# A Brief Lecture on Submodular Functions

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**Definition 1.** Let $N = \{1, \ldots, n\}$. A *set function* is a function $f : 2^N \to \mathbb{R}$. Throughout this lecture, we assume $f(\varnothing) = 0$. A set function $f$ is *submodular* if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \forall\ S, T \subseteq N.$$

A function is *supermodular* if its negation is submodular, and it is *modular* if it is both super- and submodular.

The *submodular polyhedron* associated with a submodular function $f$ is given by

$$P(f) = \{x \in \mathbb{R}^N : x(S) \leq f(S), \forall\ S \subseteq N\},$$

where $x(S) = \sum_{i \in S} x_i$, and the *base polyhedron* of $f$ is

$$B(f) = \{x \in P(f) : x(N) = f(N)\}.$$

Why should we care about submodular functions? The next two examples give some ideas.

**Example 1.** A *matroid* on $N$ is a collection $\mathcal{I} \subseteq 2^N$ satisfying the following three axioms:

*i)* $\varnothing \in \mathcal{I}$.

*ii)* If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

*iii)* For any $S \subseteq N$, all maximal subsets of $S$ contained in $\mathcal{I}$ have the same cardinality.

A member of $\mathcal{I}$ is called an *independent set*. Two textbook examples of matroids are the collection of edge sets of sub-forests of a graph and the collection of linearly independent sets of columns of a matrix.

By axiom (*iii*), we can define a function $r : 2^N \to \mathbb{Z}_+$ as

$$r(S) = \max\big\{|A| : A \subseteq S, A \in \mathcal{I}\big\}, \forall\ S \subseteq N.$$

The function $r$ is known as the *rank function* of the matroid $\mathcal{I}$, and it is submodular. In fact, matroids can equivalently be defined in terms of rank functions. Let $r : 2^N \to \mathbb{Z}_+$ be a submodular function that satisfies the following additional conditions:

- $r(S) \leq |S|, \forall\, S \subseteq N$.
- If $S \subseteq T$, then $r(S) \leq r(T)$.

Then $r$ defines a matroid by setting $\mathcal{I} = \big\{A \subseteq N : r(A) = |A|\big\}$.

**Example 2.** A *cooperative game* is a set $N$ of *players* and a *characteristic function* $c : 2^N \to \mathbb{R}$. For any $S \subseteq N$, the function value $c(S)$ represents the cost that the player set $S$ incurs when working together. In cooperative game theory, we assume the entire set of players $N$, called the *grand coalition*, decides to cooperate and incur cost $c(N)$. The main question is how to allocate this cost among the player set so that no subset of players has incentive to leave the grand coalition. One such *solution concept* is the *core* of the game, given by the set

$$\text{core}(N, c) = B(c).$$

Intuitively, a cost allocation in the core splits $c(N)$ so that no subset $S \subseteq N$ of players pays more than $c(S)$, which is the cost it would incur by leaving the grand coalition.

When $c$ is submodular, the game $(N, c)$ is called *convex*, and it has many nice properties. Among them, the core is always non-empty, and an allocation in the core can be found in polynomial time using the *greedy algorithm*.

**Theorem 1.** *Let $f : 2^N \to \mathbb{R}$ be submodular. Then $B(f) \neq \varnothing$.*

*Proof.* Let $S_0 = \varnothing$, $S_i = \{1, \dots, i\}, \forall\, i = 1, \dots, n$, and define

$$x_i = f(S_i) - f(S_{i-1}), \forall\, i = 1, \dots, n.$$

Clearly, $x(N) = f(N)$. So let $T \subseteq N$; we prove $x(T) \leq f(T)$ by induction on $|T|$. The base case ($T = \varnothing$) is trivial, so let $T \neq \varnothing$ and let $i \in N$ be the minimal element that satisfies $S_i \supseteq T$. Then

$$\begin{aligned} f(T) &\geq f(T \cup S_{i-1}) + f(T \cap S_{i-1}) - f(S_{i-1}) \\ &= f(S_i) - f(S_{i-1}) + f(T \setminus \{i\}) \\ &= x_i + f(T \setminus \{i\}) \\ &\geq x_i + x(T \setminus \{i\}) = x(T), \end{aligned}$$

where the first inequality is a consequence of the submodularity of $f$, and the second follows from the induction hypothesis. □

