

# Decomposition-Based Approximation Algorithms for the One-Warehouse Multi-Retailer Problem with Concave Batch Order Costs

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## Abstract

We study the *one-warehouse multi-retailer* (OWMR) problem under deterministic dynamic demand and concave batch order costs, where order batches have an identical capacity and the order cost function for each facility is concave within the batch. Under appropriate assumptions on holding cost structure, we obtain lower bounds via a decomposition that splits the two-echelon problem into single-facility subproblems, then propose approximation algorithms by judiciously recombining the subproblem solutions. For piecewise linear concave batch order costs with a constant number of slopes we obtain a constant-factor approximation, while for general concave batch costs we propose an approximation within a logarithmic factor of optimality. We also extend some results to subadditive order and/or holding costs.

## 1 Introduction

The *one-warehouse multi-retailer* (OWMR) problem involves a two-echelon supply chain system with a central warehouse and  $N$  downstream retailers. The warehouse orders a single product from an external supplier to fulfill the orders of the retailers, who then use their inventories to meet demand. An order cost is incurred whenever a facility (the warehouse or a retailer) places an order, and holding costs are charged for inventory. The goal is to minimize the total order and holding cost while satisfying all demand without backlogs. The problem generalizes many fundamental inventory models (Levi et al. 2008), including the single-item *lot sizing problem* (LSP), the multi-item LSP, as well as the *joint replenishment problem* (JRP). It has therefore drawn broad attention

in the operations research, discrete optimization and algorithms communities, and seen numerous applications in production planning, inventory management and distribution logistics.

This paper investigates the OWMR problem in a discrete-time finite horizon of  $T$  periods, where the demands  $d_i^t$  ( $i = 1, \dots, N, t = 1, \dots, T$ ) vary over time but are deterministic, and backlogs or lost sales are not allowed. Order lead times are zero without loss of generality, as we can always shift order times backwards in the zero-lead-time model by the lead time to get a positive-lead-time solution. The classical literature makes two cost assumptions: order costs are fixed and independent of order quantities in each period, and holding costs are non-negative, linear in inventory quantities, and additive over time. We assume the same holding cost structure, but consider a more complex order cost structure, which can model order discounts, various transportation modes, heterogeneous machines, etc. Specifically, we examine the class of *concave batch costs*, where batches have an identical capacity  $Q$ , and the order cost is a concave function of the volume on  $[0, Q]$  and repeats itself on  $(Q, 2Q]$ ,  $(2Q, 3Q]$ , etc. We also extend some results to general non-decreasing *subadditive* order and holding costs.

Our study is motivated by agricultural supply chains (Nguyen et al. 2013; 2014, Zhang et al. 2018), where local growers send product via short-haul transportation services to a consolidation center, which then delivers it to a destination via long-haul transportation services. Both echelons can choose from multiple shipping options, e.g. fixed *full-truckload* (FTL) rates, linear *less-than-truckload* (LTL) rates and courier rates, and thus the resulting transportation cost function exhibits economies of scale captured by *piecewise linear* (PWL) concave batch costs. This application represents a “reversed” OWMR system, i.e., the growers, the consolidation center and the destination correspond to the downstream retailers, the warehouse and the upstream supplier respectively, and a product ready for shipment in period  $t$  corresponds to demand due by period  $(T - t + 1)$  in a conventional OWMR problem. However, the models are interchangeable in terms of modeling and solution techniques, and we will henceforth focus on the conventional variant.

Our cost assumptions are general enough to model many pricing schemes used in practice, among which an important application is order discounts that incentivize larger order quantities. According to Nahmias (2001), there are three major types of discount schemes, namely *all-unit discounts*, *incremental discounts* and *truckload discounts*. The all-unit discount scheme applies non-increasing rates to all units of an order as the total volume increases. The incremental discount scheme applies non-increasing unit rates to incremental ranges of the order quantity. The truckload discount scheme charges an LTL rate linearly in terms of the volume until a threshold is reached when the customer is willing to pay a fixed FTL rate, and the pattern repeats itself once a truck is full. All three types result in PWL order costs, where all-unit discounts are subadditive and special cases can be approximated by PWL concave batch costs, see e.g. Hu (2016); incremental discounts are PWL concave costs, which correspond to PWL concave batch costs when the batch capacity exceeds the total demand, i.e.,  $Q \geq \sum_{i=1}^N \sum_{t=1}^T d_i^t$ ; and truckload discounts are special PWL concave batch cost functions which contain exactly one positive slope on  $[0, Q]$ . See Archetti et al. (2014), Chan et al. (2002), Hu (2016) and Appendix A for more details.

## 1.1 Literature Review

The OWMR problem is  $\mathcal{NP}$ -hard under dynamic demands even if cost parameters are static, order costs are quantity-independent, and holding costs are homogeneous for all facilities (Arkin et al.

1989). In fact, Chan et al. (2002) prove that when retailers’ order costs are time-varying, the model is at least as hard as the set cover problem, which cannot be approximated in polynomial time to within  $\mathcal{O}(\log N)$ -optimality unless  $\mathcal{P} = \mathcal{NP}$  (Feige 1998). Therefore, most research on heuristics with theoretical solution guarantees assumes static retailer order costs. Most of these results are approximation algorithms that efficiently compute solutions whose total costs are guaranteed to be within a multiplicative factor of the optimal cost, and this factor is called the *approximation ratio*. The state of the art for the standard OWMR system is the 1.8-approximation algorithm in Levi et al. (2008), which uses randomized rounding techniques to construct feasible solutions of a *mixed integer programming* (MIP) formulation based on the *linear programming* (LP) relaxation. In the same paper, the authors utilize dual information to give a 3.6-approximation for *fixed batch order costs*, a special case of concave batch costs where the cost per batch is independent of the quantity. These authors’ holding cost assumptions are more general than the standard structure, but maintain classic optimality properties like *zero-inventory ordering* (ZIO) and *first-in-first-out* (FIFO). ZIO refers to policies where a facility does not place an order until its inventory level drops to zero, whereas FIFO prioritizes orders that arrive earlier to deplete inventory and meet demand. For the special case of the JRP, Bienkowski et al. (2013) improved the approximation ratio to 1.791.

There is comparatively less literature on OWMR models with more general order costs, in part because ZIO may no longer be optimal (see e.g. Appendix B for a counterexample), which implies significant additional difficulty in these variants. This contrasts with single-echelon LSP and its generalizations, where polynomial-time optimal algorithms have been designed under many complicated cost assumptions (Akbalik and Rapine 2012, Anily et al. 2009, Archetti et al. 2014, Jin and Muriel 2009, Koca et al. 2014, Lippman 1969). Restricting the warehouse holding cost rates to be lower than the retailers’, Jin and Muriel (2009) propose a nested *dynamic programming* (DP) algorithm under fixed batch order costs whose runtime is polynomial for a fixed number of retailers, but exponential otherwise. However, the DP counts on structural properties that do not hold if the warehouse holding cost rate does not satisfy this assumption (see Appendix B for another counterexample). Chan et al. (2002) study an OWMR problem where retailers have *modified all-unit discount* order cost functions, which diminish in marginal costs and alternate flat segments with non-flat segments (see Appendix A). Assuming incremental order discounts at the warehouse, the authors obtain approximation ratios of 1.33 and 1.22 for ZIO policies under time-varying and static costs, respectively. Finding the best ZIO policy under these assumptions is  $\mathcal{NP}$ -hard, so the authors propose LP-based heuristics that give near-optimal ZIO solutions in their computational experiments. Shen et al. (2009) generalize the order cost functions to a class of PWL costs, and propose an approximation scheme with an LP relaxation of a large-scale concave cost network flow reformulation for the JRP. The algorithm yields  $(2(1+\epsilon)\log(NT))$ -optimal solutions in polynomial time with respect to the input size and  $1/\epsilon$ , for  $\epsilon > 0$ . Recent papers (Cheung et al. 2015, Nagarajan and Shi 2016) derive approximation algorithms for the submodular JRP, where the joint order cost is a submodular set function of the items and thus reflects a natural economy of scale structure. All of these algorithms are based on LP models, partly since PWL concave and submodular costs give rise to strong LP relaxations. In contrast, PWL concave batch costs can lead to arbitrarily weak LP relaxations that result in linear order costs at the lowest possible unit rates (Croxtton et al. 2003). Therefore, an LP-based approach is likely ineffective for our problem.

Another research stream concerns decomposition-based algorithms, which use the solutions of

more tractable subproblems to construct a solution for the entire problem. Stauffer et al. (2011) apply the idea to the OWMR model in Levi et al. (2008) under slightly more relaxed holding cost assumptions, and obtain a 2-approximation algorithm in  $\mathcal{O}(NT)$  time by recombining optimal single-facility subproblem solutions. Gayon et al. (2016a) extend the technique to allow demand shortages, and attain a 3-approximation for the OWMR problem with backlogs, a 2-approximation for the JRP with backlogs, and a 2-approximation for the OWMR problem with lost sales. Most recently, Gayon et al. (2016b) apply the technique to several OWMR variants, including extending the Stauffer et al. (2011) 2-approximation to the fixed batch order cost model and studying some order cost functions similar to ours. However, their results rely on warehouse order costs being “linearly sandwiched”, meaning they can be approximated within a constant error by affine functions; our results here use different assumptions and apply to any concave functions. This research stream in general is most relevant to our work here, since we also decompose the OWMR problem to provide a lower bound; nevertheless, the generality of our order functions necessitates new recombination techniques.

## 1.2 Our Contribution

Relatively little is known for the OWMR problem when the order cost function is concave, or when each batch’s order cost is concave. Our techniques provide the first known constant approximation algorithm when the batch order cost function is PWL concave with a fixed number of slopes on  $[0, Q]$ , assuming only that the subproblems are solvable in polynomial time, which occurs in realistic special cases, e.g. retailer holding costs are higher than the warehouse’s, or homogeneous; see Observation 3.1 below for the precise condition. For general concave batch costs and subadditive costs, our algorithms yield  $\mathcal{O}(\log T)$ -approximations in polynomial time if the subproblems are efficiently solvable. By taking a large enough batch size, our results directly apply also to PWL concave and general concave order costs without batches. These results indicate that some models in Chan et al. (2002) and Shen et al. (2009), among others, are polynomial-time approximable to within either constant or logarithmic factors of optimality. In addition, if the subproblems have an approximation algorithm, our results naturally extend by scaling the approximation ratio up by this algorithm’s guarantee.

Table 1 summarizes our main contributions with a comparison to the OWMR literature on similar models. For PWL concave batch order costs,  $K$  represents the number of positive slopes in the warehouse’s order cost function.

The remainder of the paper is organized as follows. We formally define the OWMR problem in §2. We present the subproblems and discuss polynomially solvable cases in §3. Then we outline the decomposition-based OWMR solution approach in §4. Next, we develop the approximation algorithm for PWL concave batch order costs in §5. Afterwards, we establish the approximation algorithms for concave batch order costs in §6, and extend some of our results to subadditive costs in §7. We summarize results of a computational study on the PWL concave batch order cost algorithm in §8, then discuss potential future research and conclude in §9. The appendix contains additional details and technical proofs not included in the body of the paper.

Papers	Order cost structure	Results
Levi et al. (2008)	Fixed costs	1.8-approximation
Gayon et al. (2016b)	Fixed batch costs, linearly sandwiched warehouse costs	2-approximation
Chan et al. (2002)	Retailer modified all-unit discounts, warehouse incremental discounts	1.22-optimality of ZIO policies
Shen et al. (2009)	Special PWL costs, JRP	$(2(1 + \epsilon) \log(NT))$ -approximation
Our results	PWL concave batch costs	$(K + 3)$ -approximation
	Concave batch costs	$(\lceil \log T \rceil + 2)$ -approximation
	Subadditive costs	$(2\lceil \log T \rceil)$ -approximation

Table 1: Comparison of our study to the OWMR literature, focusing on order costs.

## 2 OWMR Formulation

We are given  $N$  retailers,  $T$  time periods, and the demands for each retailer due in each period,  $d_i^t, i = 1, \dots, N, t = 1, \dots, T$ . The objective is to satisfy all demands  $d_i^t$  by their due times at minimum total cost, which consists of order and holding cost. For facility  $i = 0, \dots, N$ , where 0 represents the warehouse, let  $q_i^t$  and  $I_i^t$  be the order and ending inventory quantities in each period, and let  $C_i(\cdot), H_i(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the order and holding cost functions, respectively. The cost for each facility in each period is then  $C_i(q_i^t) + H_i(I_i^t)$ . Throughout most of the paper we assume linear holding costs at unit rates  $h_i$ , yielding  $H_i(I_i^t) = h_i I_i^t, i = 0, \dots, N, t = 1, \dots, T$ ; this ensures FIFO holds at optimality.

We study order cost functions with the following particular structure. The basic form is a PWL concave batch cost (Figure 1(a)), where the cost within a batch depends on the volume, and contains  $K$  segments  $[M_{k-1}, M_k]$  with decreasing positive slopes before reaching a flat segment  $[M_K, Q]$ . The cost function can be discontinuous at volumes  $0, Q, 2Q, \dots$  if there is a fixed cost per batch. We assume without loss of generality that the function contains a final flat segment, i.e.,  $Q > M_K$ . If this assumption does not hold, every unit of demand passing through the corresponding facility must pay a minimum per unit cost equal to the final (positive) slope, and this cost can be considered sunk; we can therefore decrease every slope by this amount to get an equivalent problem with a flat segment in the batch order cost. Let  $c_i(\cdot)$  be facility  $i$ 's cost function on  $[0, Q]$ ; then

$$C_i(q) = c_i(q - Q\lfloor q/Q \rfloor) + \lfloor q/Q \rfloor c_i(Q), \quad \forall i = 0, \dots, N, \quad (1)$$

where

$$c_i(q) = \begin{cases} 0, & q = 0 \\ \rho_i + \sum_{j=1}^{k-1} \eta_{i,j} (M_{i,j} - M_{i,j-1}) + \eta_{i,k} (q - M_{i,k-1}), & M_{i,k-1} < q \leq M_{i,k}, k = 1, \dots, K \\ c_{i,F}, & M_{i,K} < q \leq Q, \end{cases} \quad (2)$$

$\eta_{i,0} = 0, M_{i,0} = 0, \rho_i \geq 0$  is a fixed charge to set up a batch,  $\eta_{i,k}$  is the slope of the  $k$ -th segment,  $\eta_{i,1} > \dots > \eta_{i,K} > 0, c_{i,F}$  is the cost of ordering a full batch, and  $M_{i,K}$  is the breakpoint between segments with positive slopes and the flat segment,  $\forall i = 0, \dots, N$ . When  $K = 0$ , this reduces to

a fixed batch cost model, previously studied in Gayon et al. (2016b), Levi et al. (2008). More generally,  $c_i(\cdot)$  can be an arbitrary concave function on  $[0, Q]$  (Figure 1(b)).

