# Dynamic Linear Programming Games with Risk-Averse Players

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**Abstract**

Motivated by situations in which independent agents wish to cooperate in some uncertain endeavor over time, we study *dynamic linear programming games*, which generalize classical linear production games to multi-period settings under uncertainty. We specifically consider that players may have risk-averse attitudes towards uncertainty, and model this risk aversion using coherent conditional risk measures. For this setting, we study the *strong sequential core*, a natural extension of the core to dynamic settings. We characterize the strong sequential core as the set of allocations that satisfy a particular finite set of inequalities that depend on an auxiliary optimization model, and then leverage this characterization to establish sufficient conditions for emptiness and non-emptiness. Qualitatively, whereas the strong sequential core is always non-empty when players are risk-neutral, our results indicate that cooperation in the presence of risk aversion is much more difficult. We illustrate this with an application to cooperative newsvendor games, where we find that cooperation is possible when it least benefits players, and may be impossible when it offers more benefit.

**Keywords:** cooperative game, stochastic linear program, risk measure

## 1 Introduction

In many situations, a group of individuals, or players, may benefit from cooperating and participating in a joint enterprise, even if they do not share a common objective. For example, a set of retailers may take advantage of economies of scale by jointly managing their inventory and demand. In order for such a partnership to be successful, the participants must agree on how to share the costs they incur together. Cooperative game theory provides a mathematical framework for determining "fair" ways of sharing these costs of cooperation.

Applications of cooperative game theory in operations research and management science often assume that the costs of cooperation are static and deterministic. For many settings, this is enough:

when cooperating agents must pay for a one-time task, the costs need to be allocated only once, and the corresponding uncertainty may be negligible. However, in multi-period situations, such as supply chain management problems involving dynamic inventory, capacity or resource planning, costs are uncertain, must be allocated over time, and may be viewed as risky by the cooperating agents. While static cooperative game-theoretic models have been successfully applied in many supply chain management settings to design fair methods for sharing costs (Cachon and Netessine 2006, Nagarajan and Sošić 2008), depending on the context, accounting for dynamics, uncertainty and risk in cooperation may be more realistic, and even critical for practitioners. In this work, we study a widely applicable class of cooperative games, which we call *dynamic linear programming games*, in which the costs of cooperation are uncertain and evolve over time, and the agents involved may have risk-averse attitudes towards these costs.

In cooperative game theory, a *solution concept* is a method for allocating costs. One of the most well-studied solution concepts is the *core* (Gillies 1959), the set of allocations that are both economically efficient (i.e. exactly distribute the costs incurred by all the players) and stable against coalitional defections (i.e. the cost allocated to every subset of players is less than what it would incur on its own). Various refinements and relaxations of the core have also been proposed and studied, such as the *least core* (Maschler et al. 1979), the *$\alpha$-core* (Faigle and Kern 1993) and the *nucleolus* (Schmeidler 1969). Perhaps because of its simple definition and intuitive appeal, the core has been routinely studied as a method to allocate costs in supply chain management and other applications. A typical approach is to define a cooperative version of a classical model, design an allocation, and prove that it lies in the core. Such a result is thought to be a good indication that cooperation in the model is possible.

The core is also the starting point for our analysis. In particular, our goal is to adapt the notion of the core to the more general setting in which costs are dynamic and uncertain. To capture the possibility that the players are risk-averse and may have different attitudes towards risk, we use dynamic risk measures formed by the composition of *conditional risk mappings* (Ruszczyński and Shapiro 2006) to model a player's preferences for stochastic costs over time. This is an alternative to the traditional expected utility approach (von Neumann and Morgenstern 1944), and consistent with the dual theory of choice (Yaari 1987). Using this representation of player risk aversion, we study the *strong sequential core* (see e.g. Kranich et al. 2005) of dynamic linear programming games, the set of allocations that distribute costs as they are incurred and are stable against coalitional defections at any point in the time horizon.

## 1.1 Previous Related Work

There has been a considerable amount of research applying cooperative game theory to problems in operations research and management science. For example, in recent years, there has been an emerging body of literature on cooperative games arising from inventory management. In an *inventory centralization* or *newsvendor game* (e.g. Chen and Zhang 2009, Hartman and Dror 1996, Montrucchio and Scarsini 2007, Özen et al. 2008), a set of retailers face one period of uncertain demand and share the expected cost of a joint procurement and inventory scheme. In an *economic lot-sizing game* (e.g. Chen and Zhang 2006, Gopaladesikan et al. 2012, Toriello and Uhan 2014, van den Heuvel et al. 2007), a set of retailers face deterministic demand over a finite number of time periods and share the cost of jointly managing their collective inventory and demand over the entire time horizon. Other examples of inventory-related cooperative games include *inventory routing games* (Özener et al. 2013), *joint replenishment games* (He et al. 2012, Zhang 2009), and *production-inventory games* (Guardiola et al. 2008; 2009). For other applications of cooperative game theory to supply chain management, we refer the reader to Cachon and Netessine (2006) and Nagarajan and Sošić (2008).

The class of games that we focus on here – dynamic linear programming games – generalize the classic *linear production game*, originally proposed by Owen (1975), who used a simple and elegant argument based on linear programming (LP) duality to give a constructive proof that the core of these games is always non-empty. Linear production games have been extensively applied to operations research and management science problems, as many cooperative settings that can be modeled as an LP fall into this category. Particular examples include assignment games (Shapley and Shubik 1971), maximum flow games (Kalai and Zemel 1982), and network synthesis games (Tamir 1991), which include minimum cost spanning tree games (Granot and Huberman 1981) as a special case. Linear production games have also stimulated subsequent study (e.g. Flåm 2002, Granot 1986, Samet and Zemel 1984, van Gellekom et al. 2000). Furthermore, several researchers have since exploited the connection between other cooperative games and linear production games to establish computationally efficient and economically fair ways of allocating costs (e.g. Deng et al. 1999, Goemans and Skutella 2004, Toriello and Uhan 2013).

A few of the works cited above have mentioned issues related to using static allocations in dynamic, uncertain environments. For example, Chen and Zhang (2009) discussed how to modify their allocation of expected cost into an allocation of realized cost. Özener et al. (2013) designed some inventory routing cost allocations that depend on costs incurred per route. Flåm (2002) defined a static game in a multi-period stochastic setting, and discussed how successive static subgames have non-empty cores when optimal solutions are implemented. Nevertheless, the majority of the literature on cost sharing problems that arise in operations research and management science does not directly address the possibility that the costs of cooperation are stochastic and evolve dynamically.

Various researchers have studied how to incorporate uncertainty and dynamics into cooperative game-theoretic models and solution concepts. These models and solution concepts, however, have not enjoyed the same amount of attention as their static and deterministic counterparts. A number of researchers have considered models and solution concepts for cooperative games that allocate deterministic costs over time (e.g. Elkind et al. 2013, Filar and Petrosjan 2000, Habis and Herings 2010, Kranich et al. 2005, Lehrer and Scarsini 2013). Others have focused on models and solution concepts under uncertainty that is realized once (e.g. Alparslan-Gök et al. 2009, Chalkiadakis et al. 2007, Charnes and Granot 1973, Habis and Herings 2011a;b, Ieong and Shoham 2008, Myerson

2007, Predtetchinski et al. 2002, Suijs and Borm 1999, Suijs et al. 1999b) as well as uncertainty that evolves over time (e.g. Avrachenkov et al. 2013, Gale 1978, Petrosjan 2002, Predtetchinski 2007, Predtetchinski et al. 2004, Xu and Veinott 2013).

Some of the research mentioned above is especially relevant to our work. In particular, Kranich et al. (2005) studied the strong sequential core of a dynamic cooperative game, which we adapt to the setting we study here. The strong sequential core has been adapted and studied for other uncertain and dynamic settings, including two-period exchange economies (Habis and Herings 2011a, Predtetchinski et al. 2002), infinite-horizon dynamic exchange economies (Predtetchinski et al. 2004), and infinite-horizon stationary non-transferable utility cooperative games (Predtetchinski 2007). Xu and Veinott (2013) considered the strong sequential core in a setting similar to ours under the assumption that players are risk-neutral. In fact, some of our results in Section 4 can be seen as generalizations of their work's main theorem; we discuss this paper and its relation to our work in more detail in Section 4.3.

Our choice to represent the risk aversion of players with dynamic risk measures is also closely tied to some of the work cited above. Suijs and Borm (1999) studied so-called *stochastic cooperative games* with players whose attitudes towards risk before any uncertainty is realized are represented by static risk measures. Based on this idea, under the assumption that player preferences are represented by static convex risk measures, Uhan (2015) examined the core of *stochastic linear programming games*, a class of stochastic cooperative games that can be viewed as a special case of the dynamic linear programming games studied here. While these studies by Suijs and Borm (1999) and Uhan (2015) use risk measures to represent player preferences, they do not consider stochastic costs that evolve over time or how to represent player preferences in such a dynamic setting, like we do in this work.

Researchers in the operations research community have used risk measures to capture a decision-maker's attitude towards risk in the objective function of a variety of stochastic optimization problems. Their focus has typically been on coherent risk measures, since their supporting axioms are arguably reasonable from an economic perspective, and their convexity and robust representation yield nice structural and computational properties. Some have focused on specific coherent risk measures, such as the worst-case risk measure and conditional value-at-risk (e.g. Chan et al. 2014, Gotoh and Takano 2007, Rockafellar and Uryasev 2000, Wu et al. 2013, Yu 1998). Others have focused on more general classes of coherent risk measures (e.g. Ahmed et al. 2007, Choi et al. 2011). There has also been significant recent interest in constructing or approximating the uncertainty sets that define these measures, (e.g. Bertsimas and Brown 2009, Iancu et al. 2014). We refer the interested reader to Shapiro et al. (2014, Chapter 6) for additional references and a longer treatment.

### 1.2 Our Contributions

Our contributions can be summarized as follows. We introduce *dynamic linear programming games*, a class of cooperative games whose costs are uncertain and evolve over time. These games are a stochastic multi-period generalization of the classic linear production games (Owen 1975). We model player preferences on stochastic costs over time using dynamic risk measures formed from conditional risk mappings (Ruszczyński and Shapiro 2006). We focus on the *strong sequential core* of these games, the set of allocations that distribute costs as they are incurred and are stable against coalitional defections at any point in the time horizon (e.g. Kranich et al. 2005). In particular:

1. We characterize the strong sequential core as the set of allocations that satisfy a finite collection of inequalities. This characterization generalizes the well-known representation used for static games (Gillies 1959) and for risk-neutral dynamic games (e.g. Xu and Veinott 2013). The inequalities depend on optimization models that determine the minimum total risk a coalition incurs with a feasible allocation. The result also shows that the strong sequential core is convex if the players' risk measures are convex.

2. We explore the connection between the dual multipliers associated with the risk-minimizing optimization models mentioned above and the strong sequential core. Using this connection, we give sufficient conditions for the strong sequential core to be empty, as well as sufficient conditions for a dual-based allocation in the spirit of Owen (1975) and Xu and Veinott (2013) to be in the strong sequential core. Our results indicate that unlike the risk-neutral case, where the strong sequential core is always non-empty, when players are risk-averse the strong sequential core can easily be empty, intuitively suggesting that cooperation is more difficult when players account for risk.

3. We apply the results to newsvendor games where player preferences are represented by comonotonic risk measures. We provide examples that highlight how difficult it may be for risk-averse individuals to cooperate.

In Section 2, we introduce notation, necessary concepts and other preliminaries. We then formulate dynamic linear programming games as well as define and characterize the strong sequential core in Section 3. In Section 4, we use duality to give various conditions that determine whether the strong sequential core is empty or non-empty. Next, we apply our results to newsvendor games with risk-averse retailers in Section 5. Finally, in Section 6, we conclude and outline future avenues for research.

# 2 Preliminaries

## 2.1 Preferences over Stochastic Costs and Risk Measures

Consider a stochastic process whose evolution is represented by a finite-horizon scenario tree $\mathcal{T}$ with root node 1. For any node $t \in \mathcal{T}$, we denote $t$'s parent by $a(t)$ and its children by $\mathcal{D}_t$. We let $a(1) := 0$ represent the immediate past, i.e. incoming information at node 1. For any $t \in \mathcal{T}$, given appropriate probabilities, $\mathcal{D}_t$ can be viewed as a probability space and any vector $X \in \mathbb{R}^{\mathcal{D}_t}$ can be seen as a random variable. We will frequently consider the conditional event of reaching node $r \in \mathcal{T}$. To facilitate this, we denote the subtree rooted at node $r \in \mathcal{T}$ by $\mathcal{T}_r$, and for any pair of nodes $r \in \mathcal{T}$ and $t \in \mathcal{T}_r$, we let $\mathcal{P}(r,t)$ be the unique path from node $r$ to node $t$ in $\mathcal{T}$. In our setting, the component of $X \in \mathbb{R}^{\mathcal{D}_t}$ corresponding to node $\tau \in \mathcal{D}_t$ is a scalar that represents either an actual cost incurred at $\tau$ or a measure of risk based on the costs incurred throughout subtree $\mathcal{T}_\tau$.