**Example 3.** Let $G = (V, E)$ be an undirected graph, and let $c \in \mathbb{R}_+^E$ be a *capacity* vector. For any $S \subseteq V$, define

$$\delta(S) = \{e \in E : e \cap S = 1\}.$$

Then the function $S \mapsto c(\delta(S))$, known as the *cut function*, is submodular. Note that the non-negativity of the capacity is necessary for the cut function to be submodular.

**Definition 2.** Let $f : 2^N \to \mathbb{R}$, $z \in \mathbb{R}^N$. Let $\{1, \dots, n\} = N$ be an ordering of the elements of $N$ that satisfies $z_1 \geq \cdots \geq z_n$, and define $S_i = \{1, \dots, i\}, \forall\, i \in N$. The *Lovász extension* of $f$ at $z$ is defined by

$$\hat{f}(z) = \sum_{i=1}^{n-1} (z_i - z_{i+1}) f(S_i) + z_n f(S_n).$$

Note that $\hat{f}$ is well-defined, positively homogeneous, and satisfies

$$\hat{f}(e_S) = f(S), \forall\, S \subseteq N,$$

where $e_S \in \{0, 1\}^N$ is the characteristic vector of $S$.

**Theorem 2.** *Let $f : 2^N \to \mathbb{R}$, and let $\hat{f}$ be the Lovász extension of $f$. Then $f$ is submodular iff $\hat{f}$ is convex.*

*Proof.* ($\Leftarrow$) Let $\hat{f}$ be convex, and let $S, T \subseteq N$. Then

$$\frac{1}{2}\hat{f}(e_S) + \frac{1}{2}\hat{f}(e_T) \geq \hat{f}\left(\frac{1}{2}(e_S + e_T)\right),$$

or equivalently,

$$f(S) + f(T) \geq \hat{f}(e_S + e_T).$$

Moreover, we have

$$(e_S + e_T)_i = \begin{cases} 2, & i \in S \cap T \\ 1, & i \in S \setminus T \text{ or } i \in T \setminus S \\ 0, & \text{otherwise,} \end{cases}$$

which by the definition of $\hat{f}$ implies $\hat{f}(e_S + e_T) = f(S \cap T) + f(S \cup T)$.

($\Rightarrow$) We consider the dual pair of LP's

$$\begin{aligned} \max\ & z^\mathsf{T} x \\ \text{s.t. } & x(S) \le f(S), \forall\ S \subsetneq N \\ & x(N) = f(N), \end{aligned} \tag{P$_1$}$$

and

$$\begin{aligned} \min\ & \sum_{S \subseteq N} y_S f(S) \\ \text{s.t. } & \sum_{S \subseteq N} y_S e_S = z \\ & y_S \ge 0, \forall\ S \subsetneq N. \end{aligned} \tag{D$_1$}$$

Define $S_i, \forall\ i \in N$ as in Definition 2, let

$$x_i^* = f(S_i) - f(S_{i-1}), \forall\ i \in N,$$

and let

$$y_S^* = \begin{cases} z_i - z_{i+1}, & S = S_i, \forall\ i = 1, \ldots, n-1 \\ z_n, & S = N \\ 0, & \text{otherwise.} \end{cases}$$

By construction, $x^*$ is feasible for (P$_1$) and $y^*$ is feasible for (D$_1$). Moreover, we have

$$\begin{aligned} z^\mathsf{T} x^* &= \sum_{i \in N} z_i \big( f(S_i) - f(S_{i-1}) \big) \\ &= \sum_{i=1}^{n-1} (z_i - z_{i+1}) f(S_i) + z_n f(S_n) \\ &= \sum_{S \subseteq N} y_S^* f(S) = \hat{f}(z). \end{aligned}$$