Finally, a subadditive order cost is given by any function satisfying  $C_i(q + q') \leq C_i(q) + C_i(q')$ ,  $\forall q, q' \in \mathbb{R}_+$ . We also assume the cost function to be non-decreasing without loss of generality,  $C_i(q) \geq C_i(q') \geq 0$ ,  $\forall q \geq q' \geq 0$ . Hence it subsumes all the aforementioned cost structures, among others; see Appendix A for other special cases that have appeared in the literature.

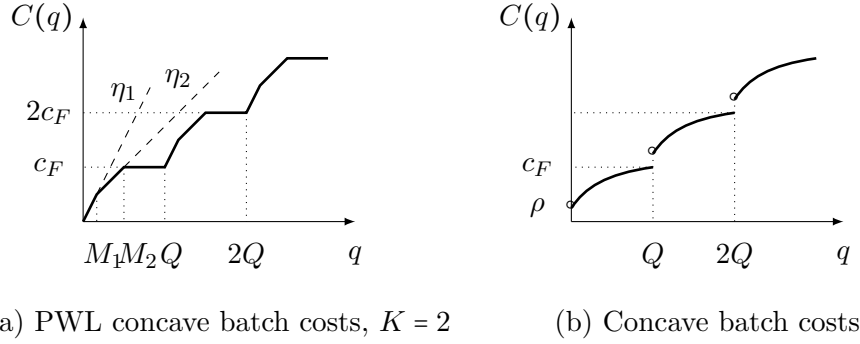


Figure 1: Order cost functions studied in this paper

We formulate the OWMR problem as

$$(OWMR) \quad \min \sum_{t=1}^T \sum_{i=0}^N C_i(q_i^t) + \sum_{t=1}^T \sum_{i=0}^N H_i(I_i^t) \quad (3a)$$

$$\text{s.t. } q_0^t + I_0^{t-1} - I_0^t - \sum_{i=1}^N q_i^t = 0, \quad \forall t = 1 \dots T \quad (3b)$$

$$q_i^t + I_i^{t-1} - I_i^t = d_i^t, \quad \forall t = 1 \dots T, i = 1, \dots, N \quad (3c)$$

$$I_i^0 = 0, \quad \forall i = 0, \dots, N \quad (3d)$$

$$I_i^T = 0, \quad \forall i = 0, \dots, N \quad (3e)$$

$$q \geq 0, I \geq 0, \quad (3f)$$

where the objective (3a) is to minimize the total order and holding cost over the planning horizon. Constraints (3b)-(3c), (3d)-(3e), (3f) specify inventory flow balance, inventory boundary conditions and the domain, respectively.

Model (3) differs from the MIP formulations used in Levi et al. (2008) and Stauffer et al. (2011). Most importantly, we explicitly model the order and inventory quantities in each period, rather than using assignment variables. Under standard order costs, ZIO implies that exactly one order is placed at a facility for each demand  $d_i^t$ , and thus a solution can be specified by giving for each demand the respective periods in which the warehouse and retailer order it. For our problem, an optimal solution may need to split the demand into multiple orders, which prevents us from making the same structural assumption without loss of optimality.

### 3 Single-Echelon Subproblems

In this section, we decompose the OWMR model into single-facility subproblems, and discuss these problems' respective solutions as well as underlying structural properties.

The *long-haul problem* (LHP) refers to the echelon from the supplier to the warehouse, where we assume demand  $d_i^t$  is due at the warehouse in period  $t$ . This subproblem is derived by dropping variables  $I_i^t, q_i^t$  and constraints (3c) from the OWMR model (3), disaggregating  $I_0^t, q_0^t$  by retailers, and replacing  $H_0$  with  $H_{0,i}$ , yielding

$$\text{(LHP)} \quad \min \sum_{t=1}^T C_0 \left( \sum_{i=1}^N q_{0,i}^t \right) + \sum_{t=1}^T \sum_{i=1}^N H_{0,i}(I_{0,i}^t) \quad (4a)$$

$$\text{s.t. } q_{0,i}^t + I_{0,i}^{t-1} - I_{0,i}^t = d_i^t, \quad \forall i = 1, \dots, N, t = 1 \dots T \quad (4b)$$

$$I_{0,i}^0 = 0, \quad \forall i = 1, \dots, N \quad (4c)$$

$$I_{0,i}^T = 0, \quad \forall i = 1, \dots, N \quad (4d)$$

$$q \geq 0, I \geq 0, \quad (4e)$$

where  $H_{0,i}(I_{0,i}^t) = h_{0,i}I_{0,i}^t$ . For reasons explained below, we revise the warehouse holding cost rates so they differentiate inventory by retailer, and set them to  $h_{0,i} := 0.5 \min\{h_0, h_i\}$ . To avoid confusion with the other models, we use the notation  $(0, i)$  to denote variables or parameters associated with retailer  $i$ 's inventory in this model, though the index 0 is not strictly necessary.

LHP (4) is a multi-item LSP where each retailer represents a separate item. Nevertheless, if two retailers  $i, j$  have the same revised holding cost rate,  $h_{0,i} = h_{0,j}$ , they are indistinguishable in the problem and can be combined into one item, implying the following observation.

**Observation 3.1.** *LHP reduces to a single-item LSP if  $h_{0,i} = h_{0,j}, \forall i, j \in \{1, \dots, N\}$ .*

The condition in Observation 3.1 is not uncommon, and we assume it to make our decomposition run in polynomial time. In practice, holding costs typically increase to reflect added value as a product moves downstream along a supply chain, which would give  $h_{0,i} = 0.5h_0, \forall i = 1, \dots, N$ . Alternately, when purchasing or transportation costs heavily depend on retailers' negotiation power or their distance from the warehouse but holding costs do not vary much, i.e.,  $h_i \approx h_j, \forall i, j \in \{1, \dots, N\}$ , it may be reasonable to consider echelon-wide homogeneous holding costs, and the condition would then also be satisfied.

The *short-haul problem* (SHP) refers to the echelon from the warehouse to the retailers, where we assume all product is available at the warehouse from the start of the planning horizon. This subproblem is obtained by dropping variables  $q_0^t, I_0^t$  and constraints (3b) from the OWMR model, yielding

$$\text{(SHP)} \quad \min \sum_{t=1}^T \sum_{i=1}^N C_i(q_i^t) + \sum_{t=1}^T \sum_{i=1}^N H_{i,i}(I_i^t) \quad (5a)$$

$$\text{s.t. } q_i^t + I_i^{t-1} - I_i^t = d_i^t, \quad \forall i = 1, \dots, N, t = 1 \dots T \quad (5b)$$

$$I_i^0 = 0, \quad \forall i = 1, \dots, N \quad (5c)$$

$$I_i^T = 0, \quad \forall i = 1, \dots, N \quad (5d)$$

$$q \geq 0, I \geq 0, \quad (5e)$$

where  $H_{i,i}(I_i^t) = h_{i,i}I_i^t$ . Similarly to LHP, we revise the retailer holding cost rates to be  $h_{i,i} := 0.5h_i$ , using the duplicated notation  $(i, i)$  to differentiate SHP costs from the originals. The following observation is immediate.

**Observation 3.2.** *SHP can be decomposed without loss of optimality into  $N$  single-item LSP's where each retailer is handled as an individual item.*

For concave batch order costs, the single-item LSP can be solved in  $\mathcal{O}(T^4)$  time using the DP algorithm in Akbalik and Rapine (2012) (see also Lippman (1969)). This implies that LHP under Observation 3.1 and SHP can both be solved in polynomial time. For the special case of concave order costs, this complexity is reduced to  $\mathcal{O}(T^2)$ ; see e.g. Aggarwal and Park (1993). In addition, our approximation results rely on structural properties for LSP with concave batch order costs similar to those shown in Anily and Tzur (2005). We next state a regeneration property.

**Proposition 3.3** (LHP regeneration property.). *Under concave batch order costs, there exists an optimal LHP solution such that:*

- i) in each period, there is at most one partially filled batch;*
- ii) for any retailer  $i$ , between two consecutive order periods  $1 \leq \tau_1 < \tau_2 \leq T$  where the corresponding warehouse orders contain partially filled batches, i.e.,*

$$\begin{aligned} q_{0,i}^{\tau_1} > 0, \quad q_{0,i}^{\tau_2} > 0, \\ q_0^{\tau_1} \bmod Q > 0, \quad q_0^{\tau_2} \bmod Q > 0, \end{aligned}$$

*there is a regeneration point, i.e.,  $I_{0,i}^\iota = 0$  for some  $\tau_1 \leq \iota < \tau_2$ ;*

- iii) for any retailer  $i$ , in each regeneration interval, i.e., between two consecutive regeneration points  $1 \leq \iota < \iota' \leq T$ , there is at most one period  $\tau \in [\iota, \iota')$  where the order quantity of retailer  $i$  is positive but the total warehouse order contains a partially filled batch.*

Note that for Proposition 3.3, the order of retailer/item  $i$  refers to the part of order placed by warehouse with holding cost  $h_{0,i}$ . For part *ii)*, the regeneration points of different retailers may be different.

*Proof.* Under concave batch order costs, part *i)* is trivial by definition of the order cost function.

We consider the network flow representation of LHP. Because the total demand,  $\sum_{t=1}^T \sum_{i=1}^N d_i^t$ , is finite, we can bound the number of batches in period  $t$  by  $\gamma_t = \lceil (\sum_{i=1}^N d_i^t) / Q \rceil$ . Introduce variables  $q_{0,i,j}^t$  to denote the order quantity of retailer  $i$  carried by  $j$ -th batch in period  $t$ . Therefore, LHP can be reformulated by letting:

$$\begin{aligned} q_{0,i}^t &= \sum_{j=1}^{\gamma_t} q_{0,i,j}^t, \quad \forall i = 1, \dots, N, \quad t = 1, \dots, T, \\ C_0 \left( \sum_{i=1}^N q_{0,i}^t \right) &= \sum_{j=1}^{\gamma_t} c_0 \left( \sum_{i=1}^N q_{0,i,j}^t \right), \quad \forall t = 1, \dots, T, \end{aligned} \tag{6}$$

where  $q_{0,i,j}^t \leq Q$ .

As shown in Figure 2, each node represents a period  $t$  or a retailer  $i$  and each period  $t \in \{1, \dots, T\}$  is duplicated with an identical copy,  $t'$ . There are five types of arcs in Figure 2:

1. a solid arc  $(0, t)$ , representing the total order flow for all retailers in period  $t$ ;



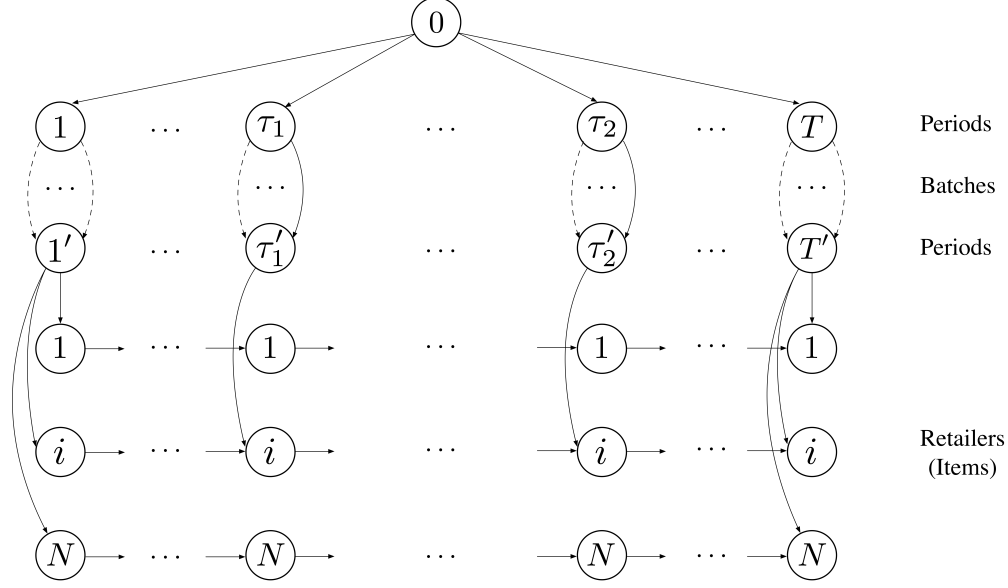


Figure 2: Network flow representation of LHP

2. a dashed arc  $(t, t')$ , representing one fully filled batch with batch capacity  $Q$ ;
3. a solid arc  $(t, t')$ , representing a partially filled batch with order quantity less than  $Q$ ;
4. a solid arc  $(t', i)$ , representing a non-empty order of retailer  $i$  in period  $t$ ;
5. a solid arc connecting the same retailer in different periods represents the inventory flow of that retailer.

The network flow representation implies that the extreme flows are acyclic: if all the solid arcs  $(0, \tau_1)$ ,  $(\tau_1, \tau_1')$ ,  $(\tau_1', i)$ ,  $(0, \tau_2)$ ,  $(\tau_2, \tau_2')$ ,  $(\tau_2', i)$  exist for some  $\tau_1, \tau_2, i$ , there is a period  $\iota$ ,  $\tau_1 \leq \iota < \tau_2$ , ending with zero inventory. The existence of such an optimal solution follows by concavity of the reformulated objective function (6).  $\square$

Another property that we apply in later sections follows intuitively.

**Proposition 3.4.** *An optimal SHP or LHP solution never places an order which does not contain any demand due in the current period.*

Gayon et al. (2016b) name this the *positive consumption ordering* property.

*Proof.* Assume that the condition is violated somewhere in an optimal subproblem solution, e.g. facility  $i$  places an order in period  $t$  that only covers demand in periods after  $t$ . We construct another feasible solution by postponing the entire order to the first period when a portion of the demand fulfilled by this order is due. The new solution incurs lower holding cost and no higher order cost for the subproblem, which is a contradiction.  $\square$

The combined subproblem solutions could of course be infeasible for OWMR if some retailer order occurs before the corresponding warehouse order. However, a feasible OWMR solution cannot cost less than the total optimal subproblem objective values.

**Proposition 3.5** (Gayon et al. (2016b), Stauffer et al. (2011)). *Let  $Z^*(P)$  be the optimal objective function value of problem  $P$ ; then*

$$Z^*(LHP) + Z^*(SHP) \leq Z^*(OWMR).$$

## 4 OWMR Approximation

Since the subproblems are tractable and Proposition 3.5 provides a lower bound for the OWMR problem, we are motivated to construct OWMR solutions from the subproblem solutions. If we can show that the total cost of the resulting solution is less than or equal to the sum of the subproblem objective values multiplied by some factor, we obtain an OWMR approximation with that guarantee. We next sketch a unified algorithmic framework to approximate the OWMR problem under any of our order cost assumptions. Our algorithms consist of two phases. In the *decomposition phase*, we solve LHP and SHP and obtain optimal solutions for both subproblems. In the *recombination phase*, we convert the subproblem solutions into a feasible OWMR solution.

