

$$\left(\alpha_i \mathbf{e}_i + (1-\alpha_i)\mathbf{e}_m, \mathbf{e}_i + \mathbf{e}_m + \sum_{u\in U} \mathbf{e}_u\right), \qquad \forall\ i \in S_j, \forall\ U \subseteq S^+ \setminus (T \cup \{i\}), \tag{15c}$$

Proof By inspection. Substitute each extreme point of $\mathrm{conv}(X_{m-1,1})$ into the lifted variable upper bound facet defined by $j \in S^+$ and the subset $T \subseteq S_{j-1}$ and verify that the facet is only satisfied at equality by the above extreme points. $\square$

Proof of Theorem 4. We show that for any cost vector $(\mathbf{c}, \mathbf{f}) \in \mathbb{R}^{m\times m}, (\mathbf{c}, \mathbf{f}) \neq (\mathbf{0}, \mathbf{0})$, the set $M(\mathbf{c}, \mathbf{f})$ of optimal solutions to the problem $\max\{\mathbf{c}^T\mathbf{x} - \mathbf{f}^T\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in X_{m-1,1}\}$ coincides with at least one of the hyperplanes associated with an inequality defining P (see, e.g., Approach 6 on p.146 of [24]). Since the inequalities defining P are all facets of $\mathrm{conv}(X_{m-1,1})$, P is a minimal polyhedral representation of $\mathrm{conv}(X_{m-1,1})$. The proof, which is outlined in Figure 1, proceeds by partitioning the space of cost vectors and by gradually eliminating cost vectors from consideration. Initially, cost vectors that lead to optimal solutions that lie on one of the trivial or formulation facets are considered. Finally, cost vectors that lead to the case in which we are indifferent between sending product exclusively from a single good supplier and from a good supplier and the bad supplier are considered. The following notation will be used:

- $\alpha_i = \frac{r_m}{q_i+r_m}, (1-\alpha_i) = \frac{q_i}{q_i+r_m}, \forall\ i \in S^+$
- $g_i = \alpha_i c_i + (1-\alpha_i)c_m - (f_i + f_m), \forall\ i \in S^+$
- $CF = \arg\max\{c_i - f_i : (\mathbf{x}, \mathbf{y}) \in X_{m-1,1}\}$

- $G = \arg\max\{g_i : (\mathbf{x}, \mathbf{y}) \in X_{m-1,1}\}$

Note that CF and G are sets, not indices. Here $c_i - f_i$ denotes the cost of sending all supply exclusively from good supplier $i \in S^+$, whereas g_i denotes the cost of sending a nontrivial convex combination of supply from supplier i and the lone bad supplier m so that $\sum_{i \in S} x_i = 1$ and $\sum_{i \in S} p_i x_i = 0$. We say that g_i is the cost associated with a "blended" solution. Each bullet below corresponds to a branch in the tree presented in Figure 1.

- If $f_i < 0$ for some $i \in S$, then $y_i = 1$ in every optimal solution, i.e., $M(\mathbf{c}, \mathbf{f}) = \{(\mathbf{x}, \mathbf{y}) \in X_{m-1,1} : y_i = 1\}$. Thus, we may assume that $f_i \geq 0, \forall i \in S$.
- If $c_m < 0$, then $x_m = 0$ in every optimal solution, i.e., $M(\mathbf{c}, \mathbf{f}) = \{(\mathbf{x}, \mathbf{y}) : x_m = 0\}$. Thus, we may assume that $c_m \geq 0$.
- If $c_m = 0$, then
  - if $c_i - f_i < 0$ for some $i \in S^+$, then $x_m = 0$ in every optimal solution. Thus, we may assume that $c_i - f_i \geq 0, \forall i \in S^+$.
  - if $c_i - f_i > 0$ for some $i \in S^+$, then $x_i = 0$ in every optimal solution. Thus, we may assume that $c_i - f_i = 0, \forall i \in S^+$.
  - if $c_i - f_i = 0, \forall i \in S^+$, then $x_i = y_i$ in every optimal solution.

  Thus, we may assume that $c_m > 0$. In the remainder of the proof, we omit the statement "Thus, we may assume ..." to refer to the complement case as the details are shown in the tree structure of Figure 1.
- If $g_j < 0, \forall j \in G$, then $x_m = 0$ in every optimal solution.
- If $c_i - f_i > g_j, \forall i \in CF, \forall j \in G$, then $\sum_{i \in S} x_i = 1$ and $x_m = 0$ in every optimal solution.
- If $c_i - f_i < 0, \forall i \in CF$, then a "blended" solution is always optimal in which case $\sum_{i \in S} p_i x_i = 0$ in every optimal solution.
- Similarly, if $c_i - f_i < g_j, \forall i \in CF, \forall j \in G$, then a "blended" solution is always optimal in which case $\sum_{i \in S} p_i x_i = 0$ in every optimal solution.
- If $c_i - f_i > 0, \forall i \in CF$, then a solution in which all product is sent exclusively from a good supplier is optimal in which case $\sum_{i \in S} x_i = 1$ in every optimal solution.
- If $i \notin CF \cup G$, then $x_i = 0$ in every optimal solution.