Therefore, $x^*$ and $y^*$ are primal and dual optimal, respectively, and $\hat{f}(z)$ is convex since it is the support function of the set $B(f)$. $\square$

**Corollary 3.** *The greedy algorithm yields an optimal solution to* $\max\{z^\mathsf{T} x : x \in B(f)\}$, *for any submodular set function $f$ and any $z \in \mathbb{R}^N$.*

**Corollary 4.** *For any submodular set function $f$, the system of inequalities that defines $B(f)$ is TDI.*

We next turn our attention to pairs of submodular functions. The following result is a generalization of Edmonds' Matroid Intersection Theorem.

**Theorem 5.** *Let $f_1, f_2 : 2^N \to \mathbb{R}$ be submodular. Then*

$$\max\{x(N) : x \in P(f_1) \cap P(f_2)\} = \min\{f_1(S) + f_2(N \setminus S) : S \subseteq N\}.$$

*Proof.* We consider the dual pair of LP's

$$\begin{aligned} \max\ & z^\mathsf{T} x \\ \text{s.t. } & x(S) \le f_1(S), \forall\ S \subseteq N \\ & x(S) \le f_2(S), \forall\ S \subseteq N, \end{aligned} \tag{P$_2$}$$

and

$$\begin{aligned} \min\ & \sum_{S \subseteq N} \big(y_S^1 f_1(S) + y_S^2 f_2(S)\big) \\ \text{s.t. } & \sum_{S \subseteq N} (y_S^1 + y_S^2) e_S = z \\ & y \ge 0. \end{aligned} \tag{D$_2$}$$

Clearly, (P$_2$) is always feasible, so choose $z$ to make it finite.

*Claim.* (D$_2$) has an optimal solution $y^*$ where $\mathcal{C}_i = \{S \subseteq N : y_S^{i*} > 0\}$ is a chain, for $i = 1, 2$.

*Proof of claim.* Consider any feasible $y$ with $y_S^i \ge y_T^i > 0$ and $S \setminus T \ne \varnothing$, $T \setminus S \ne \varnothing$. Then

$$\begin{aligned} y_S^i e_S + y_T^i e_T &= (y_S^i - y_T^i) e_S + y_T^i (e_S + e_T) \\ &= y_T^i e_{S \cap T} + (y_S^i - y_T^i) e_S + y_T^i e_{S \cup T}, \end{aligned}$$

and, by the submodularity of $f_i$,

$$\begin{aligned} y_S^i f_i(S) + y_T^i f_i(T) &= (y_S^i - y_T^i) f_i(S) + y_T^i \big(f_i(S) + f_i(T)\big) \\ &\ge y_T^i f_i(S \cap T) + (y_S^i - y_T^i) f_i(S) + y_T^i f_i(S \cup T). \end{aligned}$$

Therefore, we can define a new solution $\tilde{y}^i$ as

$$\tilde{y}_U^i = \begin{cases} y_U^i + y_T^i, & U = S \cap T, S \cup T \\ y_S^i - y_T^i, & U = S \\ 0, & U = T \\ y_U^i, & \text{otherwise.} \end{cases}$$

The new solution is feasible and has an objective value equal or better than the original solution. In particular, we can apply this procedure repeatedly to an optimal solution to obtain an optimal solution of the required form. $\square$

Let $y^*$ be an optimal solution satisfying the claim's condition. It can be shown that the restriction of the constraint matrix of ($D_2$) to the columns indexed by $\mathcal{C}_1, \mathcal{C}_2$ is totally unimodular. Therefore, for $z \in \mathbb{Z}^N$, $y^*$ is an integral vector. In particular, if $z = e_N$, then $y^*$ is $\{0,1\}$-valued, implying that $\mathcal{C}_1 = \{S\}$ and $\mathcal{C}_2 = \{N \setminus S\}$, for some $S \subseteq N$. $\square$