Inventory quantities are uniquely determined once order quantities are known, so we interchangeably refer to the values of the  $q$  variables in models (3) through (5) as the models' solutions. Given optimal LHP and SHP solutions (quantities  $q_{0,i}^t$  and  $q_i^t$  for (4) and (5)) satisfying Propositions 3.3 and 3.4, we construct an OWMR solution  $\hat{q}_i^t$  for (3). Any tuple of LHP and SHP subproblem solutions induces an OWMR solution, where the short-haul quantities are  $\hat{q}_i^t = q_i^t$  and the long-haul quantities are  $\hat{q}_0^t = \sum_{i=1}^N q_{0,i}^t$ . However, the subproblem solutions are *globally feasible* for the OWMR problem if and only if the warehouse orders arrive in time for the retailer orders,

$$\sum_{\tau=1}^t q_{0,i}^\tau \geq \sum_{s=1}^t q_i^s, \quad \forall i = 1, \dots, N, \quad t = 1, \dots, T.$$

Given LHP and SHP subproblem solutions, we can apply FIFO to determine the amount of demand  $d_i^t$  ordered by the warehouse and retailer in periods  $\tau$  and  $s$ , respectively. This portion of demand is called a *split demand* and is denoted by  $d_{i,\tau,s}^t$ , where  $\sum_{\tau=1}^t \sum_{s=1}^t d_{i,\tau,s}^t = d_i^t$ . For instance, if  $q_{0,i}^1 = 1, q_{0,i}^2 = 3, q_i^1 = 0, q_i^2 = 4$  and  $d_i^1 = 0, d_i^2 = d_i^3 = 2$  for some retailer  $i$ , the induced split demands are  $d_{i,1,2}^2 = d_{i,2,2}^2 = 1, d_{i,2,2}^3 = 2$ . Using this notation, the global feasibility condition is then

$$d_{i,\tau,s}^t = 0, \quad \forall 1 \leq s < \tau \leq t, \quad i = 1, \dots, N \quad t = 1, \dots, T.$$

Applying FIFO to the subproblem solutions, we can identify the warehouse orders that fulfill  $q_i^t$ , and the retailer orders that consume  $q_{0,i}^t$ . Let

$$\Xi_{i,s} := \{\xi_{i,s}^{\min}, \dots, \xi_{i,s}^2, \xi_{i,s}^1, \tilde{\xi}_{i,s}^1, \tilde{\xi}_{i,s}^2, \dots, \xi_{i,s}^{\max}\} \subseteq \{1, \dots, T\}$$

be the list of warehouse order periods associated with the retailer's order in period  $s$ , where  $\xi_{i,s}^k$  is the  $k$ -th largest element less than or equal to  $s$ , and  $\tilde{\xi}_{i,s}^k$  is the  $k$ -th smallest element greater than  $s$ . Intuitively, the  $\xi$  indices represent warehouse orders that are on time for the retailer's order in  $s$ , while the  $\tilde{\xi}$  indices represent orders that are too late. To construct this list, simply compare  $\sum_{\tau=1}^{s-1} q_i^\tau$  and  $\sum_{\tau=1}^s q_i^\tau$  with  $\sum_{\tau=1}^\tau q_{0,i}^\tau, \forall \tau = 1, \dots, T$ . Letting  $\xi_{i,s}^{\min} := \min\{\tau : \sum_{\tau=1}^\tau q_{0,i}^\tau > \sum_{\tau=1}^{s-1} q_i^\tau, 1 \leq \tau \leq T\}$  and  $\xi_{i,s}^{\max} := \min\{\tau : \sum_{\tau=1}^\tau q_{0,i}^\tau \geq \sum_{\tau=1}^s q_i^\tau, 1 \leq \tau \leq T\}$  be the smallest and the largest elements respectively,

$\Xi_{i,s}$  is the ordered set  $\{\tau : \xi_{i,s}^{\min} \leq \tau \leq \xi_{i,s}^{\max}, q_{0,i}^\tau > 0\}$ . For each retailer  $i$ , the construction of the sets for all  $s = 1, \dots, T$  takes  $\mathcal{O}(T)$  time since the elements in each  $\Xi_{i,s}$  are naturally ordered by FIFO and each consecutive pair of lists  $(\Xi_{i,s}, \Xi_{i,s'})$  with  $s < s'$  has at most one common element, which is known once  $\Xi_{i,s}$  is calculated, i.e.,  $\xi_{i,s'}^{\max} = \xi_{i,s}^{\min}$  if  $\sum_{\ell=1}^{\xi_{i,s}^{\max}} q_{0,i}^\ell > \sum_{\ell=1}^s q_i^\ell$ .

Now consider an arbitrary pair of given warehouse-retailer order periods  $(\tau, s)$ . If  $\tau \in \Xi_{i,s}$ , there must be a split demand  $d_{i,\tau,s}^t$  ( $t \geq \tau, s$ ) which is ordered by the warehouse in period  $\tau$  and the retailer in period  $s$ , and vice versa. We distinguish two cases:

**Case I:**  $\tau \leq s$  The warehouse order arrives in time for the retailer order.

**Case II:**  $\tau > s$  The warehouse order is too late for the retailer order.

Thus, the pair of subproblem orders are globally feasible in Case I, but cause infeasibility in Case II. To obtain a feasible OWMR solution  $\hat{q}_i^t$  of reasonable cost, we recombine the given LHP quantities  $q_{0,i}^t$  and SHP quantities  $q_i^t$  to control costs in Case I, and resolve global infeasibility while controlling costs in Case II. Define the following lists:

- $\mathcal{L}_{i,\tau}$ : list of periods where we reallocate  $q_{0,i}^\tau$ , the warehouse order quantities for retailer  $i$  initially scheduled in period  $\tau$ . Let  $\mathcal{L}_\tau := \bigcup_{i \in \{1, \dots, N\}} \mathcal{L}_{i,\tau}$  be the list of warehouse order periods associated with all retailers, i.e., where we reallocate  $q_0^\tau = \sum_{i=1}^N q_{0,i}^\tau$ , in the final OWMR solution.
- $\mathcal{S}_{i,s}$ : list of periods where we reallocate  $q_i^s$ , the retailer order quantities initially scheduled in period  $s$ .

$\mathcal{L}_\tau$  and  $\mathcal{S}_{i,s}$  consist of the periods where we split the orders initially scheduled for periods  $\tau$  or  $s$ , i.e.,  $\hat{q}_0^t > 0$  if  $t \in \bigcup_{\tau \in \{1, \dots, T\}} \mathcal{L}_\tau$ , and  $\hat{q}_i^t > 0$  if  $t \in \bigcup_{s \in \{1, \dots, T\}} \mathcal{S}_{i,s}$ ,  $\forall i = 1, \dots, N$ . To establish worst-case guarantees, we can argue that the final cost for any split demand does not exceed the corresponding initial subproblem objective values multiplied by some factor; this is a *quantity-based* approach. Alternatively, we may bound the total number of splits for an arbitrary initial order, a *time-based* approach.

Algorithm 1 outlines a unified approximation procedure. The decomposition phase is always the same, whereas the recombination phase is ad hoc. Under PWL concave batch order costs, we develop a quantity-based recombination of the initial order quantities in both Case I and Case II (Lines 4-13). Each tuple  $(i, \tau, s)$  such that  $\tau \in \Xi_{i,s}$  is associated with an element in  $\mathcal{L}_{i,\tau}$  and an element in  $\mathcal{S}_{i,s}$ . Furthermore, the target periods ( $\xi$  in Line 9) depend on the parameter  $K$  in the warehouse order cost function. We detail this in §5.

Under concave batch order costs, we use the same recombination for PWL concave batch costs in Case I (Lines 17-21), but develop a time-based recombination in Case II (Lines 22-25). The target periods ( $\iota$  in Line 23) are identical for all retailers given a pair of order periods such that  $\tau \in \Xi_{i,s}$  and  $\tau > s$ . We define  $\mathcal{L}_t$  and  $\mathcal{S}_t$  (Line 15) as the respective potential warehouse and retailer split period lists in Case II, which are directly related to  $\mathcal{L}_t$  or  $\bigcup_{i \in \{1, \dots, N\}} \mathcal{S}_{i,t}$ , detailed in §6. Under subadditive order costs, we develop a time-based recombination in both Case I and Case II, an extension of the approach for concave batch order costs in Case II. Since the decomposition phase needs to be different under subadditive holding costs, we present the algorithm for subadditive costs separately in §7.

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**Algorithm 1** OWMR approximation

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**Phase I: Decomposition**

- 1: Set  $h_{0,i} \leftarrow 0.5 \min\{h_0, h_i\}$ ,  $h_{i,i} \leftarrow 0.5h_i$ ,  $\forall i = 1, \dots, N$
- 2: Solve LHP and SHP, obtain initial order quantities  $q_{0,i}^t, q_i^t$ ,  $\forall i = 1, \dots, N$ ,  $t = 1, \dots, T$

**Phase II: Recombination**

- 3: Construct lists  $\Xi_{i,t}$ ,  $\forall i = 1, \dots, N$ ,  $t = 1, \dots, T$
  - 4: **if** PWL concave batch order costs **then**
  - 5:   **for**  $i = 1, \dots, N$ ,  $s = 1, \dots, T$ ,  $\tau \in \Xi_{i,s}$  **do**
  - 6:     **if**  $\tau \leq s$  and  $h_0 \leq h_i$  **then**
  - 7:       Use the initial order schedules for split demands  $\sum_{t=s}^T d_{i,\tau,s}^t$
  - 8:     **else**
  - 9:       Identify a common period  $\xi \in \Xi_{i,s}$  to move the order of  $d_{i,\tau,s}^t$ ,  $\forall t \geq \max\{\tau, s\}$
  - 10:       Reallocate both warehouse and retailer orders to period  $\xi$  for split demands  $\sum_{t \geq \tau, s} d_{i,\tau,s}^t$
  - 11:     **end if**
  - 12:   **end for**
  - 13: **end if**
  - 14: **if** concave batch order costs **then**
  - 15:   Construct lists  $\mathcal{L}_t$  and  $\mathcal{S}_t$ ,  $\forall t = 1, \dots, T$
  - 16:   **for**  $i = 1, \dots, N$ ,  $s = 1, \dots, T$ ,  $\tau \in \Xi_{i,s}$  **do**
  - 17:     **if**  $\tau \leq s$  and  $h_0 \leq h_i$  **then**
  - 18:       Use the initial order schedules for split demands  $\sum_{t=s}^T d_{i,\tau,s}^t$
  - 19:     **else if**  $\tau \leq s$  and  $h_0 > h_i$  **then**
  - 20:       Identify a common period  $\xi \in \Xi_{i,s}$  to move the order of  $d_{i,\tau,s}^t$ ,  $\forall t \geq \max\{\tau, s\}$
  - 21:       Reallocate both warehouse and retailer orders to period  $\xi$  for split demands  $\sum_{t=s}^T d_{i,\tau,s}^t$
  - 22:     **else**
  - 23:       Find an arbitrary common period  $\iota \in \mathcal{L}_\tau \cap \mathcal{S}_s$
  - 24:       Reallocate both warehouse and retailer orders to period  $\iota$  for split demands  $\sum_{t=\tau}^T d_{i,\tau,s}^t$
  - 25:     **end if**
  - 26:   **end for**
  - 27: **end if**
  - 28: Obtain recombined order quantities  $\hat{q}_i^t$ ,  $\forall i = 0, \dots, N$ ,  $t = 1, \dots, T$
- 

## 5 Approximation for PWL Concave Batch Costs

We first introduce some notation to describe the warehouse order cost incurred by the initial LHP solution in each period  $\tau = 1, \dots, T$ . Given order quantities  $q_0^\tau = \sum_{i=1}^N q_{0,i}^\tau > 0$ , define:

$$\rho(\tau) = \begin{cases} \rho_0, & q_0^\tau < M_{0,1} \\ \rho_0 + \sum_{j=1}^k \eta_{0,j}(M_{0,j} - M_{0,j-1}) - \eta_{0,k+1}M_{0,k}, & M_{0,k} \leq q_0^\tau < M_{0,k+1}, \quad k = 1, \dots, (K-1), \\ c_{0,F}, & q_0^\tau \geq M_{0,K} \end{cases} \quad (7a)$$

and

$$\eta(\tau) = \begin{cases} \eta_{0,k}, & M_{0,k-1} \leq q_0^\tau < M_{0,k}, \quad k = 1, \dots, K, \\ 0, & q_0^\tau \geq M_{0,K} \end{cases}, \quad (7b)$$

where  $\rho_0$ ,  $\eta_{0,k}$  and  $M_{0,k}$  are the warehouse PWL concave batch cost parameters defined in (2). The number  $\rho(\tau)$  gives the intercept of the cost function at the cost axis if we elongate the segment which contains the value of  $q_0^\tau$  in Figure 1(a), and  $\eta(\tau)$  is the corresponding slope. In particular, if  $q_0^\tau \geq M_{0,K}$ ,  $\rho(\tau)$  and  $\eta(\tau)$  indicate that the warehouse order in period  $\tau$  pays for at least one full batch. (If  $q_0^\tau > Q$ , the order cost is higher than  $c_{0,F}$ , but includes a full batch.) By the concavity of the batch cost function  $c_0(q)$ ,  $\rho(\tau) + \eta(\tau)q$  is an upper bound of the warehouse order cost for any volume  $q \in [0, Q]$ . Figure 3 is an illustration of  $\rho(\tau)$ .

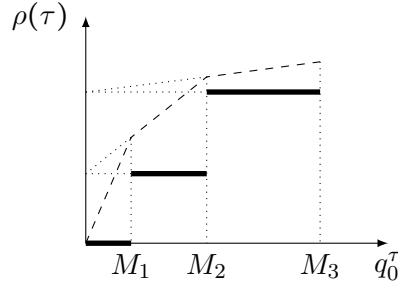


Figure 3: Illustration of  $\rho(\tau)$ .

Let  $\Theta_{i,s} := \{\underline{\theta}_{i,s}^2, \underline{\theta}_{i,s}^1, \tilde{\theta}_{i,s}^0, \tilde{\theta}_{i,s}^1, \dots\}$  be the subset of  $\Xi_{i,s}$  with

$$\begin{aligned} \underline{\theta}_{i,s}^1 &= \underline{\xi}_{i,s}^1, \\ \underline{\theta}_{i,s}^2 &= \max\{\xi \in \Xi_{i,s} : \xi < \underline{\xi}_{i,s}^1, \eta(\xi) < \eta(\underline{\theta}_{i,s}^1)\}, \\ \tilde{\theta}_{i,s}^0 &= \tilde{\xi}_{i,s}^1, \\ \tilde{\theta}_{i,s}^k &= \min\{\xi \in \Xi_{i,s} : \xi > \tilde{\theta}_{i,s}^{k-1}, \eta(\xi) < \eta(\tilde{\theta}_{i,s}^{k-1})\}, \quad \forall k = 1, \dots, K. \end{aligned} \quad (8)$$

Intuitively, period  $\underline{\theta}_{i,s}^2$  is the latest period in the list  $\Xi_{i,s}$  that is earlier than period  $\underline{\theta}_{i,s}^1$  and contains a warehouse order which costs less per unit or at least  $c_{0,F}$  in total. (Such a period may not exist, in which case  $\Theta_{i,s}$  begins with  $\underline{\theta}_{i,s}^1$ .) Similarly, period  $\tilde{\theta}_{i,s}^k$  is the first period in the list  $\Xi_{i,s}$  which is later than period  $\tilde{\theta}_{i,s}^{k-1}$  and contains a warehouse order which costs less per unit or at least  $c_{0,F}$  in total. Since the PWL concave batch cost function contains  $K$  positive slopes on  $[0, Q]$ , we observe the following.

