The Value Function of an Infinite-Horizon Single-Item Lot-Sizing Problem

Alejandro Toriello*
Daniel J. Epstein Department of Industrial and Systems Engineering
University of Southern California
toriello *at* usc *dot* edu

George Nemhauser
H. Milton Stewart School of Industrial and Systems Engineering
Georgia Institute of Technology
george *dot* nemhauser *at* isye *dot* gatech *dot* edu

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Abstract

We characterize the value function of a discounted, infinite-horizon version of the single-item lot-sizing problem. As corollaries, we show that this value function inherits several properties of finite, mixed-integer program value functions; namely, it is subadditive, lower semi-continuous and piecewise linear.

Keywords: value function, mixed-integer program, infinite optimization, production lot-sizing

1 Introduction

This paper characterizes the value function of a discounted, infinite-horizon version of the classical single-item uncapacitated lot-sizing problem (LSP). Suppose we need to manage the production schedule for a single item that experiences constant per-period demand $d > 0$. There is no production or

*Corresponding author. 3715 McClintock Avenue GER 240, Los Angeles, California, 90089.

inventory capacity, and all demand must be met each period, either with items produced that period, or items in inventory. Every period we produce, we incur a fixed cost $f > 0$ and a variable cost of $c > 0$ per unit produced. Items left over at the end of the period after demand is met incur a holding cost of $h > 0$ per unit. The initial stock on hand is $s \geq 0$. We can model this problem as

$$C(s,d) = \min \sum_{t=1}^{\infty} \gamma^{t-1}(f\delta(z_t) + cz_t + hs_t) \tag{1a}$$

$$\text{s.t. } z_t + s_{t-1} - s_t = d, \forall\ t = 1, \ldots \tag{1b}$$

$$s_0 = s \tag{1c}$$

$$z_t, s_t \geq 0, \forall\ t = 1, \ldots, \tag{1d}$$

where s_t and z_t respectively represent ending stock and production during period t, $\gamma \in [0,1)$ is a discount factor, $\delta(z) = 1$ when $z > 0$ and $\delta(z) = 0$ otherwise. We take the set of feasible solutions to be a subset of the sequences (z_t, s_t) with finitely converging objective (cf. [1].) Because the objective is a cost minimization, we use the notation C to denote the optimal value of (1) as a function of s and d; however, we refer to C as a *value function*. We include the degenerate case $d = 0$ so the domain of C is the closed cone $\mathbb{R}^2_+$.

Problem (1) and its many variations have a long history in operations research. The structure of optimal solutions in the finite, dynamic case was studied in the seminal work of Wagner and Whitin [2], and many researchers have since attempted to generalize their results for more complex models. Most authors have given results pertaining to optimal solutions' structure, and related issues such as regeneration points and replenishment intervals (see Theorem 1 below.) The interested reader may consult the texts [3, 4] and references therein.

However, in infinite-horizon problems the value function itself has received relatively little attention. One important exception is [5], where the authors give a closed-form expression for the value function of a continuous-time, average-cost EOQ model; our results yield the analogue in the discrete-time, discounted case. This knowledge about the value function is useful for exact or approximate optimization in many dynamic applications, as well as for sensitivity analysis.

From the mixed-integer programming (MIP) perspective, we show that C inherits several important characteristics from finite MIP value functions: C is subadditive, lower semicontinuous and piecewise linear. Whereas value functions of finite MIP's have been studied extensively (e.g. [6]), relatively little is known about value functions in the infinite case. Our results for the simple infinite model (1) are a first step towards addressing this gap.

2 Optimal Solutions

The following theorem summarizes the structure of optimal solutions for (1). These results are well-known throughout the mixed-integer programming, dynamic programming and inventory control community, and similar results exist for different variations of LSP. [2] has the original proofs for a finite variant of the problem; however, the arguments carry over to (1) with minor changes.

Theorem 1. *Suppose $d > 0$, and let $t^* = \lfloor \frac{s}{d} \rfloor + 1$. Any optimal solution to* (1) *satisfies the following statements.*

i) $z_t = 0, \forall\ t < t^*$.

ii) $z_{t^*} > 0$, *and* $s_{t^*-1} + z_{t^*} = k_{t^*} d$ *for some* $k_{t^*} \in \mathbb{N} = (1, 2, \dots)$.

iii) $s_{t-1} z_t = 0, \forall\ t > t^*$, *and if* $z_t > 0$, *then* $z_t = k_t d$, *for some* $k_t \in \mathbb{N}$.

