Polyhedral Approaches to Online Bipartite Matching

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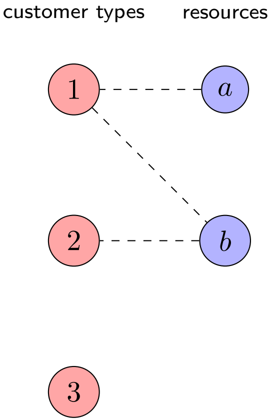
Motivation

Marketplaces and platforms in the “new” economy...



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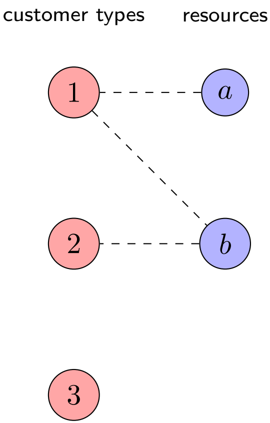

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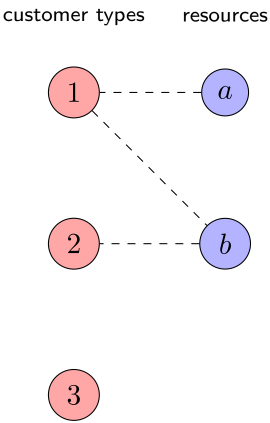

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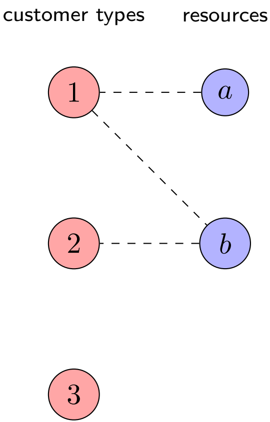

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- Web browsing: Display advertisement (banners and pop-ups).
- Ride-hailing: Driver assignment.

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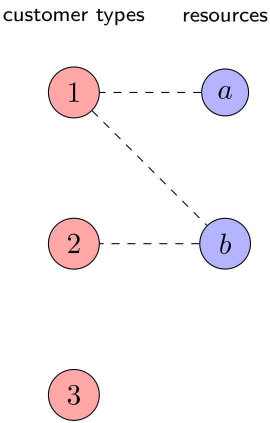
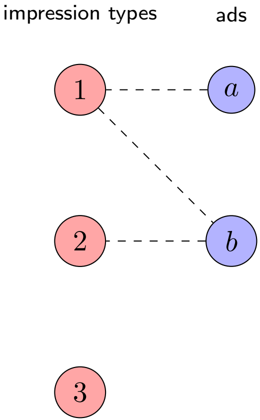


- Online search: Keyword advertisement (e.g. AdWords).
- Web browsing: Display advertisement (banners and pop-ups).
- Ride-hailing: Driver assignment.
- Many other revenue management applications.
  - Interested here in centrally managed markets.

Problem Definition



$t = 3$

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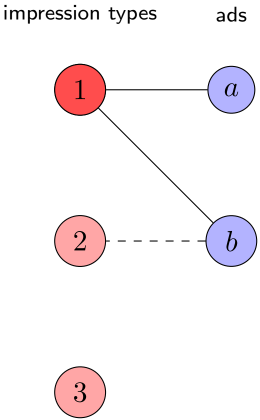

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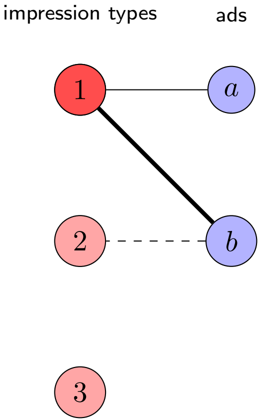