**Observation 5.1.** *The cardinality of the list  $\Theta_{i,s}$  is at most  $K + 3$ ,  $\forall i = 1, \dots, N$ ,  $s = 1, \dots, T$ .*

Since any warehouse order period  $\tau \in \Xi_{i,s}$  carries some common split demand with the retailer order in period  $s$ , periods  $\underline{\theta}_{i,s}^1$  and  $\underline{\theta}_{i,s}^2$  lie in the same LHP regeneration interval of retailer  $i$  if both exist. By Proposition 3.3, the initial LHP solution contains at most one partially filled batch in this regeneration interval; hence we derive the following observation.

**Observation 5.2.** *In the LHP solution, at most one of the warehouse orders in periods  $\underline{\theta}_{i,s}^1$  and  $\underline{\theta}_{i,s}^2$  contains a partial batch,  $\forall i = 1, \dots, N$ ,  $s = 1, \dots, T$ . Moreover, the possible partial batch will*

only happen in period  $\underline{\theta}_{i,s}^1$ . That is to say, if both  $\underline{\theta}_{i,s}^1$  and  $\underline{\theta}_{i,s}^2$  exist,  $\underline{\theta}_{i,s}^1$  will contain a partial batch and  $\underline{\theta}_{i,s}^2$  will only contain full batches. If  $\underline{\theta}_{i,s}^2$  exists, then  $\underline{\theta}_{i,s}^2 = \underline{\xi}_{i,s}^2$ .

Now we propose the recombination for PWL concave batch order costs. Given the optimal subproblem solutions, we consider a pair of warehouse-retailer order periods  $(\tau, s)$  with split demand  $d_{i,\tau,s}^t > 0$ .

**Case I:**  $\tau \leq s$  The warehouse order arrives in time to fulfill the retailer order. If  $h_i \geq h_0$ , maintain the initial schedule. For  $h_i < h_0$ , consider two possibilities: If either  $\eta(\underline{\theta}_{i,s}^1) = 0$  or  $\tau = \underline{\theta}_{i,s}^1$ , make both warehouse and retailer orders of  $d_{i,\tau,s}^t > 0$  take place in period  $\underline{\theta}_{i,s}^1$ ; otherwise, for any  $\tau < \underline{\theta}_{i,s}^1$ , make both orders take place in period  $\underline{\theta}_{i,s}^2$ . In the resulting OWMR solution the warehouse acts as a cross-dock for all demand covered by Case I when  $h_i < h_0$ .

**Case II:**  $\tau > s$  The warehouse order arrives too late for the retailer order. Move the warehouse and retailer orders of  $d_{i,\tau,s}^t$  to period  $\iota = \max\{\theta \in \Theta_{i,s} : \theta \leq \tau\}$ . By definition of  $\tilde{\theta}_{i,s}^k$ , the target period  $\iota$  must lie in the interval  $(s, \tau]$ , and hence the adjusted order periods are globally feasible. In the resulting OWMR solution, the warehouse acts as a cross-dock for all demand covered by Case II.

Algorithm 2 gives the pseudocode for this recombination. We next give a small example to illustrate it.

**Example 1.** Consider an example where  $N = 2$ ,  $T = 4$ . Demand  $d_i^t$ , warehouse order quantity  $q_{0,i}^t$  (a feasible solution to the LHP) and retailer order quantity  $q_i^t$  (a feasible solution to the SHP) are listed in Table 2. We have a batch size of  $Q = 1$  and the warehouse order cost function within a batch is the concave PWL function shown in Figure 4. We assume  $h_1 > h_0 > h_2$ .

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$d_1^t$	0.5	0.1	0.4	0.3
$d_2^t$	0.4	0.7	0.2	0.7
$q_{0,1}^t$	0.5	0.1	0.4	0.3
$q_{0,2}^t$	0.5	0.6	0.2	0.7
$q_0^t$	1.0	0.7	0.6	1.0
$q_1^t$	1.3	0	0	0
$q_2^t$	0.4	0.9	0.7	0

Table 2: An example of Algorithm 2

A simple calculation shows  $d_{1,1,1}^1 = 0.5$ ,  $d_{1,2,1}^2 = 0.1$ ,  $d_{1,3,1}^3 = 0.4$ ,  $d_{1,4,1}^4 = 0.3$ ,  $d_{2,1,1}^1 = 0.4$ ,  $d_{2,1,2}^2 = 0.1$ ,  $d_{2,2,2}^2 = 0.6$ ,  $d_{2,3,2}^3 = 0.2$ ,  $d_{2,4,3}^4 = 0.7$ , and  $d_{i,\tau,s}^t = 0$  for any other  $(i, \tau, s, t)$ . Then,  $\Xi_{1,1} = \{1, 2, 3, 4\}$ ,  $\Xi_{2,1} = \{1\}$ ,  $\Xi_{2,2} = \{1, 2, 3\}$ ,  $\Xi_{2,3} = \{4\}$ ,  $\Theta_{1,1} = \{1, 2, 4\}$ ,  $\Theta_{2,1} = \{1\}$ ,  $\Theta_{2,2} = \{1, 2, 3\}$ ,  $\Theta_{2,3} = \{4\}$  and  $\Xi_{\tau,s} = \emptyset$ ,  $\Theta_{\tau,s} = \emptyset$  for any other  $(\tau, s)$ . According to Algorithm 2, we have the recombined solution to the original OWMR as shown in Table 3.

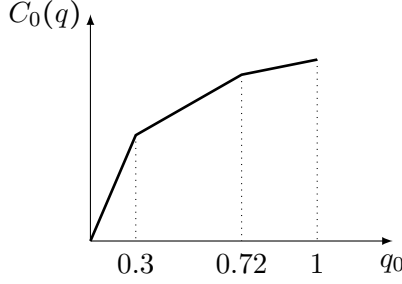


Figure 4: Warehouse order cost

	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$\hat{q}_{0,1}^t$	0.5	0.5	0	0.3
$\hat{q}_{0,2}^t$	0.5	0.6	0.2	0.7
$\hat{q}_0^t$	1	1.1	0.2	1.0
$\hat{q}_1^t$	0.5	0.5	0	0.3
$\hat{q}_2^t$	0.5	0.6	0.2	0.7

Table 3: Recombined Solution to Example 1

We next state two lemmas followed by the section’s main result.

**Lemma 5.3.** *Let  $V^a, V^b$  be the total retailer order costs incurred by the final OWMR solution and the initial SHP subproblem solution, respectively. Then  $V^a \leq (K + 3)V^b$ .*

*Proof.* Each retailer order is split into at most  $|\Theta_{i,s}|$  suborders; Observation 5.1 establishes that  $|\Theta_{i,s}| \leq K + 3$ . The proof follows from the monotonicity and subadditivity of the retailers’ order cost functions.  $\square$

**Lemma 5.4.** *Let  $H^a$  be the total inventory cost incurred by the final OWMR solution (measured based on the actual holding costs  $h_0$  and  $h_i$ ), and  $H^b$  be the total inventory cost incurred by the LHP and SHP subproblem solutions, measured with  $h_{0,i}$  and  $h_{i,i}$ . Then  $H^a \leq 2H^b$ .*

*Proof.* The lemma is proved in a similar fashion to Lemma 3 in Gayon et al. (2016b); for completeness, we include it in the Appendix.  $\square$

**Lemma 5.5.** *Let  $W^a, W^b$  be the total warehouse order costs incurred by the final OWMR solution and the initial LHP subproblem solutions, respectively. Then  $W^a \leq 3W^b$ .*

*Proof.* We consider the fully filled batches and partially filled batches for warehouse orders in the final OWMR solution separately.

Let  $c_{0,F}$  denote the warehouse order cost for a full batch. Let  $N_{0,a}$  and  $N_{0,b}$  denote the number of full batches in the final OWMR solution and the initial LHP subproblem solution, respectively. Let  $N_{0,0}$  denote the number of full batches when the entire horizon’s demands are ordered in the first period. Therefore,  $N_{0,a} \leq N_{0,0}$ ,  $N_{0,b} \leq N_{0,0}$ . Since each of the fully filled batches achieves the greatest possible batch capacity,  $N_{0,0} \cdot c_{0,F} \leq W^b$ . When we only consider the fully filled batches in

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**Algorithm 2** Recombination for PWL concave batch order costs
 

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1: Construct lists  $\Theta_{i,t}, \forall i = 1, \dots, N, t = 1, \dots, T$ 
2: for  $i = 1, \dots, N, s = 1, \dots, T, \tau \in \Xi_{i,s}$  do
3:   if  $\tau \leq s$  and  $h_0 > h_i$  then
4:     if  $\eta(\theta_{i,s}^1) = 0$  then
5:        $q_{0,i}^{\theta_{i,s}^1} \leftarrow q_{0,i}^{\theta_{i,s}^1} + \sum_{t=1}^T d_{i,\tau,s}^t, q_{0,i}^{\tau} \leftarrow q_{0,i}^{\tau} - \sum_{t=1}^T d_{i,\tau,s}^t$ 
6:        $q_i^{\theta_{i,s}^1} \leftarrow q_i^{\theta_{i,s}^1} + \sum_{t=1}^T d_{i,\tau,s}^t, q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
7:     else if  $\tau = \theta_{i,s}^1$  then
8:        $q_i^{\tau} \leftarrow q_i^{\tau} + \sum_{t=1}^T d_{i,\tau,s}^t, q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
9:     else
10:       $q_{0,i}^{\theta_{i,s}^2} \leftarrow q_{0,i}^{\theta_{i,s}^2} + \sum_{t=1}^T d_{i,\tau,s}^t, q_{0,i}^{\tau} \leftarrow q_{0,i}^{\tau} - \sum_{t=1}^T d_{i,\tau,s}^t$ 
11:       $q_i^{\theta_{i,s}^2} \leftarrow q_i^{\theta_{i,s}^2} + \sum_{t=1}^T d_{i,\tau,s}^t, q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
12:    end if
13:  else if  $\tau > s$  then
14:    Find the largest  $k$  such that  $\tau \geq \tilde{\theta}_{i,s}^k, k \leq K$ 
15:     $q_{0,i}^{\tilde{\theta}_{i,s}^k} \leftarrow q_{0,i}^{\tilde{\theta}_{i,s}^k} + \sum_{t=1}^T d_{i,\tau,s}^t, q_{0,i}^{\tau} \leftarrow q_{0,i}^{\tau} - \sum_{t=1}^T d_{i,\tau,s}^t$ 
16:     $q_i^{\tilde{\theta}_{i,s}^k} \leftarrow q_i^{\tilde{\theta}_{i,s}^k} + \sum_{t=1}^T d_{i,\tau,s}^t, q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
17:  end if
18: end for
19:  $\hat{q}_0^t \leftarrow \sum_{i=1}^N q_{0,i}^t, \hat{q}_i^t \leftarrow q_i^t, \forall t = 1, \dots, T, i = 1, \dots, N$ 

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---

the final solution, we have:  $N_{0,a} \cdot c_{0,F} \leq N_{0,0} \cdot c_{0,F} \leq W^b$ , which implies that the warehouse order cost of all full batches after recombination is bounded by  $W^b$ .

Let  $\Gamma_1 = \{\iota \in [T] \mid \hat{q}_0^\iota \bmod Q > q_0^\iota\}$  and  $\Gamma_2 = \{\iota \in [T] \mid \hat{q}_0^\iota \bmod Q \leq q_0^\iota\}$ . First consider the case where  $\iota \in \Gamma_1$ , i.e., the quantity of the partial batch for the order after recombination in period  $\iota$  is greater than the initial order quantity in period  $\iota$ ,  $q_0^\iota$ . Notice that according to the way we define the condition,  $q_0^\iota$  has to be less than a full batch capacity  $Q$ . The increments of order quantity in such periods (rather than net increments, i.e. the difference between final and initial order quantity) are caused only by Case II. This is because if we consider the rearrangements from  $(\tau, s)$  to  $(\iota, \iota)$  caused by Case I with regard to any retailer  $i$ , there are only three possible situations:

- i)  $\theta_{i,s}^1 \geq Q$ . In this case,  $\theta_{i,s}^2$  cannot exist according to our definition. There is no period  $\iota \in \{\theta_{i,s}^2, \theta_{i,s}^1\}$  with  $q_0^\iota < Q$  here.
- ii)  $\theta_{i,s}^1 < Q$ ,  $\theta_{i,s}^2$  does not exist, i.e., there is no order period in  $\Theta_{i,s}$  prior to  $\theta_{i,s}^1$ . Since  $\theta_{i,s}^1$  is the only period before  $s$ , there will be no rearrangement in this situation.
- iii)  $\theta_{i,s}^1 < Q$ ,  $\theta_{i,s}^2$  exists. The initial order quantity in  $\theta_{i,s}^2$  will be a positive integer multiple of  $Q$ . Thus,  $\theta_{i,s}^1$  will be the only period whose order quantity is less than one batch. So there is no rearrangement regarding period  $\theta_{i,s}^1$ .

Based on the above conclusion and the definition of  $\Theta_{i,s}$ , we have:  $\eta(\tau) \geq \eta(\iota)$ . This is because if  $\eta(\tau) < \eta(\iota)$ , the warehouse order quantity associated with the pair of order periods  $(\tau, s)$  and



retailer  $i$  should be rearranged to  $(\tau, \tau)$ , which contradicts with the pair  $(\iota, \iota)$ .