Proof. For $t < t^*$, if $z_t > 0$ we can always postpone production and decrease the objective. This also implies $z_{t^*} > 0$ by feasibility. For $t > t^*$, if we produce an amount that is not an integer multiple of d, we can always decrease production to the largest integer multiple of d and postpone the remaining production to the next period of positive production while improving the objective. This implies we only produce when incoming inventory equals zero. If $s - (t^* - 1)d > 0$, a similar argument shows that $s_{t^*-1} + z_{t^*}$ is an integer multiple of d. □

The theorem states that optimal solutions have a *replenishment interval* structure: Production is always equal to the cumulative demand for an interval of consecutive periods; for (1), replenishment intervals are always equal because the data is stationary, except in degenerate cases when two interval lengths are optimal.

3 The Value Function

The most basic non-trivial case occurs when the initial inventory is zero.

Proposition 2. *Suppose $d > 0$. Then*

$$C(0,d) = \min_{k \in \mathbb{N}} \left\{ \frac{1}{1-\gamma^k} \left(f + kcd + hd \sum_{\ell=1}^{k-1} \gamma^{\ell-1}(k-\ell) \right) \right\}.$$

For any d, at most two consecutive integers $k(d), k(d)+1$ minimize the quantity inside the brackets.

Proof. This proof uses standard dynamic programming techniques; see, e.g., [7]. By Theorem 1, we know that production must occur in integer multiples of d, and thus all future inventories will be integer multiples of d. We can thus consider a finite-state and -action dynamic program with state space $S = \{0, \ldots, K\}$, where the integer K is chosen large enough. The set of feasible actions are the "do nothing" action for all positive states, and the actions corresponding to a replenishment of length k, $k = 1, \ldots, K$ in the zero state. The quantity inside the minimization bracket is precisely the present value of replenishing inventory every k periods into perpetuity, and the minimum over all such quantities gives the optimal policy. Moreover, using the identity

$$\sum_{\ell=1}^{k-1} \gamma^{\ell-1}(k-\ell) = \frac{k(1-\gamma) - (1-\gamma^k)}{(1-\gamma)^2}, \forall\, k \in \mathbb{N},$$

the expression in the brackets reduces to $\frac{Ak+B}{1-\gamma^k} - C$, for appropriately chosen positive constants A, B and C. This function can easily be shown to be strictly convex for $k \geq 1$ and eventually increasing. Therefore the minimum over all natural numbers can be achieved by at most two consecutive numbers. □

This result shows that, for a fixed d, we can reformulate (1) as a MIP in a similar fashion to the finite case. We add binary decision variables $x_t \in \{0,1\}$, replace the objective (1a) with

$$\min \sum_{t=1}^{\infty} \gamma^{t-1}(f x_t + c z_t + h s_t) \tag{2}$$

and add the constraints

$$M_d x_t - z_t \geq 0, \forall\, t = 1, \ldots \tag{3}$$

As long as $M_d \geq k(d)d$, at least one optimal solution will remain feasible in the reformulation.

Corollary 3.

i) $C(s,d) = C(0,d) - cs, \forall\ 0 \leq s < d, \forall\ d > 0.$

ii) $C(s,d) = h(s-d) + \gamma C(s-d,d), \forall\ s \geq d, \forall\ d.$

Proof. (i) For simplicity, assume the optimal replenishment length is unique, and let $k(d) \in \mathbb{N}$ be the length. It suffices to prove that any optimal solution satisfies $z_1 = k(d)d - s$. Suppose not; by Theorem 1, $z_1 = k'd - s$, for some $k' \neq k(d)$. Then

$$f + c(k'd - s) + hd\sum_{\ell=1}^{k'-1}\gamma^{\ell-1}(k' - \ell) + \gamma^{k'}C(0,d) \leq$$
$$f + c(k(d)d - s) + hd\sum_{\ell=1}^{k(d)-1}\gamma^{\ell-1}(k(d) - \ell) + \gamma^{k(d)}C(0,d)$$

However, s can be eliminated from both sides, and the resulting relation implies that k' is an optimal replenishment length, contradicting our assumption.

(ii) Follows directly from Theorem 1(i). □

The next theorem summarizes the preceding results into a single formula.