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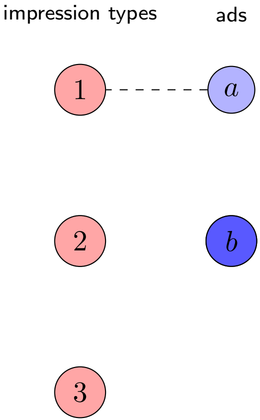
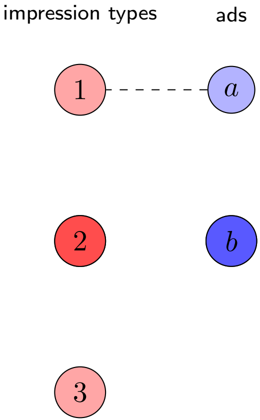


$t = 2$

Problem Definition



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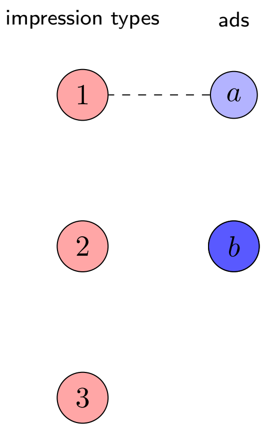
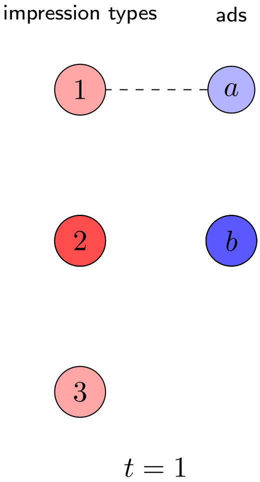


$t = 1$

Problem Definition



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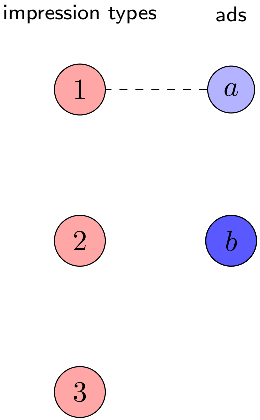
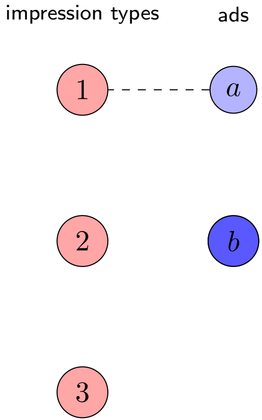


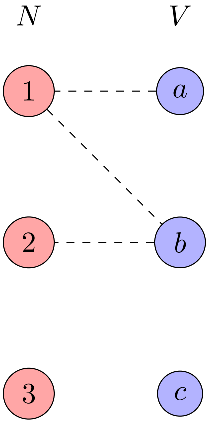
$t = 0$

Problem Definition



- Arrival models:
  - Adversarial (most conservative)
  - Unknown permutation
  - Unknown i.i.d.
  - Known i.i.d. (least conservative)

Problem Definition



- Bipartite graph with
  - V: right side (ads), fixed and known;
  - N: left side (impressions), "types" that may appear.

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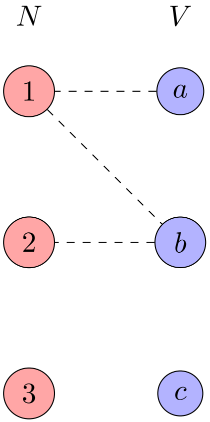

- Bipartite graph with
  - V: right side (ads), fixed and known;
  - N: left side (impressions), "types" that may appear.
- We sequentially observe T i.i.d. uniform samples drawn from N.
  - An appearing node must be matched or discarded.
  - For this talk, assume $|V| = |N| = T = n$.
- Objective is maximizing expected number of matches, or matching weight.