For any warehouse order period  $\iota \in \Gamma_1$  we define the total increments caused by the rearrangement from  $(\tau, s)$  to  $(\iota, \iota)$  as  $\tilde{D}_\iota$ . However,  $\hat{q}_0^\iota - q_0^\iota$  may not equal to  $\tilde{D}_\iota$ . By definition,  $\tilde{D}_\iota = \sum_i \sum_{U_{i,\iota}} \sum_t d_{i,\tau,s}^t$ , where  $U_{i,\iota}$  is the set of pairs of order periods  $(\tau, s)$  that will be rearranged to  $(\iota, \iota)$  for retailer  $i$ . Let  $U_\iota^0 = \{\tau \in \{\iota, \dots, T\} \mid (\tau, s) \in \cup_i U_{i,\iota}, \forall s \in \{1, \dots, \iota - 1\}\}$  be the set of warehouse order periods that will be rearranged to  $\iota$ . Define

$$\bar{\eta}_\iota = \frac{\sum_{\tau \in U_\iota^0} \eta(\tau) \cdot \{\sum_i \sum_{s: (\tau,s) \in U_{i,\iota}} \sum_t d_{i,\tau,s}^t\}}{\tilde{D}_\iota}.$$

Here, note that  $\sum_{\tau \in U_\iota^0} \sum_i \sum_{s: (\tau,s) \in M_{i,\iota}} \sum_t d_{i,\tau,s}^t = \sum_i \sum_{U_{i,\iota}} \sum_t d_{i,\tau,s}^t = \tilde{D}_\iota$ . Since  $\eta(\tau) \geq \eta(\iota)$ , we have:  $\bar{\eta}_\iota \geq \eta(\iota)$ . Therefore,

$$\begin{aligned} C_0(\hat{q}_0^\iota \bmod Q) &= \hat{\rho}(\iota) + \hat{\eta}(\iota) \cdot (\hat{q}_0^\iota \bmod Q) \\ &\leq \rho(\iota) + \eta(\iota) \cdot (\hat{q}_0^\iota \bmod Q) \\ &= c_0(q_0^\iota) + \eta(\iota) \cdot (\hat{q}_0^\iota \bmod Q - q_0^\iota) \\ &\leq c_0(q_0^\iota) + \bar{\eta}_\iota \cdot (\hat{q}_0^\iota \bmod Q - q_0^\iota). \end{aligned}$$

Summing over all  $\iota \in \Gamma_1$ , we have:

$$\sum_{\iota \in \Gamma_1} C_0(\hat{q}_0^\iota \bmod Q) \leq \sum_{\iota \in \Gamma_1} c_0(q_0^\iota) + \sum_{\iota \in \Gamma_1} \bar{\eta}_\iota \cdot (\hat{q}_0^\iota \bmod Q - q_0^\iota),$$

where

$$\sum_{\iota \in \Gamma_1} \bar{\eta}_\iota \cdot (\hat{q}_0^\iota \bmod Q - q_0^\iota) \leq \sum_{\iota \in \Gamma_1} \sum_{\tau \in M_\iota^0} \eta(\tau) \cdot \{\sum_i \sum_s \sum_t d_{i,\tau,s}^t\} \leq W^b.$$

For  $\iota \in \Gamma_2$ , we have:

$$\sum_{\iota \in \Gamma_2} C_0(\hat{q}_0^\iota \bmod Q) \leq \sum_{\iota \in \Gamma_2} c_0(q_0^\iota).$$

Therefore,

$$\begin{aligned} W^a &= N_{0,a} \cdot c_{0,F} + \sum_{\iota \in \Gamma_1} C_0(\hat{q}_0^\iota \bmod Q) + \sum_{\iota \in \Gamma_2} C_0(\hat{q}_0^\iota \bmod Q) \\ &\leq N_{0,a} \cdot c_{0,F} + \sum_{\iota \in \Gamma_1} c_0(q_0^\iota) + W^b + \sum_{\iota \in \Gamma_2} c_0(q_0^\iota) \leq 3W^b. \end{aligned} \tag{9}$$

□

**Theorem 5.6.** *Given optimal LHP and SHP solutions, applying the recombination for PWL concave batch order costs generates a  $(K + 3)$ -approximation for the OWMR problem in  $\mathcal{O}(NT)$  time, where  $K \geq 1$  is the number of positive slopes in the warehouse order cost function on  $[0, Q]$ .*

*Proof.* The running time follows because  $\sum_{s=1}^T |\Theta_{i,s}| = \mathcal{O}(T)$ , despite the need to find a complete list of  $\Theta_{i,s}$ ,  $\forall i = 1, \dots, N$ . Global feasibility will always be guaranteed by the way we construct the final solution. According to Lemma 5.3, 5.4, 5.5, the final solution is a  $(K + 3)$ -approximation for the OWMR problem. □

**Corollary 5.7.** *Given optimal LHP and SHP solutions, applying the recombination for PWL concave batch order costs generates a 2-approximation for the OWMR problem in  $\mathcal{O}(NT)$  time when the warehouse has a fixed order cost,  $C^0$ , within one batch, i.e.,  $c_0(q_0^t) = C^0 \cdot \lceil q_0^t / Q \rceil$ ,  $\forall t = 1, \dots, T$ .*

*Proof.* When each batch in warehouse orders has fixed and stationary cost, the cardinality of the list  $\Theta_{i,s}$  is at most 2, in which case  $\Theta_{i,s} = \{\underline{\theta}_{i,s}^1, \tilde{\theta}_{i,s}^0\}$ . By Lemma 5.3 and 5.4, the retailers order costs and holding costs are bounded above by  $2V^b$  and  $2H^b$ , respectively. For the full batches of warehouse orders in the final solution, the costs are bounded by  $W^b$ , which can be argued in the same way as the full batches in Lemma 5.5. The recombination algorithm guarantees  $\hat{q}_0^t = 0, \forall t \in \{\tau \in [T] \mid q_0^\tau = 0\}$ . Therefore, the number of periods with non-empty warehouse order quantity will not increase after recombination, which indicates that the costs of partial batches are bounded above by  $W^b$ . The final solution then is a 2-approximation for the OWMR problem.  $\square$

The subproblem LSP's are each solvable in  $\mathcal{O}(T^4)$  time (Akbalik and Rapine 2012) under the conditions in Observation 3.1, giving an overall running time for Algorithm 1 of  $\mathcal{O}(NT^4)$ ; this running time is reduced to  $\mathcal{O}(NT^2)$  in the case of concave order costs. We can also extend the theorem in the following manner.

**Corollary 5.8.** *Under PWL concave batch warehouse order costs, but assuming only that each retailer has non-decreasing subadditive order costs, the recombination procedure still yields a  $(K+3)$ -approximation for the OWMR problem.*

*Proof.* The proof follows because Lemma 5.3 still applies.  $\square$

Suppliers often offer volume-dependent discounts that exhibit economies of scale. Since many of the corresponding order costs can be characterized or approximated by PWL concave batch functions, our results are particularly applicable in these situations. For instance, for the OWMR problem with truckload discounts our algorithm yields a 4-approximation by setting  $K = 1$ ,  $\eta_{0,1}$  to be the unit LTL rate, and  $M_{0,1}$  to be the quantity breakpoint at which shipment costs the same whether shipped by a full truckload or with the LTL rate. For incremental discounts, it suffices to set  $K$  as the number of linear pieces in the true cost function and the batch capacity to be large enough, i.e.,  $Q \geq \sum_{i=1}^N \sum_{t=1}^T d_i^t$ . For *regular modified all-unit discounts* where each non-flat segment has an identical length and each flat segment has another identical length in the cost function (Appendix A), we can combine our algorithm with an existing technique to approximate the OWMR problem under mild conditions (Archetti et al. 2014, Hu 2016); this would imply an additional multiplicative factor of 2 in the approximation guarantee. When the warehouse order cost has incremental discounts and  $K$  is fixed, this provides a polynomial-time constant approximation for the corresponding special case of Chan et al. (2002). Furthermore, recent results show how to efficiently solve LSP models with general modified all-unit discounts (e.g. Koca et al. 2014), hence we can obtain constant approximations for more variants of the model in Chan et al. (2002).

## 6 Approximation for Concave Batch Costs

We now relax the PWL assumption on the order costs, and propose a recombination to attain a logarithmic approximation for the OWMR problem with general concave batch order costs.

### 6.1 Recombination

Given optimal LHP and SHP subproblem solutions, again consider an arbitrary pair of warehouse-retailer order periods  $(\tau, s)$  covering some split demand  $d_{i,\tau,s}^t$ . To avoid trivial cases, we assume

$T \geq 3$ .

**Case I:**  $\tau \leq s$  The warehouse order arrives in time to fulfill the retailer order. Redefine  $\eta(\tau)$  in (7b) as

$$\eta'(\tau) = \begin{cases} c_{0,F}/Q, & 0 < q_0^\tau < Q \\ 0, & q_0^\tau \geq Q \end{cases}, \quad \forall \tau \in \bigcup_{i=1}^N \Xi_{i,s},$$

where we only distinguish initial warehouse order periods that contain full batches from the others. Replacing  $\eta(\cdot)$  with  $\eta'(\cdot)$  in the definition of  $\underline{\theta}_{i,s}^1$  and  $\underline{\theta}_{i,s}^2$  in (8), Observation 5.2 remains valid for general concave  $c_0(\cdot)$  by Proposition 3.3. Hence we apply the same recombination we use for PWL concave batch order costs here as well.

**Case II:**  $\tau > s$  The warehouse order arrives too late for the retailer order. Here we take a time-based approach for the recombination. Consider the following *binary representation* of an arbitrary positive integer number  $x$ ,

$$B(x) := (b_n, \dots, b_1, b_0),$$

where  $n = \lceil \log_2 x \rceil$  is the *highest bit*,  $b_j \in \{0, 1\}$  is the value of bit  $j$ , and  $x = \sum_{j=0}^n b_j 2^j$ . Thus,  $b_n = 1$  by definition. For ease of reading, in the remainder of the paper we drop the base 2 in the logarithmic notation. Let  $J(x) := \min\{j : b_j = 1, b_j \in B(x)\}$  be the *lowest bit with value 1* of number  $x$ .

Consider the list of potential retailer order periods

$$\mathcal{S}_s = \{\phi_s^1, \phi_s^2, \dots\}, \quad s = 1, \dots, (T-1), \quad (10)$$

where  $\phi_s^1 = s$ ,  $\phi_s^2 = s + 1$ , and

$$\phi_s^{k+1} = \phi_s^k + 2^\alpha \leq T - 1, \quad \alpha = J(\phi_s^k), \quad \forall k = 2, 3, \dots$$

In other words, the increasing list  $\mathcal{S}_s$  is independent of the retailer  $i$ , and each pair of consecutive elements is separated by an increasing power of 2. It starts with periods  $s$  and  $s + 1$ , with the remaining periods defined so that  $\phi_s^{k+1}$  differs from  $\phi_s^k$  by the power of 2 corresponding to  $J(\phi_s^k)$ . This continues until  $\phi_s^{k+1}$  would go beyond  $T - 1$ , where the list ends.

For example, when  $T = 130$ ,  $s = 100 = 2^6 + 2^5 + 2^2$  is equivalent to  $B(100) = (1, 1, 0, 0, 1, 0, 0)$ , where we have  $\phi_s^1 = s = 100$  and  $\phi_s^2 = s + 1 = 101$ . Since  $B(101) = (1, 1, 0, 0, 1, 0, 1)$ ,  $B(102) = (1, 1, 0, 0, 1, 1, 0)$ ,  $B(104) = (1, 1, 0, 1, 0, 0, 0)$ ,  $B(112) = (1, 1, 1, 0, 0, 0, 0)$ ,  $B(128) = (1, 0, 0, 0, 0, 0, 0)$ , etc.; hence  $\mathcal{S}_{100} = \{100, 101, 102, 104, 112, 128\}$ .

Similarly, consider the list of potential warehouse order periods

$$\mathcal{L}_\tau = \{\psi_\tau^1, \psi_\tau^2, \dots\}, \quad \forall \tau = 2, \dots, T - 1, \quad (11)$$

where  $\psi_\tau^1 = \tau$ ,  $\psi_\tau^2 = \tau - 1$ , and

$$\psi_\tau^{k+1} = \psi_\tau^k - 2^\beta \geq 2^{\lceil \log \tau \rceil - 1}, \quad \beta = J(\psi_\tau^k), \quad \forall k = 2, 3, \dots$$

When  $\tau = T$ , let  $\mathcal{L}_T = \mathcal{L}_{T-1}$ . Therefore, the decreasing list  $\mathcal{L}_\tau$  also contains elements separated by powers of 2. It starts with period  $T - 1$  if  $\tau = T$  and otherwise  $\tau$  followed by period  $\tau - 1$ , with the remaining periods defined so that  $\psi_s^{k+1}$  decreases  $\psi_s^k$  by the power of 2 corresponding to  $J(\psi_s^k)$ . This continues until  $\psi_\tau^k$  becomes the largest power of 2 smaller than  $\tau$ , where the list ends.

For example, when  $T = 130$ ,  $\tau = 120$ , we have  $\psi_\tau^1 = \tau = 120$ ,  $\psi_\tau^2 = \tau - 1 = 119$ , where  $B(120) = (1, 1, 1, 1, 0, 0, 0)$ ,  $B(119) = (1, 1, 1, 0, 1, 1, 1)$ . According to the definition of  $\mathcal{L}_\tau$ , we then have:  $B(118) = (1, 1, 1, 0, 1, 1, 0)$ ,  $B(116) = (1, 1, 1, 0, 1, 0, 0)$ ,  $B(112) = (1, 1, 1, 0, 0, 0, 0)$ ,  $B(96) = (1, 1, 0, 0, 0, 0, 0)$ ,  $B(64) = (1, 0, 0, 0, 0, 0, 0)$ , etc.; hence  $\mathcal{L}_{120} = \{120, 119, 118, 116, 112, 96, 64\}$ .

Next we prove useful structural properties of the lists  $\mathcal{S}_s$  and  $\mathcal{L}_\tau$ .

**Lemma 6.1.** *Given an arbitrary pair of periods  $(\tau, s)$  such that  $1 \leq s < \tau \leq T$ , there exists at least one common element in the lists  $\mathcal{S}_s, \mathcal{L}_\tau$  given by (10)-(11).*

*Proof.* If  $s = T - 1$ , then  $\tau = T$  and  $s = T - 1 \in \mathcal{S}_s \cap \mathcal{L}_\tau$ . Below we assume  $\tau \leq T - 1$ .

Consider the binary vectors  $B(s) = (b_n, \dots, b_0)$ ,  $B(\tau) = (a_m, \dots, a_0)$ , then  $n = \lfloor \log s \rfloor \leq \lfloor \log \tau \rfloor = m$ . Let  $\tilde{B}(s) = (b_m, \dots, b_n, \dots, b_0)$  be the augmented vector of  $B(s)$  such that the dimension is the same as  $B(\tau)$  and the additional bits take value zero, i.e.,  $b_{n+1} = \dots = b_m = 0$ . We have  $\sum_{j=0}^m b_j \cdot 2^j = \sum_{j=0}^n b_j \cdot 2^j = s$ , so  $s$  can be represented by  $\tilde{B}(s)$  as well. Compare  $b_j$  and  $a_j$  from  $j = m$  to 0 until  $b_\ell \neq a_\ell$  for some  $\ell$ , the highest bit which takes different values in  $\tilde{B}(s)$  and  $B(\tau)$ . Since  $s < \tau$ , we have  $b_\ell = 0 < a_\ell = 1$ . Let  $\zeta_{\tau,s}$  be the integer represented by binary vector  $(a_m, \dots, a_\ell, 0, \dots, 0) = (b_m, \dots, b_{\ell+1}, 1, 0, \dots, 0)$ , i.e., the bits  $j = \ell, \dots, m$  in  $B(\zeta_{\tau,s})$  take values  $a_j$  whereas the bits  $j = 0, \dots, (\ell - 1)$  all equal zero. Since  $a_j = b_j$  for  $j = (\ell + 1), \dots, m$ ,  $a_\ell > b_\ell$ , and  $a_j \geq 0$  for  $j = 0, \dots, (\ell - 1)$ , we have  $s < \zeta_{\tau,s} \leq \tau$ .