To model a decision maker's preferences over stochastic costs that evolve over time, we use a *dynamic risk measure*. We construct this dynamic risk measure using *conditional risk mappings* (Ruszczyński and Shapiro 2006). First, we associate every node $t \in \mathcal{T}$ satisfying $\mathcal{D}_t \neq \varnothing$ with a *risk measure*, a function $\rho^t : \mathbb{R}^{\mathcal{D}_t} \to \mathbb{R}$ that satisfies the following axioms:

(M1) Monotonicity: If $X \geq Y$, then $\rho^t(X) \geq \rho^t(Y)$ for all $X, Y \in \mathbb{R}^{\mathcal{D}_t}$;

(M2) Translation invariance: $\rho^t([d + X^\tau]_{\tau \in \mathcal{D}_t}) = d + \rho^t(X)$ for all $X \in \mathbb{R}^{\mathcal{D}_t}$ and $d \in \mathbb{R}$.

In particular, we will focus on *coherent* risk measures $\rho^t$ for $t \in \mathcal{T}$ that satisfy the following additional properties (Artzner et al. 1999):

(M3) Positive homogeneity: $\rho^t(\alpha X) = \alpha \rho^t(X)$ for all $X \in \mathbb{R}^{\mathcal{D}_t}$ and $\alpha \geq 0$;

(M4) Convexity: $\rho^t(\alpha X + (1-\alpha)Y) \leq \alpha \rho^t(X) + (1-\alpha)\rho^t(Y)$ for all $X, Y \in \mathbb{R}^{\mathcal{D}_t}$ and $\alpha \in [0,1]$.

Assuming (M1) through (M3), convexity is equivalent to

(M4′) Subadditivity: $\rho^t(X+Y) \leq \rho^t(X) + \rho^t(Y)$ for all $X, Y \in \mathbb{R}^{\mathcal{D}_t}$.

It is well-known that coherent risk measures have a dual representation (e.g. Huber 1981); specifically, the coherent risk measures $\rho^t$ for $t \in \mathcal{T}$ can be represented as

$$\rho^t(X) = \max_{q \in \mathcal{Q}^t} \mathbb{E}_q[X] = \max_{q \in \mathcal{Q}^t} \sum_{\tau \in \mathcal{D}_t} q^\tau X^\tau \quad \text{for } X \in \mathbb{R}^{\mathcal{D}_t}, \tag{1}$$

where $\mathcal{Q}^t \subseteq \mathbb{R}^{\mathcal{D}_t}$ is a closed, convex set of probability measures on $\mathcal{D}_t$.

Next, given coherent risk measures $\rho^t$ for $t \in \mathcal{T}$, we define the *composite risk measure* $\phi^r : \mathbb{R}^{\mathcal{T}_r} \to \mathbb{R}$ for each node $r \in \mathcal{T}$ recursively as follows:

$$\phi^r([X^t]_{t \in \mathcal{T}_r}) = \begin{cases} X^r & \text{if } \mathcal{D}_r = \varnothing, \\ X^r + \rho^r\Big(\big[\phi^t([X^\tau]_{\tau \in \mathcal{T}_t})\big]_{t \in \mathcal{D}_r}\Big) & \text{if } \mathcal{D}_r \neq \varnothing. \end{cases} \tag{2}$$

When the conditional risk measures $\rho^t$ for $t \in \mathcal{T}$ are coherent, the composite risk measures $\phi^r$ for $r \in \mathcal{T}$ are also coherent: it is straightforward to see that $\phi^r$ for $r \in \mathcal{T}$ satisfies analogues of the four defining axioms. The composite risk measure $\phi^r$ represents a decision maker's preferences over streams of stochastic costs in a natural way:

$$(X^t)_{t \in \mathcal{T}_r} \text{ is as good as } (Y^t)_{t \in \mathcal{T}_r} \quad \Leftrightarrow \quad \phi^r([X^t]_{t \in \mathcal{T}_r}) \leq \phi^r([Y^t]_{t \in \mathcal{T}_r}).$$

In several places below, we pay close attention to the *risk-neutral* case, in which each set $\mathcal{Q}^t$ for $t \in \mathcal{T}$ is a singleton. This corresponds to the scenario tree being governed by a probability distribution, where the probability of reaching $t \in \mathcal{T}$ is $p^t := \prod_{\tau \in \mathcal{P}(1,t)\setminus\{1\}} q^\tau$; here, $q^\tau$ is the $\tau$-th coordinate of the unique element of $\mathcal{Q}^{a(\tau)}$, i.e. the conditional probability of reaching $\tau$ given that we have reached its parent. We may assume without loss of generality that $p^t > 0$ for all $t \in \mathcal{T}$. We denote the conditional probabilities by $p^{t|r} := p^t/p^r$ for $r \in \mathcal{T}$ and $t \in \mathcal{T}_r$. In this case, each conditional risk mapping $\rho^t$ is a conditional expectation, and the composite risk measure $\phi^1$ is the expectation over the entire scenario tree.

## 2.2 Cooperative Games and Linear Production Games

Cooperative games are a natural way to model situations in which a set of agents with possibly conflicting interests can collaborate to perform a common task at a lower cost. Traditionally, this task's cost is static (i.e. incurred once) and deterministic. Formally, a cooperative game is defined by a set $N := \{1, \dots, n\}$ of *players* (e.g. people or companies) and a function $f : 2^N \to \mathbb{R}$ that specifies the cost any subset of $N$, or *coalition*, incurs when performing the task, where $f(\varnothing) = 0$ by convention. The set of all players $N$ is called the *grand coalition.* Cooperative game theory is largely focused on the following question: assuming that the grand coalition agrees to cooperate and incur cost $f(N)$, how can this cost be allocated among the players in a "fair" way?

An *allocation* is a vector $\chi \in \mathbb{R}^N$ that assigns cost $\chi_i$ to player $i \in N$. Based on different notions of fairness, a *solution concept* determines a set of allocations it deems "fair" for a game. One of the most widespread solution concepts is the *core* (Gillies 1959), which imposes two conditions on an allocation. First, an allocation should at least cover the cost incurred: an allocation $\chi$ is *feasible* for a coalition $U \subseteq N$ if $\sum_{i \in U} \chi_i \geq f(U)$. Second, no coalition should have the incentive to defect, or leave the grand coalition, in the form of another allocation that results in all of its members being better off. Formally, the core consists of allocations $\chi$ satisfying the following two properties:

(C1) Feasibility: $\chi$ is feasible for the grand coalition $N$; i.e., $\sum_{i \in N} \chi_i \geq f(N)$.

(C2) Stability: For any coalition $\varnothing \neq U \subseteq N$, there does not exist an allocation $\xi \in \mathbb{R}^U$ that is feasible for $U$ and also satisfies $\xi_i < \chi_i$ for each $i \in U$.

This second condition is expressed in terms of Pareto optimality: the core is the set of non-dominated feasible allocations. It is simple to show that the core under this definition is equivalent to the polyhedron of allocations given by the inequalities

$$\sum_{i \in N} \chi_i \geq f(N), \tag{3a}$$

$$\sum_{i \in U} \chi_i \leq f(U) \qquad \text{for } U \subseteq N, \tag{3b}$$

which simply states that allocations in the core are feasible and allocate to each coalition a quantity not exceeding the coalition's stand-alone cost. In fact, this equivalence is so well-known and widespread that many researchers, especially in the operations research community, directly define the core as this polyhedron. However, we use the original Pareto definition, because the analogue of (3) is not as straightforward in our risk-averse setting.

*Linear production games* are a widely applicable class of cooperative games, first studied by Owen (1975). Given a common technology matrix $A$, a cost vector $c$ and a requirement or demand vector $d_i$ for each player $i \in N$, the cost function of the cooperative game is defined by a linear program

$$\begin{aligned} f(U) := \min_{x} \quad & cx \\ \text{s.t.} \quad & Ax = \sum_{i \in U} d_i \\ & x \geq 0. \end{aligned}$$

We assume the linear program above is feasible and bounded for any coalition $\varnothing \neq U \subseteq N$. Applying duality, we obtain

$$\begin{aligned} f(U) = \max_{\lambda} \quad & \lambda \sum_{i \in U} d_i \\ \text{s.t.} \quad & \lambda A \leq c. \end{aligned}$$

One of Owen's (1975) main results is that the allocation $\hat{\chi}_i := \hat{\lambda} d_i$ for every player $i \in N$ is in the core, where $\hat{\lambda}$ is a dual optimal solution when $U = N$. The proof follows directly from strong and weak duality.

**Notation.** To alleviate the notational burden and improve readability, in the remainder of the paper we use subsets as indices to indicate a vector's restriction. For example, given an allocation $\chi \in \mathbb{R}^N$, $\chi_U := (\chi_i)_{i \in U} \in \mathbb{R}^U$ is its restriction to the coalition $U \subseteq N$, and thus $\chi_N = \chi$. Similarly, for a random variable $X \in \mathbb{R}^{\mathcal{T}}$, $X^{\mathcal{T}_r} := (X^t)_{t \in \mathcal{T}_r}$ indicates its restriction to the subtree rooted at node $r$, and $X^{\mathcal{D}_r} := (X^t)_{t \in \mathcal{D}_r}$ is its restriction to the children of node $r$.

# 3 Dynamic Linear Programming Games

A set of players $N := \{1, \ldots, n\}$ cooperates in some endeavor over time in the presence of uncertainty. We model this uncertainty with a finite-horizon scenario tree $\mathcal{T}$, using the concepts and notation introduced in Section 2.1. At any node $t \in \mathcal{T}$, each player $i \in N$ has an initial state given by a vector $s_i^{a(t)}$ and a set of possible actions or decisions; these actions are represented by a vector $x_i^t$. A fixed vector $d_i^t$ represents player $i$'s demand or requirements at node $t$. If a coalition $U \subseteq N$ chooses to cooperate, the initial states, actions, and requirements of the players $i \in U$ determine the ending state vector $s_i^t$ for each player $i \in U$ through the linear system dynamics

$$\sum_{i \in U} A_i^t x_i^t + \sum_{i \in U} B_i^t s_i^{a(t)} - \sum_{i \in U} C_i^t s_i^t = \sum_{i \in U} d_i^t,$$

where the dimensions of all vectors and matrices are matched appropriately, and there are no redundant equations (i.e. the system has full row rank). Given initial states $s_i^{a(t)}$ and actions $x_i^t$, if more than one assignment of values to the $s_i^t$ vectors satisfies the equation, we assume the players choose one. For example, the dynamics may represent the coalition's inventory flow balance at some review period, in which case the $s_i^t$ vectors determine how the ownership of ending inventory is spread among the individual players in the coalition. The *set of feasible solutions* for coalition $U \subseteq N$ starting at node $r \in \mathcal{T}$ with initial state $\hat{s}_U^{a(r)}$ is

$$\mathcal{S}_U^r(\hat{s}_U^{a(r)}) := \left\{ \begin{array}{ll} (x,s) : \sum_{i\in U} A_i^r x_i^r - \sum_{i\in U} C_i^r s_i^r = \sum_{i\in U} d_i^r - \sum_{i\in U} B_i^r \hat{s}_i^{a(r)}, & \\ \qquad \sum_{i\in U} A_i^t x_i^t + \sum_{i\in U} B_i^t s_i^{a(t)} - \sum_{i\in U} C_i^t s_i^t = \sum_{i\in U} d_i^t & \text{for } t \in \mathcal{T}_r \setminus \{r\}, \\ \qquad x_i^t \geq 0, s_i^t \geq 0 & \text{for } i \in U, t \in \mathcal{T}_r \end{array} \right\}. \tag{4}$$

Let $\mathcal{S}_N := \mathcal{S}_N^1(\hat{s}^0)$, where $\hat{s}^0 = \hat{s}_N^0$ represents the players' initial state vectors at the start of the time horizon. We assume $\mathcal{S}_i^1(\hat{s}_i^0)$ is non-empty for each player $i \in N$, which implies $\mathcal{S}_U^1(\hat{s}_U^0)$ is non-empty for any coalition $U \subseteq N$. This assumption also implies that for each node $r \in \mathcal{T}$ and coalition $U \subseteq N$, $\mathcal{S}_U^r(\hat{s}_U^{a(r)})$ is non-empty for every state vector $\hat{s}_U^{a(r)}$ that is part of a feasible solution in $\mathcal{S}_U^1(\hat{s}_U^0)$, i.e. for every state reachable from $\hat{s}_U^0$. The cost incurred by coalition $U \subseteq N$ at node $t \in \mathcal{T}$ is $\sum_{i\in U}(c_i^t x_i^t + h_i^t s_i^t)$, where $c_i^t$ and $h_i^t$ are cost vectors of appropriate dimension. Note that we can easily work with discounted costs by incorporating a discount factor appropriately into the cost vectors $c_i^t$ and $h_i^t$.

We represent player preferences over random costs by dynamic risk measures, again using notation and concepts from Section 2.1. In particular, at each node $t \in \mathcal{T}$ with $\mathcal{D}_t \neq \varnothing$, each player $i \in N$ has a coherent risk measure $\rho_i^t : \mathbb{R}^{\mathcal{D}_t} \to \mathbb{R}$ represented by a closed, convex set of probability measures $\mathcal{Q}_i^t$, and the collection of these measures in any subtree $\mathcal{T}_r$ define a coherent dynamic risk measure $\phi_i^r : \mathbb{R}^{\mathcal{T}_r} \to \mathbb{R}$ for $r \in \mathcal{T}$.