Online Bipartite Matching

Brief review of related work

- Karp/Vazirani/Vazirani (90)
  - Adversarial model (no distribution). Randomized ranking policy is optimal, with $1 - 1/e$ competitive ratio.
- Applications in online search and revenue management renew CS interest. First studied model is i.i.d.
  - Feldman/Mehta/Mirrokni/Muthukrishnan (09): First algorithm guarantee that "beats" $1 - 1/e$.
  - Improvements and extensions: Bahmani/Kapralov (10), Haeupler/Mirrokni/Zadimoghaddam (11), Manshadi/Oveis Gharan/Saberi (12)...
- Jaillet/Lu (13): Currently best-known guarantee.
- Our goal: Upper bounds via polyhedral relaxations, employ in policy design.

Outline

An Initial Relaxation

Feldman/Mehta/Mirrokni/Muthukrishnan (09)

- z_{ij}: Probability policy ever matches impression i to ad j.

$$\max_{z \geq 0} \sum_{i,j} w_{ij} z_{ij} \qquad (w \text{ encodes compatibility})$$

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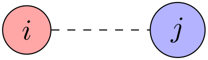
$$\sum_{j} z_{ij} \leq T/n = 1 \qquad (\text{expect to see } i \text{ once})$$

- Bipartite matching polytope for "expected graph".
  - Simple policy: Solve for max-weight matching, match only these edges.
  - Each edge matched w.p. $1 - (1 - 1/n)^n \approx 1 - 1/e$.

Improving the Relaxation

- Idea: Add additional valid inequalities satisfied by feasible policies.
  - Similar to *achievable region* approach in applied probability, e.g. Bertsimas/Niño Mora (96), Coffman/Mitrani (80).



- First example (Haeupler/Mirrokni/Zadimoghaddam, 11): Upper bounds

$$z_{ij} \leq 1 - (1 - 1/n)^n$$

are valid.
  - Impression i never appears w.p. $(1 - 1/n)^n$.

Facial Dimension

Proposition

The upper bound inequalities

$$z_{ij} \le 1 - (1 - 1/n)^n$$

are facet-defining for the polytope of achievable probabilities.*

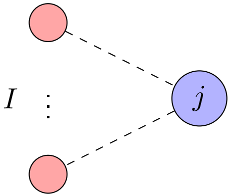
Proof sketch.

Consider the following n^2 "affinely independent" policies:

- (i, j) Match edge when possible, nothing else.
- (i', j') Match either edge when possible, nothing else.
- (i, j') Match i to j on first appearance, then to j' on second; nothing else.
- (i', j) Match (i, j) when possible; in last stage match i' instead if it appears.

* Yes, it's a full-dimensional polytope.

Right Star Inequalities

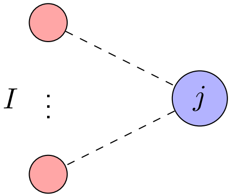


- Any set of neighbors I of ad j will never appear w.p. $(1 - |I|/n)^n$, thus

$$\sum_{i \in I} z_{ij} \leq 1 - (1 - |I|/n)^n$$

is valid.
  - Separate greedily in poly-time.

Right Star Inequalities



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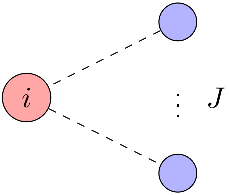
Proposition

If $I \neq N$, the inequality has facial dimension $n^2 - |I|$.

Proposition

If $I = N$, the inequality corresponds to j's degree constraint, and is facet-defining.

Left Star Inequalities



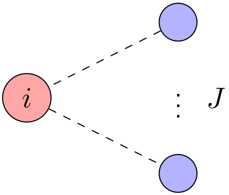
- For any set of neighbors J of impression i,

$$\sum_{j \in J} z_{ij} \leq \mathbb{E}[\min\{|J|, B(n, 1/n)\}]$$

  is valid, where B is a binomial r.v.

  - Same greedy separation.

Left Star Inequalities



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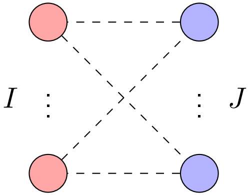
  - Same greedy separation.

