Dynamic Cost Allocation for Economic Lot Sizing Games

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Abstract

We consider a cooperative game defined by an economic lot sizing problem with concave ordering costs over a finite time horizon, in which each player faces demand for a single product in each period and coalitions can pool orders. We show how to compute a dynamic cost allocation in the strong sequential core of this game, i.e. an allocation over time that exactly distributes costs and is stable against coalitional defections at every period of the time horizon.

1 Introduction and Motivation

Production and inventory management are areas in which cooperation among independent agents has an intuitive appeal. For example, independent retailers sharing a warehouse may want to combine orders from a common supplier to enjoy economies of scale derived from larger order quantities. Viewed from a system-wide perspective, combining orders lowers the total cost, but the order consolidation cannot be achieved unless the independent retailers can agree on a "fair" way to split this cost. Over the last decade or more, cooperative game theory research in production, distribution and inventory models has endeavored to determine fair cost allocations under a variety of settings.

The economic lot sizing problem is one of the canonical production and inventory models studied in operations research and management science, almost since the field's inception (Wagner and Whitin 1958). In a cooperative setting, this model would capture the situation described above, with each retailer facing its own demand over the planning horizon that must be satisfied with product orders or inventory, but with coalitions of retailers being able to order product together. The cooperative game derived from this model was proposed in van den Heuvel et al. (2007) and subsequently studied in Chen and Zhang (2006), Gopaladesikan et al. (2012). However, all of these papers approach the problem from a static perspective; that is, the authors assume an optimal solution is implemented over the entire horizon, and seek an up-front static cost allocation in the core of this cooperative game.

Unfortunately, a static cost allocation – even one in the core – suffers from a number of significant drawbacks. First, it assumes each retailer would be able to cover its portion of the entire planning horizon's cost up front, a significant financial burden that is unrealistic in many settings. Second, once the allocated costs are collected from the participating retailers, there may be incentives for individual retailers or coalitions to defect later on in the planning horizon if they find themselves in an advantageous position. Finally, a static allocation ignores the typical rolling horizon approach to these models, in which one or a few periods' solutions are implemented, and then a new model is formulated with updated parameters. Within a rolling horizon approach, it is unclear how a static allocation could be implemented.

Our goal in this paper is to develop dynamic cost allocations for the economic lot sizing problem. Unlike their static counterparts, dynamic allocations can be constrained to allocate costs as they are incurred. They can also be designed so that no coalition ever has an incentive to defect throughout the entire planning horizon. Furthermore, because the costs are allocated as they are incurred, dynamic allocations can also be incorporated into a rolling horizon framework. Our allocation depends conceptually on extending the core concept to dynamic settings (e.g. see Kranich et al. 2005), and continues our work in Toriello and Uhan (2013) for multi-period models with linear costs.

Cooperative game theory has been successfully applied in many related models, and we cannot hope to adequately review them here. Instead, we refer the interested reader to Cachon and Netessine (2006), Nagarajan and Sošić (2008). Two in-depth references on lot sizing and more general production planning and inventory management are Pochet and Wolsey (2006), Zipkin (2000). This paper has two remaining sections. In Section 2, we introduce the model and review some relevant results. Then, in Section 3, we define the *strong sequential core*, the dynamic version of the core, and show how to compute such a dynamic allocation.

2 The Economic Lot Sizing Game

We consider the following deterministic economic lot sizing setting with multiple retailers. A set of retailers $N := \{1, \dots, n\}$ faces deterministic demand for a single product over a finite discrete time horizon. In particular, each retailer $i \in N$ needs to satisfy demand d_i^t in periods $t = r, \dots, T$; we start in period r instead of period 1 because we will eventually vary the starting period. In every period $t = r, \dots, T$, each retailer can order x units at a total cost of $c_t(x)$; the function c_t is concave and nondecreasing with $c_t(0) = 0$. Each retailer can also hold one unit of product in inventory in period $t = r, \dots, T$ at a cost of h_t. Finally, each retailer $i \in N$ has an initial inventory of $\hat{s}_i^{r-1}$. For notational convenience, we define

$$d_R^{[j,k]} := \sum_{i \in R} \sum_{t=j}^{k} d_i^t \quad \text{for } R \subseteq N,\ r \le j \le k \le T, \qquad\qquad \hat{s}_R^{r-1} := \sum_{i \in R} \hat{s}_i^{r-1}.$$

Here and throughout, we take a summation in which the initial index is greater than the ending index to be vacuous and equal to zero; for example, $d_R^{[j,j-1]} = 0$.