Next we show that both lists  $\mathcal{S}_s$  and  $\mathcal{L}_\tau$  contain  $\zeta_{\tau,s}$ .

If  $\ell = 0$ , it is trivial as  $s = \tau - 1$ , where  $\tau$  is odd and  $s$  is even. According to the definition of set  $\mathcal{S}_s$  and  $\mathcal{L}_\tau$ ,  $\zeta_{\tau,s} = \tau$  is the common period. If  $\ell = 1$ ,  $\zeta_{\tau,s} = \tau - a_0 = (s + 1) + (1 - b_0)$  is the common period.

If  $\ell \geq 2$ ,

$$\zeta_{\tau,s} = (s + 1) + \sum_{j=0}^{\ell-1} (1 - b_j) \cdot 2^j = \phi_s^2 + \sum_{j=2}^{\nu+1} 2^{J(\phi_s^j)} = \phi_s^{\nu+2},$$

where  $\nu$  is the number of bits in  $b_0, \dots, b_{\ell-1}$  that take value 0 in  $B(s)$ . Thus,  $\zeta_{\tau,s}$  is the  $(\nu + 2)$ -th element in  $\mathcal{S}_s$ . Similarly,

$$\zeta_{\tau,s} = \tau - \sum_{j=0}^{\ell-1} a_j \cdot 2^j = \begin{cases} \phi_s^{\nu'+2}, & \tau \text{ is even} \\ \phi_s^{\ell+1-\nu}, & \tau \text{ is odd} \end{cases},$$

where  $\nu'$  is the number of bits in  $a_0, \dots, a_{\ell-1}$  that take value 1 in  $B(\tau - 1)$ . Therefore,  $\zeta_{\tau,s} \in \mathcal{S}_s \cap \mathcal{L}_\tau$ .  $\square$

**Lemma 6.2.** *For the lists  $\mathcal{S}_s, \mathcal{L}_\tau$  given by (10)-(11),*

$$|\mathcal{S}_s| \leq \lfloor \log T \rfloor, \quad \forall s = 1, \dots, (T - 1); \quad |\mathcal{L}_\tau| \leq \lfloor \log T \rfloor, \quad \forall \tau = 2, \dots, T.$$

*Proof.* The list  $\mathcal{S}_s$  given by (10) consists of two subsets:  $\underline{\mathcal{S}}_s = \{\phi \in \mathcal{S}_s : J(\phi) \leq \lfloor \log s \rfloor\}$  and  $\tilde{\mathcal{S}}_s = \{\phi \in \mathcal{S}_s : J(\phi) \geq \lfloor \log s \rfloor + 1\}$ . Since  $s = \phi_s^1 < \phi_s^2 < \dots < T$  and  $0 \leq J(\phi_s^2) < J(\phi_s^3) < \dots$ , we first have:

i)  $\underline{\mathcal{S}}_s = \{s\} \cup \{\phi \in \mathcal{S}_s : J(\phi) \leq \lfloor \log s \rfloor - 1\}$  and so  $|\underline{\mathcal{S}}_s| \leq \lfloor \log s \rfloor + 1$ ;

ii)  $\tilde{\mathcal{S}}_s = \{2^{\lfloor \log s \rfloor + 1}, 2^{\lfloor \log s \rfloor + 2}, \dots, 2^{\lfloor \log T \rfloor}\}$  and so  $|\tilde{\mathcal{S}}_s| \leq \lfloor \log T \rfloor - \lfloor \log s \rfloor$ .

Since  $T \notin \mathcal{S}_s$ , we further have  $|\tilde{\mathcal{S}}_s| \leq \lfloor \log T \rfloor - \lfloor \log s \rfloor - 1$ . Therefore,  $|\mathcal{S}_s| = |\underline{\mathcal{S}}_s| + |\tilde{\mathcal{S}}_s| \leq \lfloor \log T \rfloor$ .

Now consider the list  $\mathcal{L}_\tau$  given by (11). We first have  $|\mathcal{L}_\tau| \leq \lfloor \log \tau \rfloor + 1 \leq \lfloor \log T \rfloor + 1$  since  $2^{\lfloor \log \tau \rfloor - 1} = \psi_\tau^{\lfloor \mathcal{L}_\tau \rfloor} < \dots < \psi_\tau^2$  and  $J(\psi_\tau^{\lfloor \mathcal{L}_\tau \rfloor}) > \dots > J(\psi_\tau^2) \geq 0$ . For  $\tau > 4$ , we have  $J(\psi_\tau^2) \geq 1$  if  $\tau$  is an odd number and  $J(\psi_\tau^3) \geq 2$  if  $\tau$  is an even number; hence  $|\mathcal{L}_\tau| \leq \lfloor \log T \rfloor$ . Since we assume  $T \geq 3$  and  $T \notin \mathcal{L}_\tau$ , we further have  $\mathcal{L}_2 = \{2, 1\}$ ,  $\mathcal{L}_3 \subseteq \{3, 2\}$ ,  $\mathcal{L}_4 = \{3, 2\}$  if  $T = 4$ , and  $\mathcal{L}_4 = \{4, 3, 2\}$  if  $T \geq 5$ ; again,  $|\mathcal{L}_\tau| \leq \lfloor \log T \rfloor$  for  $2 \leq \tau \leq 4$ .  $\square$

The recombination simply moves any globally infeasible warehouse-retailer order pair for periods  $\tau > s$  to any common period, say  $\iota$ , in sequences (10)-(11). For instance, if  $(\tau, s) = (120, 100)$ , then we move the warehouse and retailer orders of  $d_{i,\tau,s}^t$  to period  $\iota = 112$ . As with the previous recombinations, for Case II the warehouse serves as a cross-dock in the resulting OWMR solution. Algorithm 3 gives the pseudocode; below we analyze the complexity and solution quality.

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**Algorithm 3** Recombination for concave batch order costs

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1: for  $i = 1, \dots, N$ ,  $s = 1, \dots, T$ ,  $\tau \in \Xi_{i,s}$  do
2:   if  $\tau \leq s$  and  $h_0 > h_i$  then
3:     if  $\eta(\xi_{\underline{\mathcal{S}}_s}^1) = 0$  then
4:        $q_{0,i}^{\xi_{\underline{\mathcal{S}}_s}^1} \leftarrow q_{0,i}^{\xi_{\underline{\mathcal{S}}_s}^1} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_{0,i}^\tau \leftarrow q_{0,i}^\tau - \sum_{t=1}^T d_{i,\tau,s}^t$ 
5:        $q_i^{\xi_{\underline{\mathcal{S}}_s}^1} \leftarrow q_i^{\xi_{\underline{\mathcal{S}}_s}^1} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
6:     else if  $\tau = \xi_{\underline{\mathcal{S}}_s}^1$  then
7:        $q_i^\tau \leftarrow q_i^\tau + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
8:     else
9:        $q_{0,i}^{\xi_{\underline{\mathcal{S}}_s}^2} \leftarrow q_{0,i}^{\xi_{\underline{\mathcal{S}}_s}^2} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_{0,i}^\tau \leftarrow q_{0,i}^\tau - \sum_{t=1}^T d_{i,\tau,s}^t$ 
10:       $q_i^{\xi_{\underline{\mathcal{S}}_s}^2} \leftarrow q_i^{\xi_{\underline{\mathcal{S}}_s}^2} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
11:    end if
12:   else if  $\tau > s$  then
13:     Find some  $\iota \in \mathcal{L}_\tau \cap \mathcal{S}_s$ 
14:      $q_{0,i}^\iota \leftarrow q_{0,i}^\iota + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_{0,i}^\tau \leftarrow q_{0,i}^\tau - \sum_{t=1}^T d_{i,\tau,s}^t$ 
15:      $q_i^\iota \leftarrow q_i^\iota + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$ 
16:   end if
17: end for
18:  $\hat{q}_0^t \leftarrow \sum_{i=1}^N q_{0,i}^t$ ,  $\hat{q}_i^t \leftarrow q_i^t$ ,  $\forall t = 1, \dots, T$ ,  $i = 1, \dots, N$ 

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**Example 2.** Consider a simple illustrative example where  $N = 1$ ,  $T = 7$ , and we assume  $h_1 > h_0$ . We include one period's demand and the corresponding LHP and SHP orders that meet this demand; Table 4 lists these amounts.

We have  $d_{1,3,5}^7 = 0.2$ ,  $d_{1,4,5}^7 = 0.4$ ,  $d_{1,7,5}^7 = 0.7$  and  $d_{1,\tau,s}^t = 0$  for any other  $(\tau, s, t)$ . Then,  $\Xi_{1,5} = \{3, 4, 7\}$  and  $\Xi_{1,s} = \emptyset$  for any other  $s$ . Since  $\mathcal{S}_5 = \{5, 6\}$  and  $\mathcal{L}_7 = \{7, 6, 4\}$ , we have  $\mathcal{S}_5 \cap \mathcal{L}_7 = \{6\}$ , and Algorithm 3 gives the recombined solution shown in the bottom two rows of Table 4.

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$
$d_1^t$	0	0	0	0	0	0	1.3
$q_0^t$	0	0	0.2	0.4	0	0	0.7
$q_1^t$	0	0	0	0	1.3	0	0
$\hat{q}_0^t$	0	0	0.2	0.4	0	0.7	0
$\hat{q}_1^t$	0	0	0	0	0.6	0.7	0

Table 4: An example of Algorithm 3.

**Theorem 6.3.** *Given optimal subproblem solutions, applying the recombination for concave batch order costs in Algorithm 1 generates a  $(\lceil \log T \rceil + 2)$ -approximation for the OWMR problem in  $\mathcal{O}(NT \log T)$  time.*

*Proof.* We again use the notation  $W^a, V^a, H^a$  to represent the total warehouse order cost, total retailer order cost and total holding cost, respectively, of the resulting OWMR solution; and  $W^b, V^b, H^b$  for the combined LHP and SHP subproblem costs incurred by the given solutions. Holding cost patterns are the same as in the PWL case and thus Lemma 5.4 applies, yielding  $H^a \leq 2H^b$ .

Lemma 6.2 ensures that each retailer order is split into at most  $\lceil \log T \rceil + 2$  suborders, thus  $V^a \leq (\lceil \log T \rceil + 2)V^b$ . The warehouse order cost in Case I is bounded by  $2W^b$  since among the (at most) two suborders when  $h_0 > h_i$ , one does not contribute to extra quantities in the reallocated period, and the other is located in a period which initially contains a full batch (see the proof of Theorem 5.6). In Case II, we use identical potential order periods for all  $i = 1, \dots, N$  at the warehouse, so Lemma 6.2 ensures that each warehouse order is split into at most  $\lceil \log T \rceil$  suborders, which together with Case I gives  $W^a \leq (\lceil \log T \rceil + 2)W^b$ ; we thus obtain  $Z^a(\text{OWMR}) \leq (\lceil \log T \rceil + 2)Z^*(\text{OWMR})$ .

Next we prove the claimed running time. For each retailer  $i = 1, \dots, N$ , the initial solutions imply  $\mathcal{O}(T)$  pairs of order periods  $(\tau, s)$ , which become available with the construction of the list  $\Xi_{i,s}$ . In Case I, finding  $\theta_{i,s}^1, \theta_{i,s}^2$  for all  $s$  takes  $\mathcal{O}(T)$  time for each retailer and can be done prior to the recombination. In Case II, the construction of lists  $\mathcal{S}_s$  or  $\mathcal{L}_\tau$  can also be done beforehand and each takes  $\mathcal{O}(\log T)$  time, which gives the length of the list as well. A common period for  $(\tau, s)$  can be found by comparing lists  $\mathcal{S}_s$  and  $\mathcal{L}_\tau$  from the first elements to the last, which takes  $\mathcal{O}(\log T)$  time. Therefore, the procedure finishes in  $\mathcal{O}(NT \log T)$  time.  $\square$

As with the PWL case, the subproblem LSP's are each solvable in  $\mathcal{O}(T^4)$  (Akbalik and Rapine 2012) if Observation 3.1 holds, giving an overall running time for Algorithm 1 of  $\mathcal{O}(NT^4)$ ; when order costs are simply concave, the subproblem running times each reduce to  $\mathcal{O}(T^2)$ , making the overall running time  $\mathcal{O}(NT^2)$ . The logarithmic approximation ratio provides an alternative guarantee for the PWL concave batch order costs in §5. When  $K$  is small, i.e., the PWL concave cost function contains only a few linear pieces, the previous approach is appealing because of its constant approximation guarantee. As  $K$  increases, however, the upper bound grows linearly and gradually becomes impractical; this new worst-case guarantee provides an alternate upper bound. Also, unlike the PWL recombination, this time-based approach yields the same recombination regardless of the specific concave batch order cost function.

## 7 Extension to Subadditive Costs

Our interest in subadditive costs is motivated by the fact that some real-world order cost structures are not fully captured by concave batch costs. Representative examples include the modified all-unit discounts in Chan et al. (2002), a slightly more relaxed variant in Shen et al. (2009), the general *mixed truckload costs* in Hu (2016), and the *stepwise concave production costs* in Akbalik and Rapine (2012) among others (see Appendix A), all of which are non-decreasing subadditive order cost functions.

Assume again that the subproblem solutions yield order period pair  $(\tau, s)$  with split demand  $d_{i,\tau,s}^t$ . Recall the recombination for general concave batch costs in Case II. Clearly, it applies to any pair of warehouse-retailer order periods  $(\tau, s)$  such that  $\tau > s$ . Since  $\tau < s$  is equivalent to  $T + 1 - \tau > T + 1 - s$ , we propose the following recombination for subadditive order costs.

**Case I:**  $\tau \leq s$  Do not modify the orders if  $\tau = s$  or  $h_0 \leq h_i$ . Otherwise, find a common period in the lists  $\mathcal{L}_{T+1-\tau}$  and  $\mathcal{S}_{T+1-s}$ , say  $\iota$ . Move both the warehouse and retailer order of  $d_{i,\tau,s}^t$  to period  $T + 1 - \iota$ . Since  $T + 1 - s \leq \iota \leq T + 1 - \tau$ , the new order period satisfies  $\tau \leq T + 1 - \iota \leq s$  and the modification is thus feasible.

**Case II:**  $\tau > s$  Apply the recombination for concave batch costs. In the resulting OWMR solution, the warehouse once again operates as a cross-dock in Case II.