Theorem
The inequality is facet-defining for all J.

Proof sketch.
Construct $|J|$ policies using an ordering of J. Corresponding sub-matrix is a *circulant*, thus non-singular. Apply previous argument to remaining edges.

Complete Subgraph Inequalities



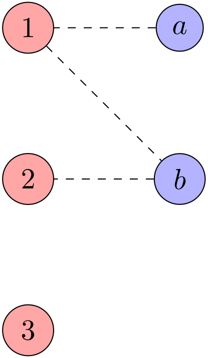
- For any $I \subseteq N$, $J \subseteq V$,

$$\sum_{ij \in I \times J} z_{ij} \leq \mathbb{E}[\min\{|J|, B(n, |I|/n)\}]$$

  is valid.
  - Greedy separation for fixed I or J, full separation with MIP.
- Inequalities are not facet-defining except in cases previously mentioned.

What More Do We Need?

Are 0-1 inequalities enough?



- Calculated this instance's convex hull with PORTA.
- The polytope has 13 non-trivial facets, only four of which are covered by our previous inequalities.
- No other facet has 0-1 structure.

Computational Experiments

Static relaxations

- Geometric mean of gap w.r.t. benchmark for different instances.

Bound/Policy	20-Cycle	200-Cycle	Small	Large Dense	Large Sparse
EG	1.27	1.27	1.32	1.00	1.22
EG + UB	1.27	1.27	1.09	1.00	1.12
EG + RS	1.13	1.10	1.05	1.00	1.08
EG + LS	1.16	1.14	1.06	1.00	1.10
EG + RS + LS	1.13	1.10	1.05	1.00	1.07
Sim. Bound	1.05	1	1.00	1	1
DP	1	-	1	-	-
Matching	0.83	0.80	0.86	0.64	0.78
2-Matching*	0.99	0.95	0.96	0.75	0.88
Best static policy	0.99	0.95	1.00	0.95	0.96

Small $\rightarrow n = 10$, large $\rightarrow n = 100$.

* From Feldman/Mehta/Mirrokni/Muthukrishnan (09)

Computational Experiments

Static relaxation takeaways

- Significant gap left to close, especially in sparser instances.
  - Right stars close more of the gap than left stars, contrasting facial dimension results.
  - We'll use EG + RS as "static" bound benchmark.

Dynamic Relaxations

- z_{ij}^t: Probability we match i to ad j in stage $t = n, \ldots, 1$.
  (Count stages down.)

- Objective becomes

$$\max \sum_{i,j,t} w_{ij} z_{ij}^t$$

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$$\sum_t z_{ij}^t = z_{ij}$$

- Use shorthand

$$Z_{IJ}^{[t_2,t_1]} := \sum_{i \in I} \sum_{j \in J} \sum_{\tau = t_1}^{t_2} z_{ij}^{\tau}$$

Proposition

$$Z_{iV}^{t} \leq 1/n, \qquad i \in N, t \in [n]$$

is valid and facet-defining.

Proof sketch.

Probability i appears in t is $1/n$.
For any (i, t) and any (k, ℓ, τ), can choose a policy with $z_{k\ell}^{\tau} = 1/n$ and equality.

Dynamic Relaxations

Simplest valid inequality

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Proof sketch.

Probability i appears in t is $1/n$.
For any (i,t) and any (k,ℓ,τ), can choose a policy with $z_{k\ell}^{\tau} = 1/n$ and equality.

Corollary

$$\sum_{j \in V} z_{ij} = Z_{iV}^{[n,1]} \leq n/n = 1,$$

i.e. implies static inequality.

Dynamic Relaxations

A more involved example

- Matching i to j in t implies two independent events:

  $\mathbb{P}(i \to j \text{ in } t) \leq \mathbb{P}(i \text{ appears in } t) \times \mathbb{P}(j \text{ not matched in } [t+1, n])$.