As in the static case outlined in Section 2.2, we assume the players in the grand coalition $N$ agree to cooperate. Unlike that case, however, here the players agree to cooperate at the start of the time horizon in node 1, but they must allocate costs throughout the time horizon in a fair way, also accounting for the temporal aspects of this allocation. A *dynamic allocation* $\chi \in \mathbb{R}^{N\times\mathcal{T}}$, also called an *allocation stream* or *cooperative payoff distribution procedure* (Avrachenkov et al. 2013, Filar and Petrosjan 2000, Kranich et al. 2005, Petrosjan 2002), assigns a cost of $\chi_i^t$ to player $i \in N$ if the process reaches node $t \in \mathcal{T}$. The allocation $\chi$ is *feasible* for coalition $U \subseteq N$ starting at node $r \in \mathcal{T}$ with initial state $\hat{s}_U^{a(r)}$ if there exists a solution $(x,s) \in \mathcal{S}_U^r(\hat{s}_U^{a(r)})$ such that

$$\sum_{i\in U} \chi_i^t \geq \sum_{i\in U}(c_i^t x_i^t + h_i^t s_i^t) \quad \text{for } t \in \mathcal{T}_r;$$

that is, the allocation covers the coalition's incurred costs as they occur. Let $\mathcal{A}_U^r(\hat{s}_U^{a(r)})$ denote the corresponding set of feasible dynamic allocations, and let $\mathcal{A}_N := \mathcal{A}_N^1(\hat{s}^0)$.

### 3.1 The Strong Sequential Core

When allocations occur over time, the notion of what is fair or desirable can be strengthened by considering temporal aspects. The strong sequential core (e.g. Kranich et al. 2005, Predtetchinski 2007, Predtetchinski et al. 2002; 2004), also referred to as the stochastic sequential core (Xu and Veinott 2013), generalizes the core's concepts of feasibility and stability in a natural way: First, allocations must cover the grand coalition's costs *as they are incurred*; this differs from other approaches to cooperative game theory under uncertainty in which a single allocation is determined at the start of the time horizon, which is unrealistic in many situations. Second, the allocation must be stable not only at the beginning of the horizon, but also throughout the horizon, so that a coalition does not find it profitable to defect as the underlying stochastic process evolves. In our case, this *time-consistent stability* must hold with respect to players' risk preferences.

**Definition 1.** A dynamic allocation $\chi \in \mathcal{A}_N$ is in the *strong sequential core* for solution $(\hat{x}, \hat{s}) \in \mathcal{S}_N$ if it satisfies the following conditions:

(SSC1) Feasibility: $\chi$ is feasible for the grand coalition $N$ with respect to $(\hat{x}, \hat{s})$, i.e.

$$\sum_{i \in N} \chi_i^t \geq \sum_{i \in N} (c_i^t \hat{x}_i^t + h_i^t \hat{s}_i^t) \quad \text{for } t \in \mathcal{T}.$$

(SSC2) Time-Consistent Stability: For any coalition $\varnothing \neq U \subseteq N$ and any node $r \in \mathcal{T}$, there does not exist a dynamic allocation $\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})$ that also satisfies $\phi_i^r(\xi_i) < \phi_i^r(\chi_i^{\mathcal{T}_r})$ for every player $i \in U$.

Intuitively, the definition of the strong sequential core stipulates that the players in the grand coalition are willing to cooperate if they can agree on (i) a set of actions to implement over the time horizon, and (ii) an allocation of the cost of these actions that covers them as they are incurred, without at any point in the process putting a coalition in a position where every coalition member would perceive lower risk by defecting. In other words, if $\chi$ is in the strong sequential core, then for any coalition $U \subseteq N$ and node $r \in \mathcal{T}$, $\chi$ must be Pareto optimal among feasible allocations for $U$ with respect to those players' risk measures.

We explore the strong sequential core of a dynamic linear programming game through the following optimization model, which finds a feasible dynamic allocation that minimizes the total risk incurred by coalition $U \subseteq N$, starting at node $r \in \mathcal{T}$ with initial state $\hat{s}_U^{a(r)}$:

$$f^r(U, \hat{s}_U^{a(r)}) := \min_{\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})} \sum_{i \in U} \phi_i^r(\xi_i^{\mathcal{T}_r}).$$

We slightly abuse notation and denote both the optimal value of the above optimization model and the model itself as $f^r(U, \hat{s}_U^{a(r)})$.

**Theorem 2.** *A dynamic allocation $\chi \in \mathcal{A}_N$ is in the strong sequential core for solution $(\hat{x}, \hat{s}) \in \mathcal{S}_N$ if and only if it satisfies*

$$\sum_{i \in N} \chi_i^t \geq \sum_{i \in N} (c_i^t \hat{x}_i^t + h_i^t \hat{s}_i^t) \quad \textit{for } t \in \mathcal{T}, \tag{5a}$$

$$\sum_{i \in U} \phi_i^r(\chi_i^{\mathcal{T}_r}) \leq f^r(U, \hat{s}_U^{a(r)}) \qquad \textit{for } \varnothing \neq U \subseteq N, r \in \mathcal{T}. \tag{5b}$$

*Furthermore, if these conditions hold, $\chi$ satisfies*

$$\sum_{i \in N} \phi_i^r(\chi_i^{\mathcal{T}_r}) = f^r(N, \hat{s}^{a(r)}) \quad \textit{for } r \in \mathcal{T}. \tag{5c}$$

*In particular, $\chi$ is optimal for $f^1(N, \hat{s}^0)$.*

*Proof.* Condition (5a) simply restates (SSC1) for completeness. Therefore, it suffices to prove the equivalence of (5b) and (SSC2) under condition (5a). First suppose $\chi \in \mathcal{A}_N$ is in the strong sequential core with solution $(\hat{x}, \hat{s})$. For the purposes of contradiction, suppose there exists $\varnothing \neq U \subseteq N$ and $r \in \mathcal{T}$ such that (5b) does not hold. Let

$$\bar{\xi} \in \arg \min_{\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})} \sum_{i \in U} \phi_i^r(\xi_i),$$

and define

$$\Delta := \sum_{i \in U} \phi_i^r(\chi_i^{\mathcal{T}_r}) - \sum_{i \in U} \phi_i^r(\bar{\xi}_i) > 0.$$

Define an alternate dynamic allocation $\bar{\chi}$ for coalition $U$ starting at node $r$: for all $i \in U$,

$$\begin{aligned} \bar{\chi}_i^r &:= \bar{\xi}_i^r + \phi_i^r(\chi_i^{\mathcal{T}_r}) - \phi_i^r(\bar{\xi}_i) - \frac{\Delta}{n}, \\ \bar{\chi}_i^t &:= \bar{\xi}_i^t && \text{for } t \in \mathcal{T}_r \setminus \{r\}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{i \in U} \bar{\chi}_i^r &= \sum_{i \in U} \bar{\xi}_i^r + \Delta - \frac{\Delta}{n}|U| \geq \sum_{i \in U} \bar{\xi}_i^r, \\ \sum_{i \in U} \bar{\chi}_i^t &= \sum_{i \in U} \bar{\xi}_i^t && \text{for } t \in \mathcal{T}_r \setminus \{r\}, \end{aligned}$$

it follows that $\bar{\chi} \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})$. Moreover, for all $i \in U$,

$$\phi_i^r(\bar{\chi}_i) = \phi_i^r(\bar{\xi}_i) + \phi_i^r(\chi_i^{\mathcal{T}_r}) - \phi_i^r(\bar{\xi}_i) - \frac{\Delta}{n} < \phi_i^r(\chi_i^{\mathcal{T}_r}),$$

which is a contradiction, since this violates (SSC2) at node $r$.

Now suppose $\chi$ satisfies (5a) and (5b), and fix a node $r \in \mathcal{T}$. For any coalition $U \subseteq N$, we have

$$\sum_{i \in U} \phi_i^r(\chi_i^{\mathcal{T}_r}) \leq f^r(U, \hat{s}_U^{a(r)}) = \min_{\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})} \sum_{i \in U} \phi_i^r(\xi_i).$$

Therefore, for any coalition $U \subseteq N$ and dynamic allocation $\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})$, we must have $\phi_i^r(\chi_i^{\mathcal{T}_r}) \leq \phi_i^r(\xi_i)$ for at least one player $i \in U$, and thus $\chi$ satisfies (SSC2).

To prove the theorem's final statement, fix a node $r \in \mathcal{T}$. Since $\chi \in \mathcal{A}_N$, we have also $\chi^{\mathcal{T}_r} \in \mathcal{A}_N^r(\hat{s}^{a(r)})$, and therefore

$$\sum_{i \in N} \phi_i^r(\chi_i^{\mathcal{T}_r}) \geq \min_{\xi \in \mathcal{A}_N^r(\hat{s}^{a(r)})} \sum_{i \in N} \phi_i^r(\xi_i) = f^r(N, \hat{s}^{a(r)}),$$

which together with (5b) gives the result. □

Note that the proof of Theorem 2 does not use the fact that the risk measures $\phi_i^r$ for $i \in N$ and $r \in \mathcal{T}$ are coherent; the only property of risk measures that the proof uses is translation invariance (M2). In addition, if the risk measures $\phi_i^r$ for $i \in N$ and $r \in \mathcal{T}$ are all convex, then Theorem 2 implies the strong sequential core is convex as well.

**Corollary 3.** *Let $r \in \mathcal{T}$ and $U \subseteq N$. Suppose there exists an optimal solution to $f^r(U, \hat{s}_U^{a(r)})$ for some initial state $\hat{s}^{a(r)}$. If some player $j \in U$ satisfies $\mathcal{Q}_j^t \subseteq \mathcal{Q}_i^t$ for all $t \in \mathcal{T}_r$ and $i \in U \setminus \{j\}$, then*

$$f^r(U, \hat{s}_U^{a(r)}) = \min_{\xi \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})} \phi_j^r\left(\sum_{i \in U} \xi_i\right).$$

*Proof.* Let $\hat{\xi} \in \mathcal{A}_U^r(\hat{s}_U^{a(r)})$ be an optimal solution to $f^r(U, \hat{s}_U^{a(r)})$. It is straightforward to see that the allocation $\bar{\xi}$ defined as

$$\bar{\xi}_j^t = \sum_{i \in U} \hat{\xi}_i^t \quad \text{for } t \in \mathcal{T}_r, \qquad \bar{\xi}_i^t = 0 \quad \text{for } i \in U \setminus \{j\}, t \in \mathcal{T}_r$$

is also in $\mathcal{A}_U^r(\hat{s}_U^{a(r)})$ and therefore is a feasible solution to $f^r(U, \hat{s}_U^{a(r)})$. Note that by construction, $\bar{\xi}_j = \sum_{i \in U} \hat{\xi}_i$. Therefore,

$$f^r(U, \hat{s}_U^{a(r)}) \leq \sum_{i \in U} \phi_i^r(\bar{\xi}_i) \stackrel{\text{(i)}}{=} \phi_j^r(\bar{\xi}_j) = \phi_j^r\left(\sum_{i \in U} \hat{\xi}_i\right) \stackrel{\text{(ii)}}{\leq} \sum_{i \in U} \phi_j^r(\hat{\xi}_i) \stackrel{\text{(iii)}}{\leq} \sum_{i \in U} \phi_i^r(\hat{\xi}_i) = f^r(U, \hat{s}_U^{a(r)}).$$

where (i) holds by construction of $\bar{\xi}$ and the positive homogeneity (M3) of $\phi_i^r$ for all $i \in U$, (ii) holds due to the subadditivity (M4′) of $\phi_j^r$, and (iii) holds because $\mathcal{Q}_j^t \subseteq \mathcal{Q}_i^t$ for all $t \in \mathcal{T}_r$ and $i \in U \setminus \{j\}$. □

Intuitively, this corollary states that if one player's level of risk aversion is less than or equal to all the others', an optimal solution of $f^r(U, \hat{s}_U^{a(r)})$ can allocate all costs to him and simply optimize with respect to his risk measure. For instance, this would apply if all players have the same attitude towards risk. The result highlights a natural difficulty in allocating costs among risk-averse players: An allocation that optimizes the total risk experienced by a coalition results in costs assigned to players who are less risk-averse, but an allocation in the strong sequential core must balance this optimality with the stability condition that would intuitively need to spread costs to other players, including more risk-averse ones. Any attempt to shift costs to more risk-averse players must do so without increasing the total risk, which may be difficult or even impossible.