**Theorem 7.1.** *Given optimal LHP and SHP subproblem solutions, applying the recombination for non-decreasing subadditive order costs to Algorithm 1 generates a  $(2\lceil \log T \rceil)$ -approximation for the OWMR problem in  $\mathcal{O}(NT \log T)$  time.*

*Proof.* The inequality  $H^a \leq 2H^b$  is again proved as in Lemma 5.4. The recombination ensures that  $\mathcal{L}_t \leq 2\lceil \log T \rceil$  and  $\mathcal{S}_{i,t} \leq 2\lceil \log T \rceil$ ,  $\forall t = 1, \dots, T$ ,  $i = 1, \dots, N$ . Therefore, both the approximation ratio and the running time are twice that of Case II for concave batch costs.  $\square$

For the overall approximation algorithm to run in polynomial time, it remains to solve the subproblems, which are typically  $\mathcal{NP}$ -hard under subadditive costs. Nevertheless, we highlight scenarios where our results apply. Akbalik and Rapine (2012) solved the LSP with capacitated stepwise concave production costs by an efficient DP, which combines our concave batch order costs with a constant order capacity limit and/or a major setup cost in each period. Hence the uncapacitated variant of the OWMR problem under their order costs can be efficiently approximated within a factor of  $\mathcal{O}(\log T)$  provided that the revised warehouse holding cost rates  $h_{0,i}$  are homogeneous. Koca et al. (2014) showed that the LSP with *piecewise concave* (PWC) order costs can be solved in polynomial time by a DP if the number of breakpoints is fixed and the breakpoints are time-invariant for the cost function on  $\mathbb{R}_+$ . In light of Theorem 7.1, this implies a  $(2\lceil \log T \rceil)$ -approximation for the corresponding OWMR problem of Chan et al. (2002) assuming homogeneous modified holding cost rates  $h_{0,i}$ .

Now that we have extended the result to subadditive order costs, a subsequent question is whether we can incorporate subadditive holding costs as well. To that end, assume without loss of generality that for any retailer  $i = 1, \dots, N$ , either  $H_i(I) \geq H_0(I)$  or  $H_i(I) < H_0(I)$  on  $I \in \mathbb{R}_+$ . That is, we can still partition the retailers into two subsets, where the retailer holding cost function is

pointwise greater than or equal to the warehouse holding cost function in one subset and lower for the other subset. Also, here we do not halve the holding cost functions in the subproblem objective functions; we simply use  $H_{0,i}(I_{0,i}^t) = \min\{H_0(I_{0,i}^t), H_i(I_{0,i}^t)\}$  and  $H_{i,i}(I_i^t) = H_i(I_i^t)$  in (4) and (5). Because of the revision, we need to verify a lower bound for the OWMR problem with the revised subproblem objective values, and verify a lower bound for the holding costs of the recombined OWMR solution with the subproblem objective values.

**Lemma 7.2.** *Under the revised holding cost functions  $H_{0,i}(\cdot)$  and  $H_{i,i}(\cdot)$ , the optimal objective values  $Z^*(P)$  are such that  $Z^*(LHP) \leq Z^*(OWMR)$  and  $Z^*(SHP) \leq Z^*(OWMR)$ . That is, the LHP and SHP subproblems each provide a lower bound for the OWMR problem.*

*Proof.* See Lemma 5.1 in Stauffer et al. (2011). This proof assumes the warehouse acts as cross-dock for the retailers with  $H_i(\cdot) < H_0(\cdot)$ . However, we can overcome this by letting the warehouse order the same quantities in the LHP problem and the OWMR problem.  $\square$

**Lemma 7.3.** *Using the same notation as before and applying the recombination for subadditive order costs, the holding costs satisfy  $H^a \leq 2\lceil \log T \rceil H^b$ .*

*Proof.* See Appendix D.  $\square$

Algorithm 4 in Appendix C gives the pseudocode to approximate the OWMR problem under subadditive costs. Finally, the result for subadditive holding costs follows.

**Theorem 7.4.** *Given optimal subproblem solutions with respect to the newly defined holding cost functions  $H_{0,i}(\cdot)$  and  $H_{i,i}(\cdot)$ , the recombination for subadditive order costs generates a  $(4\lceil \log T \rceil)$ -approximation for the OWMR problem under non-decreasing subadditive order and holding costs.*

*Proof.* We now have

$$\begin{aligned}
Z^a(\text{OWMR}) &= W^a + V^a + H^a \\
&\leq 2\lceil \log T \rceil (W^b + V^b) + H^a && \text{analogous to Theorem 7.1} \\
&\leq 2\lceil \log T \rceil (W^b + V^b + H^b) && \text{by Lemma 7.3} \\
&= 2\lceil \log T \rceil (Z^*(\text{LHP}) + Z^*(\text{SHP})) \\
&\leq 4\lceil \log T \rceil Z^*(\text{OWMR}) && \text{by Lemma 7.2.} \quad \square
\end{aligned}$$

Compared with Theorem 7.1, the approximation ratio increases by a factor of 2 because of the revised lower bounds for the OWMR problem under subadditive holding costs.

## 8 Computational Experiments

We next summarize the results of computational experiments testing Algorithms 2 and 3 on instances with PWL concave batch costs with one positive slope, which model truckload discount schemes, where the positive slope is an LTL rate and the full batch represents an FTL cost. For simplicity, we assume that the order cost function and the holding cost rate are identical for each retailer. We use a batch size  $Q = 1$  and threshold of 0.5. We generate demands independently from a uniform distribution over  $[0, 1]$ .



We conducted experiments on a Macbook Pro with OS X 10.15.2, 2.7 GHz Intel Core i7 and 16 GB of RAM. All models were implemented in Python 3.6.8 and Gurobi 8.1.1. We compare the solution generated by the algorithms to the best solution of the full OWMR model found by Gurobi in a time limit of one hour. In our implementation of the algorithms, we solve both LHP and SHP using Gurobi, and when necessary enforce a 100-second time limit for each. After solving LHP and SHP, the remaining computing time required by either algorithm to perform the order quantity recombination is negligible. Thus, even though we test two-slope instances, we expect similar solve times and performance for instances with several slopes, since the computational bottleneck is solving LHP and SHP.

We considered four cases:

1.  $c_0(Q) = 60, c_i(Q) = 40; h_0 = 40, h_i = 80$
2.  $c_0(Q) = 40, c_i(Q) = 60; h_0 = 40, h_i = 80$
3.  $c_0(Q) = 60, c_i(Q) = 40; h_0 = 80, h_i = 40$
4.  $c_0(Q) = 40, c_i(Q) = 60; h_0 = 80, h_i = 40$ .

We randomly generated demands for 10 small instances with  $N = 5$  and  $T = 25$ , and for 10 large instances with  $N = 10$  and  $T = 100$ . When combined with the four cost scenarios, this yields a total of 40 small and 40 large instances. Table 5 shows average results, where  $SC_1$  to  $SC_4$  represent the four cases with small instances, and  $LC_1$  to  $LC_4$  represent the four cases with large instances. In the table, the three OWMR columns detail what Gurobi was able to compute in one hour.  $Z^*(\text{LHP})$  and  $Z^*(\text{SHP})$  are the optimal objective values of LHP and SHP, respectively, also computed with Gurobi, while  $Z^*(\text{Alg. 2})$  and  $Z^*(\text{Alg. 3})$  are the objective values of the solutions generated by Algorithms 2 and 3.

Table 5: Performance of Algorithms 2 and 3

Cases	OWMR			$Z^*(\text{LHP})$	$Z^*(\text{SHP})$	$Z^*(\text{LHP}) + Z^*(\text{SHP})$	$Z^*(\text{Alg. 2})$	$Z^*(\text{Alg. 2})$ Best Obj.	$Z^*(\text{Alg. 3})$	$Z^*(\text{Alg. 3})$ Best Obj.	$Z^*(\text{LHP}) + Z^*(\text{SHP})$ Best Bd.
	Best Obj.	Best Bd.	Gap								
$SC_1$	7632.72	7610.71	0.29%	3900.44	3477.68	7378.12	7764.36	1.0172	9445.40	1.2374	0.9655
$SC_2$	8021.76	7997.18	0.31%	2629.14	4948.32	7577.46	8426.00	1.0502	10576.96	1.3185	0.9433
$SC_3$	7349.84	7273.06	1.04%	3900.44	3177.48	7077.92	7839.72	1.0665	9514.20	1.2945	0.9678
$SC_4$	7566.20	7487.13	1.04%	2629.14	4541.72	7170.86	8377.95	1.1070	10476.16	1.3846	0.9549
$LC_1$	60306.84	60118.92	0.31%	30590.24	27840.80	58431.04	61278.48	1.0192	72693.35	1.2054	0.9672
$LC_2$	63560.92	63194.504	0.58%	20520.24	37647.75	58167.99	66871.62	1.0582	82429.70	1.2969	0.9484
$LC_3$	58246.00	57837.59	0.70%	30590.24	25400.14	55990.38	62019.64	1.0723	73098.87	1.2550	0.9633
$LC_4$	60088.12	59565.619	0.86%	20520.24	36359.44	56879.68	66387.29	1.1146	81838.38	1.3620	0.9431

As the table indicates, Algorithm 2 generates high-quality solutions despite having short computation time limits. Interestingly, the algorithm’s best performance (within about 2% of optimal on average) comes in the most realistic scenario (cost scenario 1), where the warehouse order costs are larger but its holding costs smaller than the retailers’. We expect these cost relations to be most applicable in practice: As we discussed previously, holding costs increase as we move downstream in a supply chain, reflecting the product’s increase in value as it comes closer to the customer. Conversely, we expect higher order costs at the warehouse, as it is usually ordering product from farther away, e.g. from an overseas supplier. In general, Algorithm 2’s solution is within 10% of the best solution found by Gurobi in one hour in all cases but one,  $LC_4$ . Furthermore, the bound

generated during the course of the algorithms (by adding the objective values of LHP and SHP) is of high quality across the board, within roughly 5% or better of Gurobi’s best bound.

Algorithm 3 exhibits similar behavior relative to Algorithm 2, with its best performance in cost scenario 1, but the overall quality of the solutions is somewhat poorer, with gaps in the range of 20%-40%. This behavior is not completely surprising, since this algorithm may need to recombine quantities many more times than Algorithm 2. In other words, we pay for Algorithm 3’s generality with higher costs.

## 9 Conclusions

In this paper, we study the OWMR problem under deterministic dynamic demand and concave batch order costs. We first decompose the two-echelon problem into single-facility subproblems, which provide a lower bound with modified holding cost parameters. We then propose approximation algorithms to find OWMR solutions of guaranteed worst-case performance by judiciously recombining the subproblem solutions. Our methodology yields a constant-factor approximation for PWL concave batch order costs, and a logarithmic approximation for general concave batch order costs. We also extend our results to subadditive order and holding costs.

Our results for more general cost structures motivate questions on the LHP subproblem. When the retailers’ holding cost rates differ and are lower than the warehouse holding cost rate, the corresponding LHP problem is a variant of the multi-item LSP in Anily et al. (2009), but the LP reformulation is no longer equivalent to the original MIP model because of the revised objective function. To the best of our knowledge, there are no algorithms for this problem and its complexity is open, though a combination of existing methods (Akbalik and Rapine 2012, Anily and Tzur 2006) may perform well experimentally. Another interesting direction would be to modify our method for the infinite-horizon, constant demand rate case; there is one recent result (Stauffer 2012) that suggests this may be possible.

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## Appendix A Additional Pertinent Cost Structures

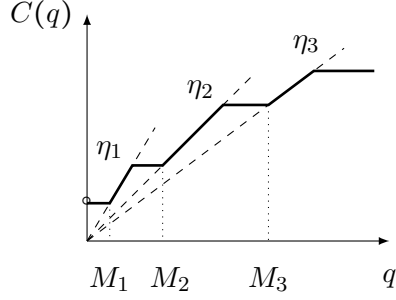
Figure 5 exemplifies several cost structures that appear in various applications. The *modified all-unit discount* (Figure 5(a)) is the retailer order cost structure considered in Chan et al. (2002) and a special case from Shen et al. (2009). Mathematically, it can be represented by the following PWL cost function:

$$C(q) = \begin{cases} 0, & q = 0 \\ \eta_1 \cdot M_1, & 0 < q < M_1 \\ \min\{\eta_k \cdot q, \eta_{k+1} \cdot M_{k+1}\}, & M_k \leq q < M_{k+1}, \quad k = 1, \dots, n-1 \\ \eta_n \cdot q, & M_n \leq q \end{cases}, \quad (12)$$

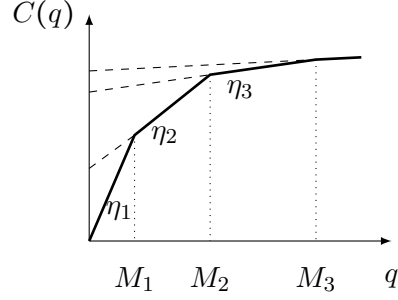
where  $\eta_1 > \eta_2 > \dots \geq 0$ ,  $M_k$  are the breakpoints between the  $k$ -th flat section and the  $k$ -th non-flat section, and  $\eta_1 \cdot M_1 \geq 0$  is a minimum charge for shipping a small volume.

Archetti et al. (2014) designed a polynomial-time DP for the LSP under an order cost structure called *regular modified all-unit discounts*, which are special modified all-unit discounts where the non-flat segments of the cost function have an identical length and so do the flat segments. Mathematically, it can be described with (12) by setting  $M_1 = M_2 - M_1 = \dots = M_n - M_{n-1}$  and  $M'_1 = M'_2 - M_1 = \dots = M'_n - M_{n-1}$ , where  $M'_{k+1} = \frac{\eta_{k+1}}{\eta_k} \cdot M_{k+1}$  are the cost breakpoints between consecutive non-flat and flat segments. In some circumstances, this cost structure can be approximated by fixed batch costs or PWL concave batch costs with  $K = 1$  (Hu 2016).

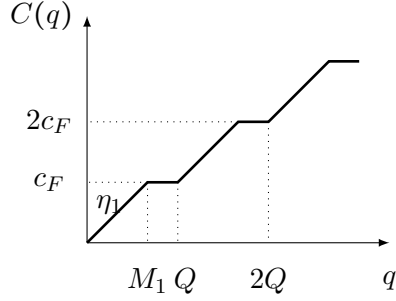
The *incremental discounts* cost function (Figure 5(b)) is the warehouse order cost structure considered in Chan et al. (2002) and a special case of Archetti et al. (2014) and Shen et al. (2009).



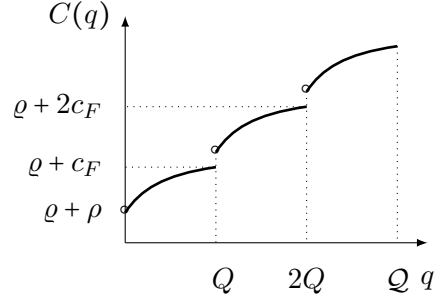
(a) Modified all-unit discounts



(b) Incremental discounts



(c) Truckload discounts



(d) Stepwise concave production costs

Figure 5: Pertinent cost structures

Mathematically, it can be described by a PWL concave function:

$$C(q) = \begin{cases} \eta_1 \cdot q, & 0 \leq q < M_1 \\ \eta_1 \cdot M_1 + \eta_2 \cdot (M_2 - M_1) + \dots + \eta_k \cdot (q - M_{k-1}), & M_{k-1} \leq q < M_k, \quad 2 \leq k < n, \\ \eta_1 \cdot M_1 + \sum_{k=2}^n \eta_k \cdot (M_k - M_{k-1}) + \eta_{n+1} \cdot (q - M_n), & M_n \leq q \end{cases}$$

where  $\eta_1 > \eta_2 > \dots > \eta_{n+1} \geq 0$ , and  $M_k$  are the breakpoints for charging new incremental discounts,  $1 \leq k \leq n$ . Our PWL concave batch costs reduce to it when  $Q \geq \sum_{i=1}^N \sum_{t=1}^T d_i^t$ .