Theorem 4. *C is given by*

$$C(s,0) = \frac{hs}{1-\gamma}, \forall\ s \geq 0 \tag{4}$$

$$C(0,d) = \min_{k\in\mathbb{N}}\left\{\frac{1}{1-\gamma^k}\left(f + kcd + hd\sum_{\ell=1}^{k-1}\gamma^{\ell-1}(k-\ell)\right)\right\}, \forall\ d > 0 \tag{5}$$

$$\begin{aligned} C(s,d) = {} & s\left(h\left(\frac{1-\gamma^k}{1-\gamma}\right) - \gamma^k c\right) - d\left(h\sum_{\ell=1}^{k}\ell\gamma^{\ell-1} - \gamma^k kc\right) \\ & + \gamma^k C(0,d), \forall\ kd \leq s < (k+1)d, \forall\ k \in \mathbb{Z}_+. \end{aligned} \tag{6}$$

Proof. The first equation is directly obvious but also follows by setting $t^* = \infty$ in the proof of Theorem 1. The second equation is a restatement of Proposition 2. The last equation follows by Corollary 3(i) and recursive applications of Corollary 3(ii). □

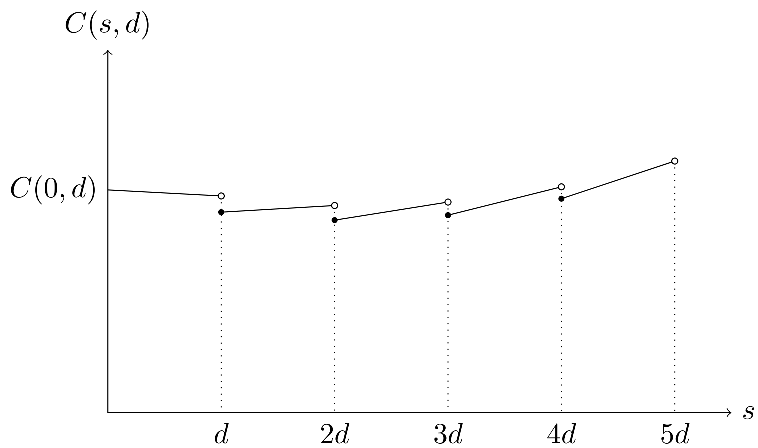

Figure 1: Single-item LSP value function for fixed $d > 0$ with $k(d) = 2$.

Figure 1 shows an example plot of $C(s, d)$ for a fixed $d > 0$ with optimal replenishment interval of two periods. The discontinuities in C occur along the lines $s = kd$, for each $k \in \mathbb{N}$. Within each region $\{(s, d) : kd \leq s < (k + 1)d\}$, $C(s, d)$ is piecewise linear and continuous. Figure 2 shows a sample plot with $s = 0$.

In the following corollary, we say that C is *subadditive* if $C(s, d) + C(s', d') \geq C(s + s', d + d'), \forall\ s, s', d, d' \geq 0$.

Corollary 5. *C satisfies the following statements:*

i) C is subadditive.

ii) C is piecewise linear, with a countably infinite number of regions in which it is affine.

iii) C is lower semicontinuous.

Before proving this result, we note that in the finite MIP case, a restriction of the value function to a bounded sub-domain is piecewise linear and defined by finitely many regions in which it is affine [6, Theorem 6.1]. Conversely, the restriction of C to a bounded subset of $\mathbb{R}^2_+$ that contains a neighborhood of a point on the ray $d = 0$ is piecewise linear but requires a countably infinite number of regions in which it is affine to define it.

For lower semicontinuity in finite MIP value functions, the interested reader may consult [9].