## 4 Dual Solutions and the Strong Sequential Core

In this section, we explore the connection between the strong sequential core and the dual solutions of the mathematical programs $f^r(U, \hat{s}_U^{a(r)})$ for every node $r \in \mathcal{T}$ and coalition $U \subseteq N$. We begin by using the robust representation of coherent risk measures $\rho_i^t$ in (1) and the recursive definition of the composite risk measures $\phi_i^t$ in (2) to reformulate $f^r(U, \hat{s}_U^{a(r)})$ as

$$\begin{aligned}
&f^r(U, \hat{s}_U^{a(r)}) \\
&\quad = \min_{\xi,z,x,s} \sum_{i \in U} z_i^r && && \text{(6a)} \\
&\qquad \text{s.t. } z_i^t - \sum_{\tau \in \mathcal{D}_t} q^\tau z_i^\tau - \xi_i^t \geq 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t \neq \varnothing, q \in \mathcal{Q}_i^t, && \text{(6b)} \\
&\qquad\quad z_i^t - \xi_i^t \geq 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t = \varnothing, && \text{(6c)} \\
&\qquad\quad \sum_{i \in U} (\xi_i^t - c_i^t x_i^t - h_i^t s_i^t) \geq 0 && \text{for } t \in \mathcal{T}_r, && \text{(6d)} \\
&\qquad\quad \sum_{i \in U} A_i^r x_i^r - \sum_{i \in U} C_i^r s_i^r = \sum_{i \in U} d_i^r - \sum_{i \in U} B_i^r \hat{s}_i^{a(r)}, && && \text{(6e)} \\
&\qquad\quad \sum_{i \in U} A_i^t x_i^t + \sum_{i \in U} B_i^t s_i^{a(t)} - \sum_{i \in U} C_i^t s_i^t = \sum_{i \in U} d_i^t && \text{for } t \in \mathcal{T}_r \setminus \{r\}, && \text{(6f)} \\
&\qquad\quad x_i^t \geq 0, s_i^t \geq 0 && \text{for } i \in U, t \in \mathcal{T}_r. && \text{(6g)}
\end{aligned}$$

Recall that the sets $\mathcal{Q}_i^t$ for all players $i \in N$ and nodes $t \in \mathcal{T}$ are closed and convex, and therefore $f^r(U, \hat{s}_U^{a(r)})$ is a linear semi-infinite program. The constraints (6b) can equivalently be written to include only the constraints for every extreme point of $\mathcal{Q}_i^t$, so when the sets $\mathcal{Q}_i^t$ are all polyhedra, the model simplifies to a (finite) linear program. Since we assume the equality constraints have no redundant equations (i.e. they have full row rank), the finite-support Haar dual – the semi-infinite analogue of the finite LP dual – is in fact a strong dual for $f^r(U, \hat{s}_U^{a(r)})$, because the coefficients of constraints (6b) are continuous functions of the sets $\mathcal{Q}_i^t$ and the model has an easily constructed Slater point (see e.g. Glashoff and Gustafson 1983, Goberna and López 1998). This dual is

$$f^r(U, \hat{s}_U^{a(r)}) = \max_{\lambda,\pi,\mu,\sigma} \sum_{t \in \mathcal{T}_r} \lambda^t \sum_{i \in U} d_i^t - \lambda^r \sum_{i \in U} B_i^r \hat{s}_i^{a(r)} \qquad \text{(7a)}$$

$$
\begin{aligned}
\text{s.t. } & \lambda^t A_i^t - \pi^t c_i^t \leq 0 && \text{for } i \in U, t \in \mathcal{T}_r, && \text{(7b)}\\
& \sum_{\tau \in \mathcal{D}_t} \lambda^\tau B_i^\tau - \lambda^t C_i^t - \pi^t h_i^t \leq 0 && \text{for } i \in U, t \in \mathcal{T}_r, && \text{(7c)}\\
& \pi^t - \sigma_i^t = 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t = \varnothing, && \text{(7d)}\\
& \pi^t - \sum_{q \in \mathcal{Q}_i^t} \mu_i^{t,q} = 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t \neq \varnothing, && \text{(7e)}\\
& \sum_{q \in \mathcal{Q}_i^r} \mu_i^{r,q} = 1 && \text{for } i \in U, && \text{(7f)}\\
& \sum_{q \in \mathcal{Q}_i^t} \mu_i^{t,q} - \sum_{q \in \mathcal{Q}_i^{a(t)}} \mu_i^{a(t),q} q^t = 0 && \text{for } i \in U, t \in \mathcal{T}_r \setminus \{r\} : \mathcal{D}_t \neq \varnothing, && \text{(7g)}\\
& \sigma_i^t - \sum_{q \in \mathcal{Q}_i^{a(t)}} \mu_i^{a(t),q} q^t = 0 && \text{for } i \in U, t \in \mathcal{T}_r \setminus \{r\} : \mathcal{D}_t = \varnothing, && \text{(7h)}\\
& \pi^t \geq 0 && \text{for } t \in \mathcal{T}_r, && \text{(7i)}\\
& \mu_i^{t,q} \geq 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t \neq \varnothing, q \in \mathcal{Q}_i^t, && \text{(7j)}\\
& \sigma_i^t \geq 0 && \text{for } i \in U, t \in \mathcal{T}_r : \mathcal{D}_t = \varnothing, && \text{(7k)}\\
& \mu \text{ has finite support.} && && \text{(7l)}
\end{aligned}
$$

The model has an intuitive probabilistic interpretation that yields interesting consequences. The variables $\pi$ can be interpreted as a probability measure over $\mathcal{T}_r$, conditional on arriving at node $r$, while the variables $\mu$ and $\sigma$ convey probabilistic information about each player's worst-case measure over $\mathcal{T}_r$; that is, the measure achieving the minimum in (2). Constraint (7f) indicates that at node $r$, each player $i \in U$ gets a unit of probability mass that must be assigned between the measures in $\mathcal{Q}_i^r$. The flow balance of probabilities down the scenario tree is enforced by (7g), while each leaf node $t$'s probability is assigned to variable $\sigma_i^t$ in (7h). Constraints (7d) and (7e) force the players' worst-case measures to coincide, equating them to $\pi$. The variables $\lambda$ correspond to the underlying linear system dynamics, and are linked to the rest of the model only through $\pi$ in constraints (7b) and (7c). We next formalize some of these structural properties.

## 4.1 Structural Properties of Dual Solutions

**Lemma 4.** *Let $r \in \mathcal{T}$ and $U \subseteq N$. For some initial state $\hat{s}^{a(r)}$, suppose $(\lambda, \pi, \mu, \sigma)$ is dual feasible for $f^r(U, \hat{s}_U^{a(r)})$. Then $\pi$ is a probability measure over $\mathcal{T}_r$; in particular, if $\pi^\ell = 0$ for some $\ell \in \mathcal{T}_r$, then $\pi^t = 0$ for all $t \in \mathcal{T}_\ell$. Moreover, for any $\ell \in \mathcal{T}_r$ with $\pi^\ell > 0$ and $\mathcal{D}_\ell \neq \varnothing$,*

$$
\left(\frac{\pi^t}{\pi^\ell}\right)_{t \in \mathcal{D}_\ell} \in \bigcap_{i \in U} \mathcal{Q}_i^\ell. \tag{8}
$$

*Proof.* The first statement directly follows from constraints (7d)-(7k). For the second, fix $\ell \in \mathcal{T}_r$ such that $\mathcal{D}_\ell \neq \varnothing$ and $\pi^\ell > 0$. By (7e)-(7g) and (7j), any dual feasible solution $(\lambda, \pi, \mu, \sigma)$ of

$f^r(U, \hat{s}_U^{a(r)})$ must satisfy

$$\begin{aligned}
\pi^\ell &= \sum_{q \in \mathcal{Q}_i^\ell} \mu_i^{\ell,q} && \text{for } i \in U, \\
\pi^t &= \sum_{q \in \mathcal{Q}_i^t} \mu_i^{t,q} = \sum_{q \in \mathcal{Q}_i^\ell} \mu_i^{\ell,q} q^t && \text{for } i \in U, t \in \mathcal{D}_\ell, \\
\mu_i^{\ell,q} &\geq 0 && \text{for } i \in U, q \in \mathcal{Q}_i^\ell.
\end{aligned}$$

Since $\pi^\ell > 0$, it follows that any dual feasible solution $(\lambda, \pi, \mu, \sigma)$ of $f^r(U, \hat{s}_U^{a(r)})$ must satisfy

$$\begin{aligned}
\sum_{q \in \mathcal{Q}_i^\ell} \frac{\mu_i^{\ell,q}}{\pi^\ell} &= 1 && \text{for } i \in U, \\
\frac{\pi^t}{\pi^\ell} = \sum_{q \in \mathcal{Q}_i^t} \frac{\mu_i^{t,q}}{\pi^\ell} &= \sum_{q \in \mathcal{Q}_i^\ell} \left(\frac{\mu_i^{\ell,q}}{\pi^\ell}\right) q^t && \text{for } i \in U, t \in \mathcal{D}_\ell, \\
\frac{\mu_i^{\ell,q}}{\pi^\ell} &\geq 0 && \text{for } i \in U, q \in \mathcal{Q}_i^\ell,
\end{aligned}$$

which is equivalent to (8). □

The next lemma establishes that optimal solutions to $f^r(U, \hat{s}_U^{a(r)})$ are time consistent.

**Lemma 5.** *Fix $r \in \mathcal{T}$ and $U \subseteq N$. Let $(\hat{\xi}, \hat{z}, \hat{x}, \hat{s})$ and $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ be primal and dual optimal solutions to $f^r(U, \hat{s}_U^{a(r)})$ for some initial state $\hat{s}^{a(r)}$, respectively. Then for any $\ell \in \mathcal{T}_r \setminus \{r\}$ such that $\hat{\pi}^\ell > 0$, $(\bar{\xi}, \bar{z}, \bar{x}, \bar{s}) = (\hat{\xi}^{\mathcal{T}_\ell}, \hat{z}^{\mathcal{T}_\ell}, \hat{x}^{\mathcal{T}_\ell}, \hat{s}^{\mathcal{T}_\ell})$ is a primal optimal solution to $f^\ell(U, \hat{s}_U^{a(\ell)})$, and $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma}) = (\hat{\lambda}^{\mathcal{T}_\ell}, \hat{\pi}^{\mathcal{T}_\ell}, \hat{\mu}^{\mathcal{T}_\ell}, \hat{\sigma}^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ is a dual optimal solution to $f^\ell(U, \hat{s}_U^{a(\ell)})$.*

*Proof.* Fix $\ell \in \mathcal{T}_r$. It is clear that $(\bar{\xi}, \bar{z}, \bar{x}, \bar{s})$ is a primal feasible solution of $f^\ell(U, \hat{s}_U^{a(\ell)})$. We show that $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ is dual feasible for $f^\ell(U, \hat{s}_U^{a(\ell)})$. The solution $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ satisfies constraints (7b)-(7e) and (7g)-(7k) in $f^r(U, \hat{s}_U^{a(r)})$, and so $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ satsifies these same constraints in $f^\ell(U, \hat{s}_U^{a(\ell)})$, since these constraints in $f^\ell(U, \hat{s}_U^{a(\ell)})$ are a subset of those in $f^r(U, \hat{s}_U^{a(r)})$ and $\hat{\pi}^\ell > 0$. From (7e), we have that

$$\sum_{q \in Q_i^\ell} \bar{\mu}_i^{\ell,q} = \sum_{q \in \mathcal{Q}_i^\ell} \frac{\hat{\mu}_i^{\ell,q}}{\hat{\pi}^\ell} = 1 \quad \text{for } i \in U,$$

and so $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ also satisfies constraints (7f) in $f^\ell(U, \hat{s}_U^{a(\ell)})$. Therefore, $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ is dual feasible for $f^\ell(U, \hat{s}_U^{a(\ell)})$.

The objective function value of the dual solution $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ in $f^\ell(U, \hat{s}_U^{a(\ell)})$, multiplied by $\hat{\pi}^\ell$, is