The *economic lot sizing problem* for a subset of retailers $R \subseteq N$ seeks to satisfy the demands $d_R^r, \dots, d_R^T$ with initial inventory $\hat{s}_R^{r-1}$ in a way that minimizes the total ordering and inventory cost. The *economic lot sizing game* is a transferable utility cooperative game $(N, f_r(\cdot, \hat{s}^{r-1}))$ in which each retailer corresponds to a player, and the cost $f_r(R, \hat{s}^{r-1})$ to a subset of players $R \subseteq N$ is the optimal value of its economic lot sizing problem. Following standard terminology in the cooperative game theory literature, we refer to a subset of players as a *coalition*, and the set of all players as the *grand coalition*.

It is well-known that any instance of the economic lot sizing problem can be transformed into an equivalent instance with zero inventory by using the initial inventory to greedily satisfy demand in the beginning time periods (Zabel 1964). It is also well-known that when the initial inventory is zero, there exists an optimal solution that satisfies the *zero-inventory property* in which orders only occur in periods when the inventory level is zero (Wagner and Whitin 1958, Wagner 1960). Therefore, when initial inventories are of the form

$$\hat{s}_i^{r-1} = d_i^{[r,\tau-1]} \quad \text{for } i \in N$$

for some $\tau \geq r$, we can model the economic lot sizing problem for coalition $R \subseteq N$ with the following linear program (Chen and Zhang 2006):

$$f_r(R, \hat{s}^{r-1}) = \sum_{t=r}^{\tau-1} h_t d_R^{[t+1,\tau-1]} + \min_x \quad \sum_{j=\tau}^{T} \sum_{k=j}^{T} \Big[c_j \big(d_R^{[j,k]} \big) + \sum_{t=j}^{k} h_t d_R^{[t+1,k]} \Big] x_{jk} \tag{1a}$$

$$\text{s.t.} \quad \sum_{j=r}^{t} \sum_{k=t}^{T} x_{jk} = 1 \qquad \text{for } t = \tau, \ldots, T, \tag{1b}$$

$$x_{jk} \geq 0 \qquad \text{for } j, k : \tau \leq j \leq k \leq T, \tag{1c}$$

where x_{jk} is a decision variable that indicates an order in period j to meet demands in periods $j, \ldots, k$, for all j, k such that $\tau \leq j \leq k \leq T$. Although the decision variables are continuous, this interpretation is well-defined: Chen and Zhang (2006) showed that there always exists an optimal solution to (1) such that $x_{jk} \in \{0, 1\}$ for all j, k. The first term of the objective (1a) is the cost of greedily using the initial inventory $\hat{s}_R^{r-1}$ to satisfy demand in periods $r, \ldots, \tau - 1$, and the coefficient of x_{jk} in the second term of the objective is the cost of satisfying demand in periods $j, \ldots, k$ with an order in period j. The constraints (1b) ensure that demand is satisfied in each period $\tau, \ldots, T$. We slightly abuse notation and use $f_r(R, \hat{s}^{r-1})$ to refer to the model (1) as well as its optimal value.

The dual of $f_r(R, \hat{s}^{r-1})$ is

$$f_r(R, \hat{s}^{r-1}) = \sum_{t=r}^{\tau-1} h_t d_R^{[t+1,\tau-1]} + \max_\alpha \quad \sum_{t=\tau}^{T} d_R^t \alpha_t \tag{2a}$$

$$\text{s.t.} \quad \sum_{t=j}^{k} d_R^t \alpha_t \leq c_j \big(d_R^{[j,k]} \big) + \sum_{t=j}^{k} h_t d_R^{[t+1,k]} \tag{2b}$$

$$\text{for } j, k : \tau \leq j \leq k \leq T.$$

Gopaladesikan et al. (2012) proposed a polynomial-time combinatorial algorithm that solves $f_r(R, \hat{s}^{r-1})$ and its dual (actually, an affine transformation of its dual), and used the primal and dual optimal solutions in conjunction with some structural properties established by Chen and Zhang (2006) to compute an allocation in the core of the economic lot sizing game $(N, f_r(\cdot, \hat{s}^{r-1}))$. We summarize these results in the lemma below.

Lemma 1 (Chen and Zhang 2006, Gopaladesikan et al. 2012)**.** *Let x^* and α^* respectively be optimal solutions to $f_r(R, s^{r-1})$ and its dual, obtained using the algorithm by Gopaladesikan et al. (2012). Then, x^* and α^* satisfy the following properties:*

(a) $\alpha_t^* + h_{t+1} \geq \alpha_{t+1}^*$ *for* $t = \tau, \ldots, T$.