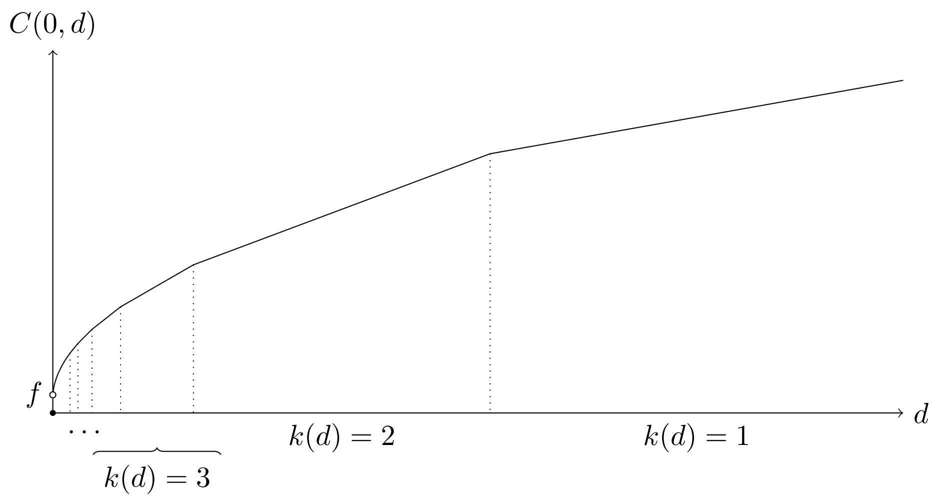

Figure 2: Single-item LSP value function for $s = 0$.

Proof. (i) This proof is almost identical to the finite case; see [8]. Let s, d and s', d' be pairs of starting inventories and demands, with respective optimal solutions (z_t, s_t) and (z'_t, s'_t). Then $(\hat{z}_t, \hat{s}_t) = (z_t, s_t) + (z'_t, s'_t)$ is feasible for (1) with starting inventory and demand given by $s + s'$, $d + d'$ respectively, and has an objective no greater than $C(s, d) + C(s', d')$.

(ii) Follows directly from Theorem 4.

(iii) First, suppose $d > 0$; discontinuities occur when s is an integer multiple of d. If $s = d$, then using the basic fact $C(0, d) \geq \frac{cd}{1-\gamma}$, we have

$$C(0, d) - cd \geq \gamma C(0, d),$$

where the quantity on the left is $\lim_{s \uparrow d} C(s, d)$ and the right-hand side is $C(d, d)$. A similar argument, also using $C(0, d) \geq \frac{cd}{1-\gamma}$, establishes lower semicontinuity for $s = kd, k \geq 2$.

Now suppose $d = 0$. The proof is trivial for $C(0, 0)$, since all other function values are positive. So suppose $s > 0$, and let $(\hat{s}, \hat{d})$ satisfy $\hat{s} > 0$, $0 < \hat{d} < \frac{\hat{s}}{2}$. Using the identity

$$\sum_{\ell=1}^{k} \ell \gamma^{\ell-1} = \frac{k\gamma^{k+1} - k\gamma^k + 1 - \gamma^k}{(1-\gamma)^2},$$

we have

$$
\begin{aligned}
C(\hat{s},\hat{d}) &= \hat{s}\Big[h\Big(\frac{1-\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}}{1-\gamma}\Big) - \gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}c\Big] + \gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}C(0,\hat{d}) \\
&\qquad - \hat{d}\Big[\frac{h}{(1-\gamma)^2}\Big(\Big\lfloor\frac{\hat{s}}{\hat{d}}\Big\rfloor\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor+1} - \Big\lfloor\frac{\hat{s}}{\hat{d}}\Big\rfloor\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor} + 1 - \gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}\Big) - \gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}\Big\lfloor\frac{\hat{s}}{\hat{d}}\Big\rfloor c\Big] \\
&= \frac{h\hat{s}}{1-\gamma} - h\hat{d}\Big(\frac{\hat{s}}{\hat{d}} - \Big\lfloor\frac{\hat{s}}{\hat{d}}\Big\rfloor\Big)\frac{\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}}{1-\gamma} + \hat{d}\Big(\frac{\hat{s}}{\hat{d}} - \Big\lfloor\frac{\hat{s}}{\hat{d}}\Big\rfloor\Big)\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}c \\
&\qquad - h\hat{d}\Big(\frac{1-\gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}}{1-\gamma}\Big) + \gamma^{\lfloor\frac{\hat{s}}{\hat{d}}\rfloor}C(0,\hat{d}) \to \frac{hs}{1-\gamma} = C(s,0),
\end{aligned}
$$

as $(\hat{s},\hat{d}) \to (s,0)$. □

4 Conclusions

We have characterized the value function of an infinite-horizon, discounted, single-item lot-sizing problem. Our results provide a link between value function theory in mixed-integer programming and dynamic programming. An obvious question is the extension of these structural results to other infinite-horizon models, either in production planning or more generally in infinite mixed-integer programming. Further work in this area could reveal links between mixed-integer programming and dynamic programming and allow us to extend known value function theory from finite mixed-integer programs into the infinite realm.

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