$$\begin{aligned}
\hat{\pi}^\ell \left[\bar{\lambda}^\ell \sum_{i \in U} (d_i^\ell - B_i^\ell \hat{s}_i^{a(\ell)}) + \sum_{t \in \mathcal{T}_\ell \setminus \{\ell\}} \bar{\lambda}^t \sum_{i \in U} d_i^t\right] &= \hat{\lambda}^\ell \sum_{i \in U} (d_i^\ell - B_i^\ell \hat{s}_i^{a(\ell)}) + \sum_{t \in \mathcal{T}_\ell \setminus \{\ell\}} \hat{\lambda}^t \sum_{i \in U} d_i^t \\
&= \hat{\lambda}^\ell \sum_{i \in U} (A_i^\ell \hat{x}_i^\ell - C_i^\ell \hat{s}_i^\ell) + \sum_{t \in \mathcal{T}_\ell \setminus \{\ell\}} \hat{\lambda}^t \sum_{i \in U} (A_i^t \hat{x}_i^t + B_i^t \hat{s}_i^{a(t)} - C_i^t \hat{s}_i^t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t\in\mathcal{T}_\ell}\sum_{i\in U} \hat{\lambda}^t A_i^t \hat{x}_i^t + \sum_{t\in\mathcal{T}_\ell}\sum_{i\in U}\left[\sum_{\tau\in\mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau - C_i^t\right]\hat{s}_i^t \overset{\text{(i)}}{=} \sum_{t\in\mathcal{T}_\ell}\sum_{i\in U} \hat{\pi}^t c_i^t \hat{x}_i^t + \sum_{t\in\mathcal{T}_\ell}\sum_{i\in U} \hat{\pi}^t h_i^t \hat{s}_i^t \\
&= \sum_{t\in\mathcal{T}_\ell} \hat{\pi}^t \sum_{i\in U}(c_i^t\hat{x}_i^t + h_i^t \hat{s}_i^t) \overset{\text{(ii)}}{=} \sum_{t\in\mathcal{T}_\ell} \hat{\pi}^t \sum_{i\in U} \hat{\xi}_i^t = \sum_{i\in U}\left[\sum_{t\in\mathcal{T}_\ell:\mathcal{D}_t\neq\varnothing} \hat{\pi}^t\hat{\xi}_i^t + \sum_{t\in\mathcal{T}_r:\mathcal{D}_t=\varnothing} \hat{\pi}^t\hat{\xi}_i^t\right] \\
&\overset{\text{(iii)}}{=} \sum_{i\in U}\left[\sum_{t\in\mathcal{T}_\ell:\mathcal{D}_t\neq\varnothing}\sum_{q\in\mathcal{Q}_i^t} \hat{\mu}_i^{t,q}\hat{\xi}_i^t + \sum_{t\in\mathcal{T}_r:\mathcal{D}_t=\varnothing} \hat{\sigma}_i^t\hat{\xi}_i^t\right] \\
&\overset{\text{(iv)}}{=} \sum_{i\in U}\left[\sum_{t\in\mathcal{T}_\ell:\mathcal{D}_t\neq\varnothing}\sum_{q\in\mathcal{Q}_i^t} \hat{\mu}_i^{t,q}(\hat{z}_i^t - \sum_{\tau\in\mathcal{D}_t} q^\tau \hat{z}_i^\tau) + \sum_{t\in\mathcal{T}_r:\mathcal{D}_t=\varnothing} \hat{\sigma}_i^t\hat{z}_i^t\right] \\
&= \sum_{i\in U}\left[\sum_{q\in\mathcal{Q}_i^\ell} \hat{\mu}_i^{\ell,q}\hat{z}_i^\ell + \sum_{t\in\mathcal{T}_\ell\setminus\{\ell\}:\mathcal{D}_t\neq\varnothing}\left(\sum_{q\in\mathcal{Q}_i^t}\hat{\mu}_i^{t,q} - \sum_{q\in\mathcal{Q}_i^{a(t)}} \hat{\mu}_i^{a(t),q} q^t\right)\hat{z}_i^t\right. \\
&\qquad \left. + \sum_{t\in\mathcal{T}_\ell\setminus\{\ell\}:\mathcal{D}_t=\varnothing}\left(\hat{\sigma}_i^t - \sum_{q\in\mathcal{Q}_i^{a(t)}} \hat{\mu}_i^{a(t),q} q^t\right)\hat{z}_i^t\right] \overset{\text{(v)}}{=} \hat{\pi}^\ell \sum_{i\in U}\hat{z}_i^\ell = \hat{\pi}^\ell \sum_{i\in U} \bar{z}_i^\ell,
\end{aligned}
$$

where (i), (ii), and (iv) follow from complementary slackness, and (iii) and (v) follow from dual feasibility. Therefore, $(\bar{\xi}, \bar{z}, \bar{x}, \bar{s})$ and $(\bar{\lambda}, \bar{\pi}, \bar{\mu}, \bar{\sigma})$ are primal and dual feasible solutions of $f^\ell(U, \hat{s}_U^{a(\ell)})$, respectively, with the same objective function value, and so they are in fact optimal solutions. □

## 4.2 Implications on the Strong Sequential Core

**Corollary 6.** *For any $r \in \mathcal{T}$ and $U \subseteq N$, let $(\hat{\xi}, \hat{z}, \hat{x}, \hat{s})$ and $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ be primal and dual optimal solutions to $f^r(U, \hat{s}_U^{a(r)})$ for some initial state $\hat{s}^{a(r)}$, respectively. Then*

$$
\sum_{i\in U} \phi_i^r(\hat{\xi}_i) = \sum_{i\in U} \mathbb{E}_{\hat{\pi}}[\hat{\xi}_i] = \sum_{i\in U}\sum_{t\in\mathcal{T}_r} \hat{\pi}^t \hat{\xi}_i^t. \tag{9}
$$

*Proof.* Follows from the proof of Lemma 5. □

By Theorem 2, if $\hat{\chi}$ is in the strong sequential core for solution $(\hat{x}, \hat{s})$, then $\hat{\chi}$ is an optimal solution for $f^1(N, \hat{s}^0)$. The above corollary states that any such allocation must be *risk-aligned*; that is, the same measure over $\mathcal{T}$ must yield the worst-case expected cost for all players. In other words, an allocation in the strong sequential core must assign costs so that all players expect the same worst-case measure over possible outcomes. Furthermore, Lemma 4 implies that, wherever it is positive, the conditional probabilities implied by this worst-case measure must lie in the intersection of all players' uncertainty sets.

The following result states that the strong sequential core must be empty if some coalition's sets of possible probability distributions do not intersect at some node.

**Corollary 7.** *Suppose there exists a coalition $U \subseteq N$, a node $r \in \mathcal{T}$ with $\mathcal{D}_r \neq \varnothing$, and a node $\ell \in \mathcal{T}_r$ with $\mathcal{D}_\ell \neq \varnothing$ such that*

(a) $\bigcap_{i\in U} \mathcal{Q}_i^\ell = \varnothing$,

(b) $q^t > 0$ *for all* $q \in \bigcap_{i \in U} \mathcal{Q}_i^{a(t)}$ *and* $t \in \mathcal{P}(r, \ell) \setminus \{r\}$.

*In addition, suppose* $\mathcal{S}_U^r(\hat{s}_U^{a(r)})$ *is non-empty for some initial state* $\hat{s}^{a(r)}$. *Then* $f^r(U, \hat{s}_U^{a(r)})$ *is unbounded from below. Furthermore, if* (a) *holds for any* $\ell \in \mathcal{T}$, *the strong sequential core is empty.*

*Proof.* Fix a coalition $U \subseteq N$, a node $r \in \mathcal{T}$ with $\mathcal{D}_r \neq \varnothing$, and a node $\ell \in \mathcal{T}_r$ with $\mathcal{D}_\ell \neq \varnothing$ such that (a) and (b) hold. We show that the dual of $f^r(U, \hat{s}_U^{a(r)})$ is infeasible by contradiction: suppose $(\lambda, \pi, \mu, \sigma)$ is a dual feasible solution of $f^r(U, \hat{s}_U^{a(r)})$. We have that

$$\pi^t = \sum_{q \in \mathcal{Q}_i^t} \mu_i^{t,q} = \sum_{q \in \mathcal{Q}_i^{a(t)}} \mu_i^{a(t),q} q^t \quad \text{for } i \in U, t \in \mathcal{P}(r, \ell).$$

Since (b) holds, we must have $\pi^t > 0$ for all $t \in \mathcal{P}(r, \ell)$, and in particular, $\pi^\ell > 0$. It follows by Lemma 4 that $(\lambda, \pi, \mu, \sigma)$ must satisfy (8) which contradicts (a). Therefore, the dual of $f^r(U, \hat{s}_U^{a(r)})$ must be infeasible, and so $f^r(U, \hat{s}_U^{a(r)})$ is unbounded from below.

When (a) alone holds, the above argument with $r = \ell$ in conjunction with Theorem 2 implies that the strong sequential core is empty. □

Under condition (a) of this corollary, a coalition completely disagrees on how likely the children of node $\ell$ will realize, conditioned upon arriving at $\ell$. This allows for infinite risk arbitrage in $f^\ell(U, \hat{s}_U^{a(\ell)})$, where each player bets an arbitrary amount on his own beliefs against scenarios he sees as impossible. In practical terms, players cannot hope to cooperate if their attitudes towards risk at some potential node are completely incompatible.

We next address how to use dual solutions to construct an allocation. These allocations depend in particular on the grand coalition's risk-aligned worst-case measure $\hat{\pi}$. The following assumption addresses what may happen when this measure gives zero probability mass to some parts of the scenario tree.

**Assumption 8.** At least one of the following holds:

(a) $q^t > 0$ for all $q \in \bigcap_{i \in N} \mathcal{Q}_i^{a(t)}$ and $t \in \mathcal{T} \setminus \{1\}$.
(b) $\{\hat{\lambda}^t : \hat{\lambda}^t A_i^t \leq 0, \hat{\lambda}^t C_i^t \geq 0\} = \{0\}$ for all $i \in N$ and $t \in \mathcal{T}$.

The first assumption forces the risk-aligned worst-case measure to have positive support over the entire scenario tree $\mathcal{T}$. The second forces any leaf node $t$ assigned zero probability mass by $\hat{\pi}$ – that is, $\hat{\pi}^t = 0$ – to in turn have its corresponding dual variable $\hat{\lambda}^t = 0$ as well. This proceeds inductively backwards through any subtree with zero probability mass. Under this assumption, the following lemma hints at how an optimal dual solution can be used to construct an allocation satisfying the conditions of Theorem 2.

**Lemma 9.** *Fix* $r \in \mathcal{T}$. *Let* $(\hat{\xi}, \hat{z}, \hat{x}, \hat{s})$ *and* $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ *respectively be primal and dual optimal solutions to* $f^r(N, \hat{s}^{a(r)})$ *for some initial state* $\hat{s}^{a(r)}$, *and suppose Assumption 8 holds. Define the* *allocation*

$$\hat{\chi}_i^t = \begin{cases} \dfrac{1}{\hat{\pi}^t}\left[\hat{\lambda}^t(d_i^t - B_i^t \hat{s}_i^{a(t)}) + \displaystyle\sum_{\tau\in\mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\right] & \text{if } \hat{\pi}^t > 0, \\ 0 & \text{if } \hat{\pi}^t = 0 \end{cases} \tag{10}$$

*for each $i \in N$ and $t \in \mathcal{T}_r$. If*

$$\sum_{i\in N} \phi_i^r(\hat{\chi}_i^{\mathcal{T}_r}) = f^r(N, \hat{s}^{a(r)}),$$

*then for every node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$,*

$$\sum_{i\in N} \hat{\chi}_i^\ell = \sum_{i\in N}(c_i^\ell \hat{x}_i^\ell + h_i^\ell \hat{s}_i^\ell), \tag{11a}$$

$$\sum_{i\in N} \phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) = f^\ell(N, \hat{s}^{a(\ell)}), \tag{11b}$$

$$\sum_{i\in U} \phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) \le f^\ell(U, \hat{s}^{a(\ell)}) \quad \text{for } U \subseteq N. \tag{11c}$$

*Proof.* First, we show that (11a) holds. At each node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$, we have

$$\begin{aligned} \hat{\pi}^\ell \sum_{i\in N} \hat{\chi}_i^\ell &= \hat{\lambda}^\ell \sum_{i\in N}(d_i^\ell - B_i^\ell \hat{s}_i^{a(\ell)}) + \sum_{t\in\mathcal{D}_\ell} \hat{\lambda}^t \sum_{i\in N} B_i^t \hat{s}_i^\ell \overset{\text{(i)}}{=} \hat{\lambda}^\ell \sum_{i\in N}(A_i^\ell \hat{x}_i^\ell - C_i^\ell \hat{s}_i^\ell) + \sum_{t\in\mathcal{D}_\ell} \hat{\lambda}^t \sum_{i\in N} B_i^t \hat{s}_i^\ell \\ &= \sum_{i\in N} \hat{\lambda}^\ell A_i^\ell \hat{x}_i^\ell + \sum_{i\in N}\left[\sum_{t\in\mathcal{D}_\ell} \hat{\lambda}^t B_i^t - \hat{\lambda}^\ell C_i^\ell\right]\hat{s}_i^\ell \overset{\text{(ii)}}{=} \hat{\pi}^\ell \sum_{i\in N} c_i^\ell \hat{x}_i^\ell + \hat{\pi}^\ell \sum_{i\in N} h_i^\ell \hat{s}_i^\ell, \end{aligned} \tag{12}$$

where (i) holds due to primal feasibility and (ii) holds due to complementary slackness in $f^r(N, \hat{s}^{a(r)})$.