(*b*) *For all* $\bar{d}_R = (\bar{d}_R^r, \ldots, \bar{d}_R^T)$ *such that* $\bar{d}_R^t \leq d_R^t$ *for* $t = r, \ldots, T$,

$$\sum_{t=j}^{k} \bar{d}_R^t \alpha_t^* \leq c_j\big(\bar{d}_R^{[j,k]}\big) + \sum_{t=j}^{k} h_t \bar{d}_R^{[t+1,k]} \quad \textit{for } j,k : \tau \leq j \leq k \leq T.$$

(*c*) *There exist* replenishment intervals $\{(a_1,b_1),\ldots,(a_m,b_m)\}$ *such that* $a_1 = \tau$, $b_m = T$, $a_\ell \leq b_\ell$ *and* $b_\ell + 1 = a_{\ell+1}$ *for all* $\ell = 1,\ldots,m$, *and*

$$x^*_{a_\ell b_\ell} = 1 \qquad \textit{for } \ell = 1,\ldots,m,$$
$$\sum_{t=a_\ell}^{b_\ell} d_R^t \alpha_t^* = c_{a_\ell}\big(d_R^{[a_\ell,b_\ell]}\big) + \sum_{t=a_\ell}^{b_\ell} h_t d_R^{[t+1,b_\ell]} \quad \textit{for } \ell = 1,\ldots,m.$$

3 Dynamic Cost Allocation

Now suppose that the time horizon starts in period $r = 1$, with each player having zero initial inventory, and the players agree to implement an optimal solution x^* to $f_1(N,0)$ with corresponding dual optimal solution α^* and replenishment intervals $\{(a_1,b_1),\ldots,(a_m,b_m)\}$ with $a_1 = 1$ and $b_m = T$, obtained by the algorithm of Gopaladesikan et al. (2012).

In previous approaches to economic lot sizing games (Chen and Zhang 2006, Gopaladesikan et al. 2012, van den Heuvel et al. 2007), the authors assume the entire cost of the optimal solution is allocated up front at the start of the horizon. Instead, we make the more realistic assumption that ordering costs must be allocated in the same period an order takes place, and holding costs for each period must be allocated in that same period. Finally, we assume the players agree that at any point in time, each player owns the inventory that was ordered to meet its own demand; in other words, for each $\ell = 1,\ldots,m$, the inventory of player $i \in N$ is

$$\hat{s}_i^{a_\ell - 1} = 0, \tag{3a}$$
$$\hat{s}_i^{r-1} = d_i^{[r,b_\ell]} \quad \text{for } r = a_\ell + 1,\ldots,b_\ell. \tag{3b}$$

Definition 2 (Kranich et al. 2005, Toriello and Uhan 2013)**.** In a *dynamic allocation* $\chi = (\chi_i^t)_{i\in N, t=1,\ldots,T}$, player $i \in N$ is allocated a cost of χ_i^t in period t. The *strong sequential core* of the economic lot sizing game $(N,(f_r)_{r=1,\ldots,T})$ is the set of dynamic allocations χ that satisfy the following conditions for some optimal solution x^* to $f_1(N,0)$:

(*a*) *Stage-wise efficiency*: For each $\ell = 1,\ldots,m$,

$$\sum_{i\in N} \chi_i^{a_\ell} = c_{a_\ell}\big(d_N^{[a_\ell,b_\ell]}\big) + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]}, \tag{4a}$$
$$\sum_{i\in N} \chi_i^t = h_t d_N^{[t+1,b_\ell]} \quad \text{for } t = a_\ell + 1,\ldots,b_\ell. \tag{4b}$$

In other words, the costs incurred in each period must be fully distributed among the grand coalition.

(*b*) *Time-consistent stability*: For each period $r = a_\ell,\ldots,b_\ell$ for some $\ell = 1,\ldots,m$,

$$\sum_{i\in R}\sum_{t=r}^{T} \chi_i^t \leq f_r(R, \hat{s}^{r-1}) \quad \text{for } R \subseteq N, \tag{4c}$$

where the initial inventories $\hat{s}^{r-1}$ are as defined in (3). In other words, at any point in the time horizon, the cost allocated to any coalition from that point forward does not exceed its cost if it abandons the grand coalition and continues on its own.