Finally, we arrive at the last black box in Figure 1 in which we only have to consider cost vectors that satisfy $\mathbf{c} \in \mathbb{R}^{m-1} \times \mathbb{R}_{++}, \mathbf{f} \in \mathbb{R}^m_+$, $0 = c_i - f_i = g_j, \forall i \in CF, \forall j \in G; CF \cup G = S^+$. Let $F_0 = \{i \in S^+ : f_i = 0\}$ and $F_+ = \{i \in S^+ : f_i > 0\}$. We now consider two cases, $f_m = 0$ and $f_m > 0$, and show that the former leads to extreme points that lie on a lifted blending facet and the latter to extreme points on a lifted variable upper bound facet.

Suppose $f_m = 0$. Set $T = CF$ and note that $f_i > 0, \forall i \in T$, i.e., $T \subseteq F_+$. This follows since for all $k \in CF \cap G$, $0 = c_k - f_k = g_k$ implies $c_k = f_k = c_m (> 0)$. Similarly, for all $i \in CF \setminus G$, we have $c_i = f_i \geq 0$ by assumption. Suppose, to the arrive at a contradiction, that $f_i = 0$. Since $0 > g_i = \alpha_i c_i + (1 - \alpha_i) c_m$ and

$(1-\alpha_i)c_m > 0$ by assumption, it must be the case that $c_i < 0$, which is a contradiction. Then, in accordance with Proposition 3, the following extreme points lie on the lifted blending facet defined by T:

$$\left(\mathbf{0}, \sum_{k \in U} \mathbf{e}_k\right), \qquad \forall\, U \subseteq F_0 \tag{16a}$$

$$\left(\mathbf{e}_i, \mathbf{e}_i + \sum_{k \in U} \mathbf{e}_k\right), \qquad \forall\, i \in CF, \forall\, U \subseteq S^+ \setminus F_+ \tag{16b}$$

$$\left(\alpha_i \mathbf{e}_i + (1-\alpha_i)\mathbf{e}_m, \mathbf{e}_i + \mathbf{e}_m + \sum_{j \in U} \mathbf{e}_j\right), \qquad \forall\, i \in G, \forall\, U \subseteq S^+ \setminus F_+. \tag{16c}$$

Suppose $f_m > 0$. Set $j = \max\{t \in G\}$ and $T = CF \cap S_{j-1}$ so that $CF \subseteq (S^+ \setminus S_{j-1}) \cup T$ and $G \subseteq S_j$. Then, in accordance with Proposition 4, the following extreme points lie on the lifted variable upper bound facet defined by j and $T \subseteq S_{j-1}$:

$$\left(\mathbf{0}, \sum_{k \in U} \mathbf{e}_k\right), \qquad \forall\, U \subseteq F_0 \tag{17a}$$

$$\left(\mathbf{e}_i, \mathbf{e}_i + \sum_{k \in U} \mathbf{e}_k\right), \qquad \forall\, i \in CF, \forall\, U \subseteq S^+ \setminus (T \cup \{i\} \cup F_+) \tag{17b}$$

$$\left(\alpha_i \mathbf{e}_i + (1-\alpha_i)\mathbf{e}_m, \mathbf{e}_i + \mathbf{e}_m + \sum_{k \in U} \mathbf{e}_k\right), \qquad \forall\, i \in G, \forall\, U \subseteq S^+ \setminus (T \cup \{i\} \cup F_+). \tag{17c}$$