Next, using the robust representation of coherent risk measures in (1) and the recursive definition of the composite risk measures in (2), we formulate $\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell})$ for each player $i \in N$ and node $\ell \in \mathcal{T}_r$ as the following primal-dual pair of linear semi-infinite programs:

$$\begin{aligned} \phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) = \min_{z_i} \quad & z_i^\ell \\ \text{s.t.} \quad & z_i^t - \sum_{\tau\in\mathcal{D}_t} q^\tau z_i^\tau \ge \hat{\chi}_i^t && \text{for } t \in \mathcal{T}_\ell : \mathcal{D}_t \neq \varnothing, q \in \mathcal{Q}_i^t, \\ & z_i^t \ge \hat{\chi}_i^t && \text{for } t \in \mathcal{T}_\ell : \mathcal{D}_t = \varnothing \\ = \max_{\mu_i,\sigma_i} \quad & \sum_{t\in\mathcal{T}_\ell:\mathcal{D}_t\neq\varnothing} \sum_{q\in\mathcal{Q}_i^t} \hat{\chi}_i^t \mu_i^{t,q} + \sum_{t\in\mathcal{T}_\ell:\mathcal{D}_t=\varnothing} \hat{\chi}_i^t \sigma_i^t \\ \text{s.t.} \quad & \sum_{q\in\mathcal{Q}_i^\ell} \mu_i^{\ell,q} = 1, \\ & \sum_{q\in\mathcal{Q}_i^t} \mu_i^{t,q} = \sum_{q\in\mathcal{Q}_i^{a(t)}} \mu_i^{a(t),q} q^t && \text{for } t \in \mathcal{T}_\ell \setminus \{\ell\} : \mathcal{D}_t \neq \varnothing, \\ & \sigma_i^t = \sum_{q\in\mathcal{Q}_i^{a(t)}} \mu_i^{a(t),q} q^t && \text{for } t \in \mathcal{T}_\ell \setminus \{\ell\} : \mathcal{D}_t = \varnothing, \end{aligned}$$

$$\mu_i^{t,q} \geq 0 \qquad \text{for } t \in \mathcal{T}_\ell : \mathcal{D}_t \neq \varnothing, q \in \mathcal{Q}_i^t,$$
$$\sigma_i^t \geq 0 \qquad \text{for } t \in \mathcal{T}_\ell : \mathcal{D}_t = \varnothing.$$

For every node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$, consider the solution $(\hat{\lambda}^{\mathcal{T}_\ell}, \hat{\mu}^{\mathcal{T}_\ell}, \hat{\sigma}^{\mathcal{T}_\ell}, \hat{\pi}^{\mathcal{T}_\ell})/\hat{\pi}^\ell$. Using a similar argument to the proof of Lemma 5, it is straightforward to see that $(\hat{\mu}_i^{\mathcal{T}_\ell}, \hat{\sigma}_i^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ is a dual feasible solution of $\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell})$ for every player $i \in N$. As a result, by looking at the objective function value of $(\hat{\mu}_i^{\mathcal{T}_\ell}, \hat{\sigma}_i^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ in the dual of $\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell})$, we obtain

$$\begin{aligned}
\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) &\geq \frac{1}{\hat{\pi}^\ell}\Bigg[\sum_{t \in \mathcal{T}_\ell : \mathcal{D}_t \neq \varnothing} \sum_{q \in \mathcal{Q}_i^t} \hat{\chi}_i^t \hat{\mu}_i^{t,q} + \sum_{t \in \mathcal{T}_\ell : \mathcal{D}_t = \varnothing} \hat{\chi}_i^t \hat{\sigma}_i^t\Bigg] \\
&\overset{\text{(iii)}}{=} \frac{1}{\hat{\pi}^\ell}\Bigg[\sum_{\substack{t \in \mathcal{T}_\ell : \mathcal{D}_t \neq \varnothing, \\ \hat{\pi}^t > 0}} \hat{\chi}_i^t \hat{\pi}^t + \sum_{\substack{t \in \mathcal{T}_\ell : \mathcal{D}_t = \varnothing, \\ \hat{\pi}^t > 0}} \hat{\chi}_i^t \hat{\pi}^t\Bigg] = \frac{1}{\hat{\pi}^\ell} \sum_{t \in \mathcal{T}_\ell : \hat{\pi}^t > 0} \hat{\chi}_i^t \hat{\pi}^t \\
&= \frac{1}{\hat{\pi}^\ell} \sum_{t \in \mathcal{T}_\ell : \hat{\pi}^t > 0} \Bigg[\hat{\lambda}^t (d_i^t - B_i^t \hat{s}_i^{a(t)}) + \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\Bigg]
\end{aligned} \tag{13}$$

where (iii) holds because $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ is feasible in the dual of $f^r(N, \hat{s}^{a(r)})$.

Suppose Assumption 8(b) holds. Fix a node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell = 0$. By Lemma 4, $\hat{\pi}^t = 0$ for all $t \in \mathcal{T}_\ell$, and so

$$\hat{\lambda}^t A_i^t \leq \hat{\pi}^t c_i^t = 0 \qquad \text{for } i \in N, t \in \mathcal{T}_\ell,$$
$$\hat{\lambda}^t C_i^t \geq \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau - \hat{\pi}^t h_i^t = \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \qquad \text{for } i \in N, t \in \mathcal{T}_\ell.$$

Applying this recursively from the leaves of $\mathcal{T}_\ell$, it follows that $\hat{\lambda}^t = 0$ for all $t \in \mathcal{T}_\ell$. Therefore, under either condition of Assumption 8, from (13) we have that for any $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$,

$$\begin{aligned}
\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) &\geq \frac{1}{\hat{\pi}^\ell} \sum_{t \in \mathcal{T}_\ell : \hat{\pi}^t > 0} \Bigg[\hat{\lambda}^t (d_i^t - B_i^t \hat{s}_i^{a(t)}) + \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\Bigg] \\
&= \frac{1}{\hat{\pi}^\ell} \sum_{t \in \mathcal{T}_\ell} \Bigg[\hat{\lambda}^t (d_i^t - B_i^t \hat{s}_i^{a(t)}) + \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\Bigg] \\
&= \frac{1}{\hat{\pi}^\ell}\Bigg[\sum_{t \in \mathcal{T}_\ell} \hat{\lambda}^t d_i^t - \sum_{t \in \mathcal{T}_\ell} \hat{\lambda}^t B_i^t \hat{s}_i^{a(t)} + \sum_{t \in \mathcal{T}_\ell} \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\Bigg] = \frac{1}{\hat{\pi}^\ell}\Bigg[\sum_{t \in \mathcal{T}_\ell} \hat{\lambda}^t d_i^t - \hat{\lambda}^\ell B_i^\ell \hat{s}_i^{a(\ell)}\Bigg].
\end{aligned} \tag{14}$$

Applying (14) first to the case where $\ell = r$ (note that $\hat{\pi}^\ell = \hat{\pi}^r = 1 > 0$), we obtain

$$\sum_{i \in N} \phi_i^r(\hat{\chi}_i^{\mathcal{T}_r}) \geq \sum_{i \in N} \Bigg[\sum_{t \in \mathcal{T}_r} \hat{\lambda}^t d_i^t - \hat{\lambda}^r B_i^r \hat{s}_i^{a(r)}\Bigg] \overset{\text{(iv)}}{=} f^r(N, \hat{s}^{a(r)}) \overset{\text{(v)}}{=} \sum_{i \in N} \phi_i^r(\hat{\chi}_i^{\mathcal{T}_r}),$$

where (iv) holds because $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ is a dual optimal solution of $f^r(N, \hat{s}^{a(r)})$, and (v) holds by assumption. Therefore, we can conclude that for each player $i \in N$, $(\hat{\mu}_i, \hat{\sigma}_i)$ is a dual optimal solution of $\phi_i^r(\hat{\chi}_i^{\mathcal{T}_r})$ for each player $i \in N$. Furthermore, it is clear from the structure of the dual of $\phi_i^r(\hat{\chi}_i^{\mathcal{T}_r})$ that the principle of optimality holds, and so $(\hat{\mu}_i^{\mathcal{T}_\ell}, \hat{\sigma}_i^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ is a dual optimal solution of $\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell})$ for every node $\ell \in \mathcal{T}_r$ with $\hat{\pi}^\ell > 0$. In other words, by (14) we have that

$$\phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) = \frac{1}{\hat{\pi}^\ell}\left[\sum_{t\in\mathcal{T}_\ell} \hat{\lambda}^t d_i^t - \hat{\lambda}^\ell B_i^\ell \hat{s}_i^{a(\ell)}\right] \quad \text{for } i \in N, \ell \in \mathcal{T}_r : \hat{\pi}^\ell > 0.$$

Putting this together, we show (11b) holds. For every node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$, we have

$$\sum_{i\in N} \phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) = \sum_{i\in N} \frac{1}{\hat{\pi}^\ell}\left[\sum_{t\in\mathcal{T}_\ell} \hat{\lambda}^t d_i^t - \hat{\lambda}^\ell B_i^\ell \hat{s}_i^{a(\ell)}\right] \stackrel{\text{(vi)}}{=} f^\ell(N, \hat{s}^{a(\ell)}),$$

where (vi) holds because $(\hat{\lambda}^{\mathcal{T}_\ell}, \hat{\mu}^{\mathcal{T}_\ell}, \hat{\sigma}^{\mathcal{T}_\ell}, \hat{\pi}^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ is a dual optimal solution of $f^\ell(N, \hat{s}^{a(\ell)})$ by Lemma 5. Finally, we show (11c) holds: for every node $\ell \in \mathcal{T}_r$ such that $\hat{\pi}^\ell > 0$, we have

$$\sum_{i\in U} \phi_i^\ell(\hat{\chi}_i^{\mathcal{T}_\ell}) = \sum_{i\in U} \frac{1}{\hat{\pi}^\ell}\left[\sum_{t\in\mathcal{T}_\ell} \hat{\lambda}^t d_i^t - \hat{\lambda}^\ell B_i^\ell \hat{s}_i^{a(\ell)}\right] \stackrel{\text{(vii)}}{\leq} f^\ell(U, \hat{s}_U^{a(\ell)}) \quad \text{for } U \subseteq N,$$

where (vii) holds because $(\hat{\lambda}^{\mathcal{T}_\ell}, \hat{\mu}^{\mathcal{T}_\ell}, \hat{\sigma}^{\mathcal{T}_\ell}, \hat{\pi}^{\mathcal{T}_\ell})/\hat{\pi}^\ell$ is a dual *feasible* solution of $f^\ell(U, \hat{s}_U^{a(\ell)})$. □

Finally, we give the main result of this section, an algorithm that under certain sufficient conditions, uses dual optimal solutions to compute an allocation that is in the strong sequential core.

**Theorem 10.** *Let $(\hat{\xi}, \hat{z}, \hat{x}, \hat{s})$ be a primal optimal solution of $f^1(N, \hat{s}^0)$. Consider the following algorithm.*

Initialize the queue of nodes $\mathcal{R} = (1)$.
**while** $\mathcal{R} \neq \varnothing$ **do**
  Dequeue $r$ from $\mathcal{R}$.
  Solve $f^r(N, \hat{s}^{a(r)})$. Let $(\hat{\lambda}, \hat{\mu}, \hat{\sigma}, \hat{\pi})$ be a dual optimal solution.
  For every player $i \in N$ and node $t \in \mathcal{T}_r$ such that $\hat{\pi}^t > 0$, let

$$\hat{\chi}_i^t = \frac{1}{\hat{\pi}^t}\left[\hat{\lambda}^t(d_i^t - B_i^t \hat{s}_i^{a(t)}) + \sum_{\tau\in\mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\right].$$

  Find the coarsest decomposition of the remaining nodes $\{t \in \mathcal{T}_r : \hat{\pi}^t = 0\}$ into subtrees. Enqueue the root nodes of these subtrees to $\mathcal{R}$.
**end while**

*Suppose Assumption 8 holds and*

$$\sum_{i\in N} \phi_i^r(\hat{\chi}_i^{\mathcal{T}_r}) = f^r(N, \hat{s}^{a(r)}) \quad \text{for } r \in \mathcal{T}.$$

*Then the algorithm above constructs an allocation in the strong sequential core.*

*Proof.* For any iteration of the above algorithm, we can always decompose the nodes $\{t \in \mathcal{T}_r : \hat{\pi}^t = 0\}$ into subtrees, because of Lemma 4. Therefore, the algorithm is well-defined and defines an allocation $\hat{\chi}_i^t$ exactly once for every player $i \in N$ and node $t \in \mathcal{T}$. Through repeated applications of Lemma 9, it follows that $\hat{\chi}$ is a feasible allocation for the coalition $N$ starting at node 1 that satisfies (5). Therefore, by Theorem 2, the allocation $\hat{\chi}$ constructed by the algorithm above is in the strong sequential core. $\square$

Combining Theorem 2 and Theorem 10, we obtain the following corollary.

**Corollary 11.** *Let $(\hat{\xi}, \hat{z}, \hat{x}, \hat{s})$ and $(\hat{\lambda}, \hat{\pi}, \hat{\mu}, \hat{\sigma})$ be primal and dual optimal solutions to $f^1(N, \hat{s}^0)$, respectively. If $\hat{\pi}^t > 0$ for all $t \in \mathcal{T}$, the allocation*

$$\hat{\chi}_i^t = \frac{1}{\hat{\pi}^t}\left[\hat{\lambda}^t(d_i^t - B_i^t \hat{s}_i^{a(t)}) + \sum_{\tau \in \mathcal{D}_t} \hat{\lambda}^\tau B_i^\tau \hat{s}_i^t\right] \quad \textit{for } i \in N, t \in \mathcal{T}$$

*is in the strong sequential core if and only if*

$$\sum_{i \in N} \phi_i^1(\hat{\chi}_i) = f^1(N, \hat{s}^0).$$

### 4.3 Risk-Neutral Players

Consider the special case in which the players are *risk neutral*. That is, for each player $i \in N$ and node $r \in \mathcal{T}$, the risk measure $\phi_i^r$ is simply the conditional expectation with respect to a given conditional probability distribution $(p^{t|r})_{t \in \mathcal{T}_r}$ on $\mathcal{T}_r$. In this case, the optimization model $f^r(U, \hat{s}_U^{a(r)})$ simplifies to

$$\begin{aligned}
f^r(U, \hat{s}_U^{a(r)}) \ = \min_{x,s} \quad & \sum_{i \in U} \sum_{t \in \mathcal{T}_r} p^{t|r}(c_i^t x_i^t + h_i^t s_i^t) \\
\text{s.t.} \quad & \sum_{i \in U} A_i^r x_i^r - \sum_{i \in U} C_i^r s_i^r = \sum_{i \in U} d_i^r - \sum_{i \in U} B_i^r \hat{s}_i^{a(r)}, \\
& \sum_{i \in U} A_i^t x_i^t + \sum_{i \in U} B_i^t s_i^{a(t)} - \sum_{i \in U} C_i^t s_i^t = \sum_{i \in U} d_i^t \qquad \text{for } t \in \mathcal{T}_r \setminus \{r\}, \\
& x_i^t \geq 0, s_i^t \geq 0 \qquad \text{for } i \in U, t \in \mathcal{T}_r.
\end{aligned}$$

In other words, $f^r(U, \hat{s}_U^{a(r)})$ here is the minimum expected cost of a solution in $\mathcal{S}_U^r(\hat{s}_U^{a(r)})$.