**Corollary 6.** *Let $f_1, f_2 : 2^N \to \mathbb{R}$ be submodular. The system of inequalities defined by $P(f_1) \cap P(f_2)$ is TDI.*

**Corollary 7** (Frank's Discrete Separation Theorem)**.** *Let $f, g : 2^N \to \mathbb{R}$ be sub- and supermodular, respectively, and suppose they satisfy*

$$f(S) \geq g(S), \forall\ S \subseteq N.$$

*Then $\exists\ x \in \mathbb{R}^N$ satisfying*

$$f(S) \geq x(S) \geq g(S), \forall\ S \subseteq N.$$

*Moreover, if $f$ and $g$ are integer-valued, the separating vector $x$ can be chosen from $\mathbb{Z}^N$.*

*Proof.* Take

$$\begin{aligned} f_1(S) &= f(S), \forall\ S \subseteq N \\ f_2(S) &= g(N) - g(N \setminus S), \forall\ S \subseteq N. \end{aligned}$$

Then

$$\begin{aligned} \max\{x(N) : x \in P(f_1) \cap P(f_2)\} &= \min\{f_1(S) + f_2(N \setminus S) : S \subseteq N\} \\ &= \min\{f(S) + g(N) - g(S) : S \subseteq N\} \\ &= g(N) + \min\{f(S) - g(S) : S \subseteq N\} \\ &= g(N), \end{aligned}$$

where we can take $S = \varnothing$ as a minimizer on the right-hand side. Any maximizer $x$ of the left-hand side above satisfies $x(S) \leq f(S), \forall\ S \subseteq N$ and

$$x(S) \leq g(N) - g(N \setminus S) = x(N) - g(N \setminus S), \forall\ S \subseteq N,$$

which we can rewrite as $x(S) \geq g(S), \forall\ S \subseteq N$. If $f$ and $g$ are integer-valued, then by Corollary 6 we know that $P(f_1) \cap P(f_2)$ is integral, implying that $x$ can be chosen integral as well. $\square$

## Notes

For anyone interested in learning more about submodular functions, Nemhauser and Wolsey's text [NW99, Chapter III.3] is a good place to start. Unlike other texts that cover matroids, this book emphasizes the role of submodular rank functions, and many of the proofs use submodularity. With the exception of the Lovász extension and Frank's separation theorem, everything in this lecture is proved there. The ultimate authority on submodular functions is probably Fujishige, and his book on them [Fuj05] has everything, including the two combinatorial algorithms for submodular minimization (in the second edition only.) The second volume of Schrijver's set on combinatorial optimization [Sch03] is a great reference both for matroids and submodular functions, and includes Schrijver's own submodular minimization algorithm. Finally, anyone looking for an additional level of abstraction from submodular functions should look at Murota's monograph on discrete convex analysis [Mur03]. This book introduces the notion of L-convexity and M-convexity, and every result in this lecture can be generalized in this paradigm. This was also my main source for the proofs in this lecture.

Although matroids have been studied as far back as the 1930's, the polyhedral results and their extensions to general submodular functions in this lecture come mainly from the work of Edmonds (see, for example, [Edm70].)

Shapley was the first to consider convex cooperative games in [Sha71], although he considered *value games*, which have a supermodular characteristic function $v : 2^N \to \mathbb{R}$ that represents the value that a set of players can obtain by working together. (In fact, Shapley actually called these functions *convex set functions*, which unfortunately has led to some confusion.) The core is then defined with the inequalities reversed. However, cost and value games are in a sense equivalent, because we can define a submodular cost function $c(S) = v(N) - v(N \setminus S)$ that has the same core as the original value function $v$.

The Lovász extension was introduced by Lovász in [Lov83], who used it to characterize submodular functions. However, the term *Lovász extension* was coined by Fujishige, I think.

As you might expect, Frank's separation theorem was originally proved by Frank in [Fra82].

## References


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