The only fact that we need to justify is that $F_0 \subseteq S^+ \setminus T$, or, equivalently, $T \subseteq F_+$. Suppose, to arrive at a contradiction, that this is not the case, i.e., that $T \neq \emptyset$ and $\exists i \in T$ such that $f_i = 0$. Then, since $i \in CF$ and $f_i = 0$, we have $c_i - f_i = c_i = f_i = 0$ and $0 \geq g_i = \alpha_i c_i + (1-\alpha_i)c_m - f_i - f_m = (1-\alpha_i)c_m - f_m$, which implies that $f_m \geq (1-\alpha_i)c_m$. Since $j \notin T$ by construction and $1-\alpha_1 > \cdots > 1-\alpha_{m-1}$ by assumption, we see that $f_m \geq (1-\alpha_i)c_m > (1-\alpha_j)c_m$, or

$$(1-\alpha_j)c_m - f_m < 0\ . \tag{18}$$

In addition, we have $c_j - f_j \leq 0$, which means that $f_j \geq c_j$ and

$$\alpha_j c_j - f_j \leq 0\ . \tag{19}$$

It follows from inequalities (18) and (19) that

$$0 = g_j = \underbrace{\alpha_j c_j - f_j}_{\leq 0} + \underbrace{(1-\alpha_j)c_m - f_m}_{<0} < 0\ , \tag{20}$$

which is a contradiction. □

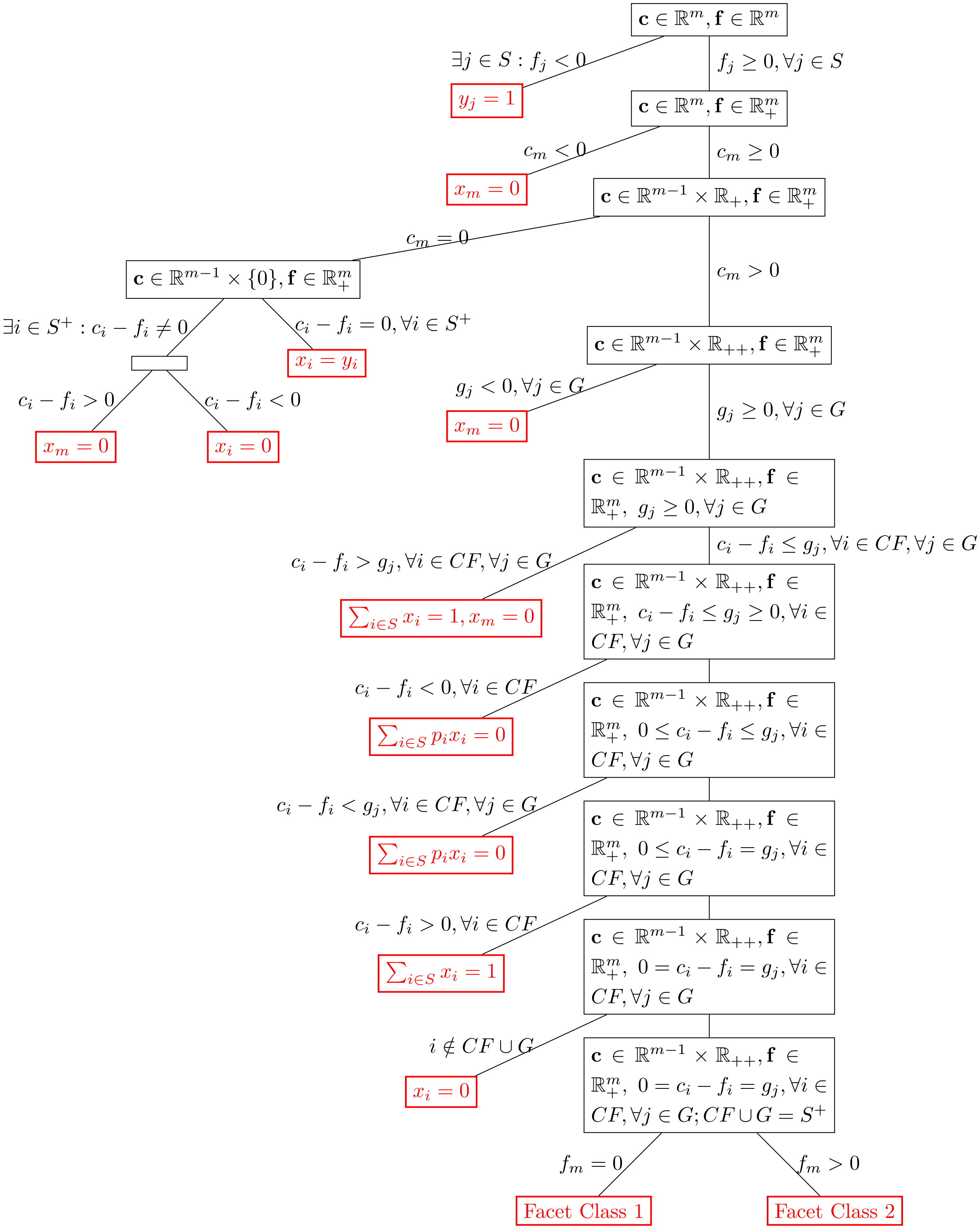

Figure 1: Proof Outline of Theorem 4