Theorem 2 implies that the strong sequential core is the set of all allocations $\chi$ that satisfy

$$\sum_{i \in N} \chi_i^r = \sum_{i \in N}(c_i^t \hat{x}_i^t + h_i^t \hat{s}_i^t) \quad \text{for } r \in \mathcal{T}, \tag{15a}$$

$$\sum_{i \in U} \sum_{t \in \mathcal{T}_r} p^{t|r} \chi_i^t \leq f^r(U, \hat{s}_U^{a(r)}) \qquad \text{for } r \in \mathcal{T}, U \subseteq N. \tag{15b}$$

The constraints (15a) require that the dynamic allocation $\chi$ exactly covers the cost incurred when node $r \in \mathcal{T}$ is realized, and the constraints (15b) require that at any node in the scenario tree, every coalition's expected cost allocation from that point forward does not exceed its expected cost if it defects from the grand coalition and continues on its own. Risk neutrality means that the conditions of Corollary 11 apply, which implies the following result.

**Corollary 12** (Xu and Veinott 2013). *When players are risk neutral, the strong sequential core is non-empty, and allocation $\hat{\chi}$ defined in* (10) *is in the strong sequential core.*

Xu and Veinott (2013) consider a class of games that generalize dynamic linear programming games, but require risk-neutral players. They study the solution concept defined by the set of allocations described in (15), which they name the *sequential stochastic core*. They also show that a dual-based dynamic allocation similar to (10) is in this set.

# 5 Application to Newsvendor Games

We next apply our results to cost sharing in newsvendor settings. The newsvendor problem and its extensions have been widely studied from a cooperative game theoretic perspective (e.g. Chen 2009, Chen and Zhang 2009, Hartman and Dror 2005, Hartman et al. 2000, Montrucchio and Scarsini 2007, Müller et al. 2002, Özen et al. 2008, Slikker et al. 2005). Similarly, several authors have studied the optimization of risk-averse newsvendor models and their generalizations (e.g. Ahmed et al. 2007, Chen et al. 2009, Choi and Ruszczyński 2008; 2011, Choi et al. 2011, Gotoh and Takano 2007, Xin et al. 2013). However, to the best of our knowledge, no one has studied cost allocation among risk-averse players in any inventory setting, including the newsvendor model.

In this setting, the players in $N$ are independent retailers, each of which is planning to meet one-time demand for a common product one period in the future by placing an order in the present period. The scenario tree $\mathcal{T}$ consists of the present node 1 and its children $\mathcal{D}_1$; every node $t \in \mathcal{D}_1$ represents a possible realization of each player's demand given by $d_i^t$. Without loss of generality, we assume scenarios are ordered such that $t \leq t'$ implies $\sum_{i \in N} d_i^t \leq \sum_{i \in N} d_i^{t'}$.

The first-period unit ordering and holding costs are given respectively by $c$ and $h$. The second-period unit backlog cost, incurred when the order does not meet demand, is $b$, whereas excess inventory above realized demand has a unit salvage value of $v$ (i.e. a cost of $-v \leq 0$). We assume $v < c + h < b$ to avoid unrealistic cases. We assume these parameters are identical for all players, and that the second-period costs $b$ and $-v$ are invariant across scenarios $t \in \mathcal{D}_1$, which is typical in newsvendor models. We also assume all players share a common risk measure $\phi$ given by a closed, convex set of probability measures $\mathcal{Q}^1$ over $\mathcal{D}_1$. Henceforth, whenever possible we suppress unnecessary indices, referring for instance to these two sets simply as $\mathcal{Q}$ and $\mathcal{D}$. Finally, we assume the players have no initial stock.

For this two-period process, we can write the dynamic risk of allocation $\xi$ to player $i \in N$ as $\phi(\xi_i) = \xi_i^1 + \rho(\xi_i^{\mathcal{D}})$, where we also suppress indices for $\phi$ and $\rho$. Furthermore, since the conditional risk measure is identical across players, Corollary 3 implies that we can simplify the primal (6) of the optimization problem $f^1(N,0)$ faced by the grand coalition in the first period to

$$\begin{aligned} f^1(N,0) = \min_{x,s} \quad & (c+h)\sum_{i\in N} x_i^1 + \max_{\pi\in\mathcal{Q}} \sum_{t\in\mathcal{D}} \pi^t \sum_{i\in N}(bx_i^t - vs_i^t) \\ \text{s.t.} \quad & \sum_{i\in N}(x_i^1 + x_i^t - s_i^t) = \sum_{i\in N} d_i^t \qquad\qquad \text{for } t\in\mathcal{D}, \\ & x, s \geq 0. \end{aligned}$$

The dual (7) reduces to

$$\begin{aligned} f^1(N,0) = \max_{\lambda,\pi} \quad & \sum_{t\in\mathcal{D}} \lambda^t \sum_{i\in N} d_i^t \\ \text{s.t.} \quad & \pi^t v \leq \lambda^t \leq \pi^t b \quad \text{for } t\in\mathcal{D}, \\ & \sum_{t\in\mathcal{D}} \lambda^t \leq c+h, \\ & \pi\in\mathcal{Q}. \end{aligned}$$

Under the change of variables $\bar{\lambda}^t = \lambda^t - \pi^t v$ for $t\in\mathcal{D}$, for fixed $\pi$ the dual is a fractional knapsack problem. By defining a critical index

$$t(\pi) := \max\left\{ t\in\mathcal{D} : \sum_{\tau\geq t} \pi^\tau \geq \frac{c+h-v}{b-v} \right\},$$

we can write a dual optimal solution for a fixed $\pi\in\mathcal{Q}$ as

$$\lambda^t(\pi) = \begin{cases} \pi^t b & \text{if } t\in\mathcal{D} : t > t(\pi), \\ \pi^t v & \text{if } t\in\mathcal{D} : t < t(\pi), \\ c+h-b\sum_{\tau>t(\pi)}\pi^\tau - v\sum_{\tau<t(\pi)}\pi^\tau & \text{if } t\in\mathcal{D} : t = t(\pi). \end{cases}$$

The dual objective value for any $\pi\in\mathcal{Q}$ is therefore

$$(c+h)\sum_{i\in N} d_i^{t(\pi)} + b\sum_{t>t(\pi)} \pi^t \sum_{i\in N}(d_i^t - d_i^{t(\pi)}) - v\sum_{t<t(\pi)} \pi^t \sum_{i\in N}(d_i^{t(\pi)} - d_i^t),$$

i.e. the expected cost (under the distribution $\pi$) of ordering the total demand for scenario $t(\pi)$ in the first period and then paying the corresponding backlog cost or receiving the corresponding salvage value in the second period. Based on the above discussion, for a fixed $\pi\in\mathcal{Q}$, there exists a primal optimal solution that satisfies

$$\sum_{i\in N} x_i^1 = \sum_{i\in N} d_i^{t(\pi)};$$

$$\sum_{i \in N} x_i^t = \begin{cases} \sum_{i \in N}(d_i^t - d_i^{t(\pi)}) & \text{if } t > t(\pi), \\ 0 & \text{otherwise;} \end{cases} \qquad \sum_{i \in N} s_i^t = \begin{cases} \sum_{i \in N}(d_i^{t(\pi)} - d_i^t) & \text{if } t < t(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{\pi}$ be optimal for the dual of $f(N, 0)$, so that the optimal first-period order is $\sum_{i \in N} x_i^1 = \sum_{i \in N} d_i^{t(\hat{\pi})}$. Let us also assume the players divide this order in the natural way, $\hat{s}_i^1 = d_i^{t(\hat{\pi})}$. Then the dynamic allocation $\hat{\chi}$ defined in Section 4 is

$$\hat{\chi}_i^1 = (c+h) d_i^{t(\hat{\pi})}, \qquad \hat{\chi}_i^t = \begin{cases} b(d_i^t - d_i^{t(\hat{\pi})}) & \text{if } t \in \mathcal{D} : t > t(\hat{\pi}), \\ -v(d_i^{t(\hat{\pi})} - d_i^t) & \text{if } t \in \mathcal{D} : t < t(\hat{\pi}), \\ 0 & \text{if } t \in \mathcal{D} : t = t(\hat{\pi}). \end{cases}$$

Suppose player $i$'s demand satisfies

$$d_i^t \geq d_i^{t(\hat{\pi})} \text{ if } t > t(\hat{\pi}) \quad \text{and} \quad d_i^t \leq d_i^{t(\hat{\pi})} \text{ if } t < t(\hat{\pi}). \tag{16}$$

Then the allocation mirrors what player $i$ would pay if he operated alone and ordered $d_i^{t(\hat{\pi})}$ in the first period. For any scenarios that do not satisfy (16), either he must pay a further cost in the salvage case, or he receives a credit in the backlog case.

If the players are risk-neutral, the results in Section 4.3 guarantee that this allocation is in the strong sequential core. In this case, the distribution $\hat{\pi}$ is in fact equal to a fixed distribution $p$, and the expected cost of any coalition's allocation does not exceed that coalition's optimal expected cost (under the same distribution $p$) if it were to defect from the grand coalition and act independently. If the players are risk-averse, however, the situation becomes more complicated; to establish the conditions of Theorem 10 we need additional assumptions about $\mathcal{Q}$.

For the remainder of this section, we assume that the risk measure $\rho$ is *comonotonic*; that is, in addition to being coherent, the measure must satisfy one additional condition:

(M5) Comonotonicity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if $X, Y \in \mathbb{R}^{\mathcal{D}}$ are comonotone; that is, if $(X^t - X^{t'})(Y^t - Y^{t'}) \geq 0$ for all $t, t' \in \mathcal{D}$.

We can fully characterize comonotonic risk measures using a *Choquet capacity* (Choquet 1955). This is a function $\kappa : 2^{\mathcal{D}} \to \mathbb{R}$ satisfying the following properties:

(CC1) Monotonicity: $\kappa(\mathcal{I}) \leq \kappa(\mathcal{I}')$ for $\mathcal{I} \subseteq \mathcal{I}' \subseteq \mathcal{D}$.

(CC2) Normalization: $\kappa(\varnothing) = 0$, $\kappa(\mathcal{D}) = 1$.

(CC3) Submodularity: $\kappa(\mathcal{I}) + \kappa(\mathcal{I}') \geq \kappa(\mathcal{I} \cap \mathcal{I}') + \kappa(\mathcal{I} \cup \mathcal{I}')$ for $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{D}$.

**Lemma 13** (Schmeidler 1986)**.** *A coherent risk measure $\rho$ is comonotonic if and only if*

$$\mathcal{Q} = \left\{ q \in \mathbb{R}^{\mathcal{D}}_{\geq 0} : \sum_{t \in \mathcal{D}} q^t = \kappa(\mathcal{D}) = 1, \sum_{t \in \mathcal{I}} q^t \leq \kappa(\mathcal{I}) \text{ for } \mathcal{I} \subseteq \mathcal{D} \right\}$$

*for some Choquet capacity $\kappa$.*

In other words, this theorem states that $\rho$ is comonotonic precisely when $\mathcal{Q}$ is the *base polyhedron* of a Choquet capacity $\kappa$. These polyhedra are well-studied and optimizing over them can be achieved with a greedy algorithm (Edmonds 1970).

Based on the discussion above, we can write the optimal value for $f^1(N,0)$ as

$$f^1(N,0) = \max_{\pi\in\mathcal{Q}} \min_{t\in\mathcal{D}} \Big\{ (c+h)\sum_{i\in N} d_i^t + b\sum_{\tau>t}\pi^\tau \sum_{i\in N}(d_i^\tau - d_i^t) + v\sum_{\tau<t}\pi^\tau\sum_{i\in N}(d_i^\tau - d_i^t)\Big\}. \tag{17}$$

Note that $t(\pi)$ is an optimal solution to the inner minimization problem. In the outer maximization problem, the objective function is concave in $\pi$ because it is the minimum of a collection of affine functions, one per $t\in\mathcal{D}$. However, for any $t\in\mathcal{D}$, because we assume that scenarios are ordered such that $\tau\le\tau'$ implies $\sum_{i\in N} d_i^\tau \le \sum_{i\in N} d_i^{\tau'}$, the coefficients $\sum_{i\in N}(d_i^\tau - d_i^t)$ of each $\pi^\tau$ are in the same non-decreasing order. So, if $\mathcal{Q}$ is a base polyhedron, the same extreme point is optimal regardless of which affine function is minimal. This extreme point can be obtained by a greedy algorithm that increases each variable $\pi^\tau$ as much as possible, in the order of non-increasing objective function coefficients.