- Equivalent to

$$z_{ij}^t \leq 1/n\left(1 - Z_{Nj}^{[n,t+1]}\right).$$

Theorem

$$Z_{Nj}^{[n,t+1]} + n z_{i_t j}^{t} \leq 1, \qquad i \in N, j \in V, t \leq n-1$$

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Theorem

These two inequality classes imply EG + RS static relaxation.

- New LP has $\Theta(n^3)$ variables and constraints, versus $\Theta(n^2)$ and $\Theta(n2^n)$.

Linking Many Periods

- Connect any number of consecutive periods:

$$Z_{Nj}^{[n,t+1]} + n z_{i_t j}^{t} \leq 1 \qquad t \leq n-1$$

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$$Z_{NJ}^{[n,t+2]} + n Z_{i_{t+1} J}^{t+1} + n^2 Z_{i_t J}^{t} \le 1 + n \qquad |J| = 2, t \le n-2$$

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$$\vdots \qquad \vdots$$

$$Z_{NJ}^{n} + \sum_{t=1}^{n-1} n^{n-t} Z_{i_t J}^{t} \leq \sum_{t=1}^{n-1} n^{t-1} \qquad |J| = n-1$$

Linking Many Periods

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$$
\begin{aligned}
Z_{Nj}^{[n,t+1]} + n z_{i_t j}^{t} &\leq 1 && t \leq n-1 \\
Z_{NJ}^{[n,t+2]} + n Z_{i_{t+1} J}^{t+1} + n^2 Z_{i_t J}^{t} &\leq 1 + n && |J| = 2, t \leq n-2 \\
\vdots \qquad & \qquad \vdots \\
Z_{NJ}^{n} + \sum_{t=1}^{n-1} n^{n-t} Z_{i_t J}^{t} &\leq \sum_{t=1}^{n-1} n^{t-1} && |J| = n-1
\end{aligned}
$$

Theorem

All such inequalities are valid and facet-defining.

- Can be generalized further... but difficult to separate and numerically unstable.

Heuristic Policies

Based on dynamic ad values

- Recall two-period inequality:

$$Z_{Nj}^{[n,t+1]} + nz_{ij}^{t} \leq 1 \qquad \leftrightarrow \qquad r_{ij}^{t}$$

Use corresponding dual multiplier, "value" of having j available in t when i appears.

Heuristic Policies

Based on dynamic ad values

- Recall two-period inequality:

$$Z_{Nj}^{[n,t+1]} + nz_{ij}^t \leq 1 \qquad \leftrightarrow \qquad r_{ij}^t$$

  Use corresponding dual multiplier, "value" of having j available in t when i appears.

- Policy maximizes net value: If i appears in stage t, choose

$$\underset{\text{available } j}{\arg\max} \left\{ w_{ij} - 1/n \sum_{k \in N} \sum_{\tau < t} r_{kj}^{\tau} \right\}$$

  or none if negative.

- Similar to dynamic bid price policies in revenue management (e.g. Adelman, 07).

Computational Experiments

Dynamic relaxation and policy

Bound/Policy	Small	Large Dense	Large Sparse
EG + RS	1.08	1.00	1.08
Dynamic LP	1.04	0.97	1.03
Sim. Bound	1.03	1	1
DP	1	-	-
Dynamic Policy	1.00	0.96	0.96
Best static policy	0.99	0.95	0.95

Small $\to n = 10$, large $\to n = 100$, 20 of each.

- Dynamic relaxation and policy improve across the board.
- Bound improves in every instance tested.

Conclusions

- Polyhedral relaxations offer way to derive bounds for dynamic matching and resource allocation.
  - Also offer insight into high-quality policy design.
- Even for bipartite matching, structure appears very complex vis-à-vis deterministic case.
  - Must combine combinatorial and probabilistic techniques.
- Exploring use in other dynamic resource allocation contexts.

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