We define the dynamic allocation $\hat{\chi}$ as follows:

$$\begin{aligned} \hat{\chi}_i^{a_\ell} &= \sum_{t=a_\ell}^{b_\ell} d_i^t \left(\alpha_t^* - \sum_{u=a_\ell}^{t-1} h_u \right) + h_{a_\ell} d_i^{[a_\ell+1,b_\ell]} \\ \hat{\chi}_i^t &= h_t d_i^{[t+1,b_\ell]} \quad \text{for } t = a_\ell+1,\ldots,b_\ell \end{aligned} \qquad \text{for } \ell = 1,\ldots,m. \tag{5}$$

Theorem 3. *The dynamic allocation $\hat{\chi}$ defined in* (5) *is in the strong sequential core of the economic lot sizing game $(N,(f_r)_{r=1,\ldots,T})$.*

Proof. First, we show that $\hat{\chi}$ is stage-wise efficient. For any $\ell = 1,\ldots,m$,

$$\begin{aligned} \sum_{i\in N} \hat{\chi}_i^{a_\ell} &= \sum_{i\in N} \left[\sum_{t=a_\ell}^{b_\ell} d_i^t \left(\alpha_t^* - \sum_{u=a_\ell}^{t-1} h_u \right) + h_{a_\ell} d_i^{[a_\ell+1,b_\ell]} \right] \\ &= \sum_{t=a_\ell}^{b_\ell} d_N^t \left(\alpha_t^* - \sum_{u=a_\ell}^{t-1} h_u \right) + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]} \\ &= \sum_{t=a_\ell}^{b_\ell} d_N^t \alpha_t^* - \sum_{t=a_\ell}^{b_\ell} \sum_{u=a_\ell}^{t-1} h_u d_N^t + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]} \\ &= \sum_{t=a_\ell}^{b_\ell} d_N^t \alpha_t^* - \sum_{t=a_\ell}^{b_\ell} \sum_{u=t+1}^{b_\ell} h_t d_N^u + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]} \\ &= \sum_{t=a_\ell}^{b_\ell} d_N^t \alpha_t^* - \sum_{t=a_\ell}^{b_\ell} h_t d_N^{[t+1,b_\ell]} + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]} \\ &\overset{\text{(i)}}{=} c_{a_\ell}\left(d^t_{[a_\ell,b_\ell]}\right) + h_{a_\ell} d_N^{[a_\ell+1,b_\ell]}, \end{aligned}$$

where (i) follows from Lemma 1(c). In addition, we have for all $\ell = 1,\ldots,m$:

$$\sum_{i\in N} \hat{\chi}_i^t = \sum_{i\in N} h_t d_i^{[t+1,b_\ell]} = h_t d_N^{[t+1,b_\ell]} \quad \text{for } t = a_\ell+1,\ldots,b_\ell.$$

Next, we show that $\hat{\chi}$ is time-consistently stable. Fix $r \in \{a_p+1,\ldots,b_p,a_{p+1}\}$, for some $p \in \{0,1,\ldots,m\}$ (ignoring $a_0+1,\ldots,b_0$ and a_{m+1}). For any $R \subseteq N$, we have that

$$\begin{aligned} \sum_{i\in R}\sum_{t=r}^{T} \hat{\chi}_i^t &= \sum_{t=r}^{b_p} h_t d_R^{[t+1,b_p]} + \sum_{\ell=p+1}^{m} \left(\sum_{t=a_\ell}^{b_\ell} d_R^t \left(\alpha_t^* - \sum_{u=a_\ell}^{t-1} h_u \right) + \sum_{t=a_\ell}^{b_\ell} h_t d_R^{[t+1,b_\ell]} \right) \\ &= \sum_{t=r}^{b_p} h_t d_R^{[t+1,b_p]} + \sum_{\ell=p+1}^{m} \left(\sum_{t=a_\ell}^{b_\ell} d_R^t \alpha_t^* - \sum_{t=a_\ell}^{b_\ell} \sum_{u=a_\ell}^{t-1} h_u d_R^t + \sum_{t=a_\ell}^{b_\ell} h_t d_R^{[t+1,b_\ell]} \right) \\ &= \sum_{t=r}^{b_p} h_t d_R^{[t+1,b_p]} + \sum_{\ell=p+1}^{m} \left(\sum_{t=a_\ell}^{b_\ell} d_R^t \alpha_t^* - \sum_{t=a_\ell}^{b_\ell} \sum_{u=t+1}^{b_\ell} h_t d_R^u + \sum_{t=a_\ell}^{b_\ell} \sum_{u=t+1}^{b_\ell} h_t d_R^u \right) \end{aligned}$$

$$= \sum_{t=r}^{b_p} h_t d_R^{[t+1,b_p]} + \sum_{\ell=p+1}^{m} \sum_{t=a_\ell}^{b_\ell} d_R^t \alpha_t^*$$
$$\overset{\text{(ii)}}{\leq} f_r(R, \hat{s}^{r-1})$$

where (ii) holds because Lemma 1(b) implies that α^* is a dual feasible solution for $f_r(R, \hat{s}^{r-1})$. □

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