**Proposition 14.** *Suppose the players' common risk measure is comonotonic, and that their demands are pairwise comonotone; in particular, $t\le t'$ implies $d_i^t \le d_i^{t'}$ for $i\in N$. In addition, suppose we obtain an optimal distribution $\hat\pi\in\mathcal{Q}$ to $f(N,0)$ as written in* (17) *using the greedy algorithm described above. Then the dynamic allocation $\hat\chi$ is in the strong sequential core.*

*Proof.* For player $i\in N$, the allocation's risk is

$$\begin{aligned}\phi(\hat\chi_i) &= \max_{\pi\in\mathcal{Q}}\Big\{(c+h)d_i^{t(\hat\pi)} + b\sum_{t>t(\hat\pi)}\pi^t(d_i^t - d_i^{t(\hat\pi)}) + v\sum_{t<t(\hat\pi)}\pi^t(d_i^t - d_i^{t(\hat\pi)})\Big\}\\ &= (c+h)d_i^{t(\hat\pi)} + b\sum_{t>t(\hat\pi)}\hat\pi^t(d_i^t - d_i^{t(\hat\pi)}) + v\sum_{t<t(\hat\pi)}\hat\pi^t(d_i^t - d_i^{t(\hat\pi)}).\end{aligned}$$

The second equality follows because the players' demand is pairwise comonotone, and thus the coefficients of each $\pi^\tau$ within the maximization are non-increasing in the same order as in (17), which then implies that $\hat\pi$ remains optimal. This in turn implies $\sum_{i\in N}\phi(\hat\chi_i) = f^1(N,0)$, i.e. the conditions of Theorem 10 hold. □

Unfortunately, the conditions that this result requires for the strong sequential core to be non-empty – particularly the comonotonicity of demand – are precisely those in which players benefit the least from cooperating. Assuming all players have access to the same newsvendor costs, the risk $\phi(\hat\chi_i)$ that player $i$ perceives from allocation $\hat\chi_i$ is precisely what his risk would be when operating alone. However, the following examples illustrate that even apparently small generalizations of these conditions can already render the strong sequential core empty.

**Example 15** (Newsvendor game with complementary demand)**.** Consider a two-player newsvendor game with ordering and holding costs $c + h = 1$, backlog cost $b$ and salvage value $v$. We have two second-period scenarios, $\mathcal{D} = \{2, 3\}$, and the two players' demands are exactly complementary: $d_1^3 = d_2^2 = 1$ and $d_1^2 = d_2^3 = 0$. The common risk measure is defined by the set $\mathcal{Q} = \text{conv}\{(q, 1-q), (1-q, q)\}$, where $q \geq 1/2$ without loss of generality. Note that any risk measure over two scenarios is comonotonic. We further assume $q \geq (1 - v)/(b - v)$.

Drawing upon the previous discussion, we can write the risk experienced by player 1 starting in the first period as

$$
\begin{aligned}
f^1(\{1\}, 0) &= \min_{0 \leq x \leq 1} \max \big\{x + q(-vx) + (1-q)\big(b(1-x)\big), x + (1-q)(-vx) + q\big(b(1-x)\big)\big\} \\
&= \min_{0 \leq x \leq 1} \max \big\{x\big(1 - qv - (1-q)b\big) + (1-q)b, x\big(1 - (1-q)v - qb\big) + qb\big\},
\end{aligned}
$$

where $x$ represents the amount that player 1 orders in the first period. It is straightforward to show that under the assumption that $q \geq (1 - v)/(b - v)$, it is optimal for player 1 to order one unit in the first period. By symmetry, the same holds for player 2. It is clear that it is also optimal for the grand coalition to order one unit in the first period.

Assuming the grand coalition orders together, let $\hat{s}_1^1 = s$ be the quantity owned by player 1 at the end of the first period, and as a result, $\hat{s}_1^2 = 1 - s$ is the quantity owned by player 2 at the end of the first period. The following table gives the values of $f^r(U, \hat{s}_U^{a(r)})$:

| $r$ | 1 | 2 | 3 |
|---|---|---|---|
| $f^r(\{1\}, \hat{s}_1^{a(r)})$ | $1 - (1-q)v$ | $-vs$ | $b(1-s)$ |
| $f^r(\{2\}, \hat{s}_2^{a(r)})$ | $1 - (1-q)v$ | $bs$ | $-v(1-s)$ |
| $f^r(\{1,2\}, \hat{s}_{\{1,2\}}^{a(r)})$ | 1 | 0 | 0 |

We will use these values to determine conditions on allocations in the strong sequential core. From the second-period optimality constraints (5c), we immediately get

$$
\chi_1^2 + \chi_2^2 = 0, \qquad \chi_1^3 + \chi_2^3 = 0;
$$

i.e. the second-period allocation is a side payment from one player to the other. The first-period time-consistent stability constraints (5b) then imply

$$
\begin{aligned}
1 \geq \phi(\chi_1) + \phi(\chi_2) &= \chi_1^1 + \max\{q\chi_1^2 + (1-q)\chi_1^3, (1-q)\chi_1^2 + q\chi_1^3\} \\
&\quad + \chi_2^1 + \max\{q\chi_2^2 + (1-q)\chi_2^3, (1-q)\chi_2^2 + q\chi_2^3\} \\
&\geq \chi_1^1 + \chi_2^1 + q(\chi_1^3 + \chi_2^2) + (1-q)(\chi_1^2 + \chi_2^3) \\
&= \chi_1^1 + \chi_2^1 + (2q-1)(\chi_1^3 + \chi_2^2).
\end{aligned}
$$

Moreover, the first-period feasibility constraint (5a) requires $\chi_1^1 + \chi_2^1 \geq 1$, which together with the previous inequality yields

$$(2q-1)(\chi_1^3+\chi_2^2) \le 1-(\chi_1^1+\chi_2^1) \le 0.$$

Therefore, if $q > 1/2$ – that is, if the players aren't neutral to risk – we get the condition $\chi_1^3+\chi_2^2 \le 0$. However, the second-period time-consistent stability constraints (5b) require

$$-\chi_1^3 = \chi_2^3 =\le -v(1-s), \quad -\chi_2^2 = \chi_1^2 \le -vs \quad \Rightarrow \quad \chi_1^3+\chi_2^2 \ge v,$$

and thus the strong sequential core is empty when $v > 0$.

This example highlights how difficult it may be for risk-averse individuals to cooperate. When there is zero salvage value, the strong sequential core is non-empty; the set will have second-period allocations equal to zero. However, by increasing salvage to any positive value $0 < v < 1$ (a change that could help both players and hurts neither), cooperation becomes impossible. Furthermore, the strong sequential core is empty for *any risk measure* in our example that is not risk neutral; if $(1-v)/(b-v) \le 1/2$, we can take $q = 1/2+\epsilon$ for an arbitrarily small $\epsilon > 0$, and the strong sequential core would still be empty.

Demand in a newsvendor setting with two players is unlikely to be purely complementary. Our next example examines a similar situation with independently distributed demand.

**Example 16** (Newsvendor game with independent demand)**.** Consider the two-player newsvendor game with cost parameters as in Example 15, where now each player has i.i.d. Bernoulli demand. We identify each of the four second-period scenarios with a two-digit binary number indicating each player's demand realization, $\mathcal{D} = \{00, 01, 10, 11\}$; for example, scenario 10 means player 1's demand occurs and player 2's does not. Suppose the backlog cost $b$ is large enough that it is optimal for the grand coalition to order two units in the first period. In addition, suppose both players hold one unit of inventory at the end of this period, i.e. $\hat{s}_1^1 = \hat{s}_2^1 = 1$.

Let $\chi$ be an arbitrary allocation in the strong sequential core. Applying conditions (5a) and (5b) to each second-period scenario, it is simple to show that a player's allocation must be zero if his demand realizes, and $-v$ if it does not. This yields the following allocations and grand coalition costs:

| $r$ | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $\chi_1^r$ | $-v$ | $-v$ | 0 | 0 |
| $\chi_2^r$ | $-v$ | 0 | $-v$ | 0 |
| $f^r(\{1,2\}, \hat{s}^1)$ | $-2v$ | $-v$ | $-v$ | 0 |

Therefore, $\chi_1^r + \chi_2^r = f^r(\{1,2\}, \hat{s}^1)$ for all $r \in \mathcal{D}$, and so

$$f^1(\{1,2\}, 0) = 2 + \rho\big([f^r(\{1,2\}, \hat{s}^1]_{r\in\mathcal{D}}\big) = 2 + \rho(\chi_1^{\mathcal{D}} + \chi_2^{\mathcal{D}}).$$

Applying (5a) in the first period, we obtain

$$\phi(\chi_1) + \phi(\chi_2) = \chi_1^1 + \chi_2^1 + \rho(\chi_1^{\mathcal{D}}) + \rho(\chi_2^{\mathcal{D}}) \geq 2 + \rho(\chi_1^{\mathcal{D}}) + \rho(\chi_2^{\mathcal{D}}).$$

Putting this all together with (5b) in the first period, it follows that any allocation $\chi$ in the strong sequential core must satisfy

$$\rho(\chi_1^{\mathcal{D}} + \chi_2^{\mathcal{D}}) \geq \rho(\chi_1^{\mathcal{D}}) + \rho(\chi_2^{\mathcal{D}}).$$

Note that $\chi_1^{\mathcal{D}}$ and $\chi_2^{\mathcal{D}}$ are *not* comonotone, and therefore by subadditivity it is possible that $\rho(\chi_1^{\mathcal{D}} + \chi_2^{\mathcal{D}}) < \rho(\chi_1^{\mathcal{D}}) + \rho(\chi_2^{\mathcal{D}})$, implying that the strong sequential core is empty. In fact, this latter inequality is strict precisely when cooperation among the two players lowers their total risk. So the strong sequential core is non-empty only when cooperation brings no benefit to the players because their risk measure perceives cooperation as equally preferable to operating alone.

As these results on newsvendor games suggest, although the strong sequential core of a dynamic linear programming game is always non-empty when the players are risk neutral, the strong sequential core may be empty when players are risk averse, and only non-empty when players least benefit from cooperation. Similar observations have been made in other contexts. For example, Predtetchinski et al. (2004) and Predtetchinski (2007) both showed that in certain infinite-horizon settings, the strong sequential core is nonempty when the discount factor is sufficiently high, which can be interpreted as when the players behave more like risk-neutral players. Habis and Herings (2011a) demonstrated that in so-called two-period finance economies, the strong sequential core is only nonempty in economically uninteresting cases.

# 6 Conclusions

We introduced the general class of dynamic linear programming games to model situations in which risk-averse agents cooperate over time. To evaluate whether cooperation in a particular setting is plausible, we generalized the strong sequential core concept into this risk-averse setting and gave a characterization using finitely many inequalities and an auxiliary optimization model that minimizes a coalition's total risk. Our results can be qualitatively summarized by observing that risk aversion often appears to be in conflict with cooperation; specifically, risk-neutral players can always construct an allocation in the strong sequential core, whereas risk aversion can imply an empty strong sequential core even in situations where the grand coalition's total risk is reduced by cooperating, and even when each player's attitude towards risk is arbitrarily close to neutral. Our exploration of newsvendor games with risk-averse players confirms this general notion: In situations where players can expect to benefit from cooperating, such as when demand is complementary or independent, the strong sequential core can easily be empty. Conversely, we can only guarantee that the strong sequential core of these newsvendor games is non-empty when demand is comonotone, precisely the setting in which players benefit the least from cooperation.

Our results motivate some interesting directions for further research. Although our study of newsvendor games illustrates the difficulty of having risk-averse players cooperate even in simple settings, one could argue that the notion of cooperation imposed by the strong sequential core is too restrictive. We can investigate whether cooperation is easier under relaxed notions of cooperation, such as the *weak sequential core* (Habis and Herings 2011b, Kranich et al. 2005), which only blocks deviations by coalitions that are not vulnerable to subsequent defections. We can also start with solution concepts for static cooperative games with relaxed notions of coalitional stability, such as the *least core* (Maschler et al. 1979) and the *$\alpha$-core* (Faigle and Kern 1993), and consider their dynamic risk-averse analogues. Alternatively, we can move away from core-like solution concepts that impose coalitional stability and explore dynamic risk-averse analogues of other kinds of solution concepts such as the Shapley value (Shapley 1953). Unfortunately, finding sensible analogues of these solution concepts is unlikely to be a clear-cut task. For instance, Timmer et al. (2003) proposed three solution concepts for stochastic cooperative games (Suijs et al. 1999a), inspired by three equivalent formulations of the Shapley value for static cooperative games; they show that when costs are stochastic, these three solution concepts can in fact be different.

Another interesting direction is to find subclasses of dynamic linear programming games with risk-averse players that are more amenable to cooperation than the newsvendor games we discussed above. Such an analysis might help to clarify when cooperation is possible. More generally, we can ask whether the tradeoff between cooperation and risk-averse optimization can be made explicit, say by establishing the required level of suboptimality for a solution with a stable allocation.

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