The *truckload discounts* cost function (Figure 5(c)) is a special case of Hu (2016) and our PWL concave batch costs where  $K = 1$ ,  $\rho = 0$ . The name is due to Archetti et al. (2014) and Nahmias (2001).

The *stepwise concave production cost* (Figure 5(d)) is the LSP order cost structure in Akbalik and Rapine (2012), which combines our concave batch costs with a major setup cost and a total order capacity limit in each period. Mathematically, it can be represented by

$$C(q) = \begin{cases} 0, & q = 0 \\ \varrho + \rho \cdot \lceil \frac{q}{Q} \rceil + \mathcal{C}(q), & 0 < q \leq Q \end{cases},$$

where  $Q$  is the production capacity in each period. The cost function contains a fixed part and a variable part. The fixed components  $\varrho$  and  $\rho$  are the setup costs for each order and each batch, respectively. The variable component  $\mathcal{C}(\cdot)$  is a continuous function on  $[0, Q]$  which is duplicated

after every segment of a fixed length  $Q$  and concave within the segment. Gayon et al. (2016b) use a similar structure, where  $\mathcal{C}$  is only required to be non-decreasing and subadditive within each segment; they name such order costs *LTL freight cost functions*.

## Appendix B Examples

**Example 3** (Suboptimality of ZIO policies for inventory models with non-concave order costs). Consider the OWMR instance listed in Table 6 under fixed batch order costs. Since there is only one retailer and  $h_0 > c_{0,F}$ , the warehouse operates as a cross-docking station and the instance reduces to an LSP with batch order cost  $c_F = c_{0,F} + c_{1,F} = 30$ . The best ZIO policy is then found with DP, which incurs a total cost of 98, higher than that for the non-ZIO solution. This also shows suboptimality of ZIO for the single-echelon LSP with non-concave order costs.

Parameters: $Q = 10$ , $c_{0,F} = 20$ , $c_{1,F} = 10$ , $h_0 = 40$ , $h_1 = 2$						
Demands	Best ZIO solution			A better non-ZIO solution		
$(d_1^1, \dots, d_1^T)$	$(q_0^1, \dots, q_0^T)$	$(q_1^1, \dots, q_1^T)$	obj.	$(q_0^1, \dots, q_0^T)$	$(q_1^1, \dots, q_1^T)$	obj.
(6,6,4,4)	(6,6,8,0)	(6,6,8,0)	98	(10,10,0,0)	(10,10,0,0)	92

Table 6: Violation of ZIO

**Example 4** (Violation of regeneration properties for OWMR problems with batch capacities). Jin and Muriel (2009) proved that the regeneration property holds at every facility for the OWMR problem with fixed batch order costs if the warehouse holding cost rate is no higher than any retailer’s, i.e.,  $h_0 \leq h_i$ ,  $\forall i = 1, \dots, N$ . Here we show this policy may be suboptimal when  $h_0 > h_i$ . Consider the instance listed in Table 7. Since  $h_0 > c_{0,F}$ , the warehouse does not hold inventory. We computed the best solution that obeys the regeneration properties (i.e., column “Best RP solution”) by enumeration. The corresponding total cost is 85, higher than 81, the objective value achieved by another feasible solution which violates regeneration properties at both retailers.

Parameters: $Q = 10$ , $c_{0,F} = 20$ , $c_{1,F} = c_{2,F} = 10$ , $h_0 = 40$ , $h_1 = 1$ , $h_2 = 3$							
Retailers	Demands	Best RP solution			A better non-RP solution		
$i$	$(d_i^1, d_i^2)$	$(q_{0,i}^1, q_{0,i}^2)$	$(q_i^1, q_i^2)$	obj.	$(q_{0,i}^1, q_{0,i}^2)$	$(q_i^1, q_i^2)$	obj.
1	(6,4)	(10,0)	(10,0)	85	(7,3)	(7,3)	81
2	(3,7)	(10,0)	(10,0)		(3,7)	(3,7)	

Table 7: Violation of regeneration properties when  $h_0 > h_i$

## Appendix C Pseudocode

Algorithm 4 gives the pseudocode for the recombination under subadditive order costs.

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**Algorithm 4** OWMR approximation for subadditive costs

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**Phase I: Decomposition**

- 1: **if** linear holding costs **then**
- 2:   Set  $h_{0,i} \leftarrow 0.5 \min\{h_0, h_i\}$ ,  $h_{i,i} \leftarrow 0.5h_i$ ,  $\forall i = 1, \dots, N$
- 3: **end if**
- 4: **if** subadditive holding costs **then**
- 5:   Set  $H_{0,i}(\cdot) \leftarrow \min\{H_0(\cdot), H_i(\cdot)\}$ ,  $H_{i,i}(\cdot) \leftarrow H_i(\cdot)$ ,  $\forall i = 1, \dots, N$
- 6: **end if**
- 7: Solve LHP and SHP, obtain initial order quantities  $q_{0,i}^t, q_i^t$ ,  $\forall i = 1, \dots, N$ ,  $t = 1, \dots, T$

**Phase II: Recombination**

- 8: Construct lists  $\Xi_{i,t}$ ,  $\forall i = 1, \dots, N$ ,  $t = 1, \dots, T$
  - 9: Construct lists  $\mathcal{L}_t$  and  $\mathcal{S}_t$ ,  $\forall t = 1, \dots, T$
  - 10: **for**  $i = 1, \dots, N$ ,  $s = 1, \dots, T$ ,  $\tau \in \Xi_{i,s}$  **do**
  - 11:   **if**  $\tau < s$  and  $H_0(\cdot) > H_i(\cdot)$  **then**
  - 12:     Find some  $\iota \in \mathcal{L}_{T+1-\tau} \cap \mathcal{S}_{T+1-s}$
  - 13:      $q_{0,i}^{T+1-\iota} \leftarrow q_{0,i}^{T+1-\iota} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_{0,i}^\tau \leftarrow q_{0,i}^\tau - \sum_{t=1}^T d_{i,\tau,s}^t$
  - 14:      $q_i^{T+1-\iota} \leftarrow q_i^{T+1-\iota} + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$
  - 15:   **else if**  $\tau > s$  **then**
  - 16:     Find some  $\iota \in \mathcal{L}_\tau \cap \mathcal{S}_s$
  - 17:      $q_{0,i}^t \leftarrow q_{0,i}^t + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_{0,i}^\tau \leftarrow q_{0,i}^\tau - \sum_{t=1}^T d_{i,\tau,s}^t$
  - 18:      $q_i^t \leftarrow q_i^t + \sum_{t=1}^T d_{i,\tau,s}^t$ ,  $q_i^s \leftarrow q_i^s - \sum_{t=1}^T d_{i,\tau,s}^t$
  - 19:   **end if**
  - 20: **end for**
  - 21:  $\hat{q}_0^t \leftarrow \sum_{i=1}^N q_{0,i}^t$ ,  $\hat{q}_i^t \leftarrow q_i^t$ ,  $\forall t = 1, \dots, T$ ,  $i = 1, \dots, N$
-

## Appendix D Proofs

### D.1 Proof of Lemma 5.4

Consider an arbitrary split demand induced by the given subproblem solutions, say  $d_{i,\tau,s}^t$ , which is a portion of  $d_i^t$  and ordered by the warehouse and the retailer in periods  $\tau, s$ , respectively. Let  $H^b(d_{i,\tau,s}^t)$  be the total subproblem holding cost for this split demand before recombination and  $H^a(d_{i,\tau,s}^t)$  the OWMR holding cost after recombination. We have  $H^b(d_{i,\tau,s}^t) = [h_{0,i} \cdot (t - \tau) + h_{i,i} \cdot (t - s)] \cdot d_{i,\tau,s}^t$ .

- i) Case I,  $h_i \geq h_0$ :  $H^a(d_{i,\tau,s}^t) = [h_0 \cdot (s - \tau) + h_i \cdot (t - s)] \cdot d_{i,\tau,s}^t \leq [h_0 \cdot (t - \tau) + h_i \cdot (t - s)] \cdot d_{i,\tau,s}^t = [2h_{0,i} \cdot (t - \tau) + 2h_{i,i} \cdot (t - s)] \cdot d_{i,\tau,s}^t = 2H^b(d_{i,\tau,s}^t)$ .
- ii) Case I,  $h_i < h_0$ : Let  $\xi$  be the period that both orders are re-assigned to. Then,  $H^a(d_{i,\tau,s}^t) = h_i \cdot (t - \xi) \cdot d_{i,\tau,s}^t = [h_i \cdot (s - \xi) + h_i \cdot (t - s)] \cdot d_{i,\tau,s}^t \leq [2h_{0,i} \cdot (t - \tau) + 2h_{i,i} \cdot (t - s)] \cdot d_{i,\tau,s}^t = 2H^b(d_{i,\tau,s}^t)$ .
- iii) Case II:  $H^a(d_{i,\tau,s}^t) = h_i \cdot (t - \iota) \cdot d_{i,\tau,s}^t \leq 2h_{i,i} \cdot (t - s) \cdot d_{i,\tau,s}^t \leq 2H^b(d_{i,\tau,s}^t)$ .

Summing  $H^a(d_{i,\tau,s}^t)$  over all split demands  $d_{i,\tau,s}^t$ , we obtain  $H^a \leq 2H^b$ .

### D.2 Proof of Lemma 7.3

We first have  $H^b = \sum_{t=1}^T \sum_{i=1}^N (H_{0,i}(I_{0,i}^t) + H_i(I_i^t))$  and  $H^a = \sum_{t=1}^T \sum_{j=0}^N H_j(\hat{I}_j^t)$ , where  $I$  and  $\hat{I}$  are the inventory quantities implied by the initial subproblem solutions and the recombined OWMR solution, respectively. Consider an arbitrary split demand  $d_{i,\tau,s}^t$  induced by the initial solutions. Let  $H^a(d_{i,\tau,s}^t)$  be the associated holding costs after recombination. Let  $\bar{H}_j(I) = H_j(I)/I$  be the average unit holding cost of carrying inventory quantities  $I \in \mathbb{R}_+$  for one period at facility  $j = 0, \dots, N$ .

- i) Case I,  $\tau = s$  or  $H_0(\cdot) \leq H_i(\cdot)$ : The initial schedules are respected, so

$$\begin{aligned} H^a(d_{i,\tau,s}^t) &= \sum_{\sigma=\tau}^{s-1} \bar{H}_0(\hat{I}_0^\sigma) \cdot d_{i,\tau,s}^t + \sum_{\sigma=s}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t \\ &\leq \sum_{\sigma=\tau}^{t-1} \bar{H}_0(\hat{I}_0^\sigma) \cdot d_{i,\tau,s}^t + \sum_{\sigma=s}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t. \end{aligned} \quad (13)$$

- ii) Case I,  $H_0(\cdot) > H_i(\cdot)$ : The common split period  $T + 1 - \iota \in [\tau, s]$  and the warehouse does not keep inventory after recombination, so

$$H^a(d_{i,\tau,s}^t) = \sum_{\sigma=T+1-\iota}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t \leq \sum_{\sigma=\tau}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t. \quad (14)$$

- iii) Case II: The common split period  $\iota \in [s, \tau]$  and the warehouse does not keep inventory after recombination, so

$$H^a(d_{i,\tau,s}^t) = \sum_{\sigma=\iota}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t \leq \sum_{\sigma=s}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t. \quad (15)$$



Summing (13)-(15) over all split demands  $d_{i,\tau,s}^t$ , we have

$$H^a \leq \sum_{t=1}^T \sum_{\tau=1}^T \sum_{s=1}^T \sum_{i=1}^N H^a(d_{i,\tau,s}^t) \quad (16a)$$

$$\begin{aligned} &\leq \sum_{1 \leq \tau \leq s \leq t \leq T} \left[ \sum_{i \in \tilde{\mathcal{N}}} \left( \sum_{\sigma=\tau}^{t-1} \bar{H}_0(\hat{I}_0^\sigma) + \sum_{\sigma=s}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \right) \cdot d_{i,\tau,s}^t + \sum_{i \in \underline{\mathcal{N}}} \sum_{\sigma=\tau}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t \right] \\ &\quad + \sum_{1 \leq s < \tau \leq t \leq T} \sum_{i=1}^N \sum_{\sigma=s}^{t-1} \bar{H}_i(\hat{I}_i^\sigma) \cdot d_{i,\tau,s}^t \end{aligned} \quad (16b)$$

$$\leq \sum_{1 \leq \tau \leq \sigma \leq s \leq t \leq T} \left( \sum_{i \in \tilde{\mathcal{N}}} d_{i,\tau,s}^t \cdot \bar{H}_0(\hat{I}_0^\sigma) + \sum_{i \in \underline{\mathcal{N}}} d_{i,\tau,s}^t \cdot \bar{H}_i(\hat{I}_i^\sigma) \right) + \sum_{\sigma=1}^T \sum_{i=1}^N \left( \sum_{s=1}^{\sigma} \sum_{t=s}^T \sum_{\tau=1}^t d_{i,\tau,s}^t \right) \cdot \bar{H}_i(\hat{I}_i^\sigma) \quad (16c)$$

$$\leq \sum_{1 \leq \tau \leq \sigma \leq T} \gamma_{0,\tau}^\sigma \cdot \sum_{i=1}^N H_{0,i}(I_{0,i}^\tau) + \sum_{i=1}^N \sum_{\sigma=1}^T \sum_{s=1}^T \gamma_{i,s}^\sigma \cdot H_i(I_i^s) \quad (16d)$$

$$\leq \sum_{t=1}^T \sum_{\sigma=1}^T \sum_{i=1}^N (\gamma_{0,t}^\sigma \cdot H_{0,i}(I_{0,i}^t) + \gamma_{i,t}^\sigma \cdot H_i(I_i^t)) \quad (16e)$$

$$\leq 2 \lceil \log T \rceil \cdot \sum_{t=1}^T \sum_{i=1}^N (H_{0,i}(I_{0,i}^t) + H_i(I_i^t)) \quad (16f)$$

$$= 2 \lceil \log T \rceil \cdot H^b, \quad (16g)$$

where  $\tilde{\mathcal{N}} = \{i : H_0(\cdot) \leq H_i(\cdot), 1 \leq i \leq N\}$  and  $\underline{\mathcal{N}} = \{i : H_0(\cdot) > H_i(\cdot), 1 \leq i \leq N\}$ .  $\gamma_{j,t}^\sigma$  is an indicator which equals 1 if an initial order placed by facility  $j \in \{0, \dots, N\}$  in period  $t$  is reallocated to period  $\sigma$ , 0 otherwise. (16a) is implied by holding cost subadditivity, (16b)-(16c) are direct consequences of (13)-(15), (16d)-(16e) follow by definition of  $\gamma_{j,t}^\sigma$ , holding cost subadditivity as well as monotonicity, and (16f)-(16g) are given by Lemma 6.2.