Linear Algebra

Linear Equations

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Systems of Linear Equations

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- systems of linear equations
- elementary row operations
- solving linear systems

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- identify coefficients and variables in a linear system
- apply elementary row operations to solve linear systems of equations

A Single Linear Equation

A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the number of variables.

For example,

- $2x_1 + 4x_2 = 4$ is one equation with two variables
- $3x_1 + 2x_2 + x_3 = 6$ is one equation with three variables

Systems of Linear Equations

When we have one or more linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\begin{aligned} x_1 + 1.5x_2 + 0.9x_3 &= 4 \\ 5x_1 + 7x_3 &= 5 \end{aligned}$$

We might want to know:

- what values of the unknowns satisfy both equations?
- what procedure can we use to identify those values?

The Solution Set

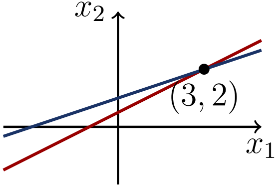
Definition: A Solution of a Linear System

The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution set** of the system. One point in the solution set is a **solution**.

Two Variable Case

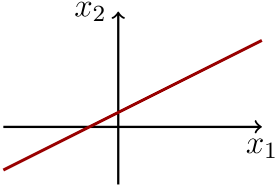
The equation of the form $a_1x_1 + a_2x_2 = b$ defines a line. How many different ways can two lines intersect?

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$



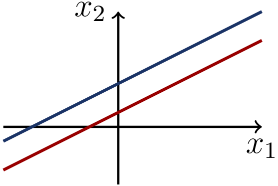
non-parallel lines
exactly one solution

$$x_1 - 2x_2 = -1$$
$$-x_1 + 2x_2 = 1$$



identical lines
infinitely many solutions

$$x_1 - 2x_2 = -1$$
$$-x_1 + 2x_2 = 3$$



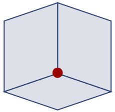
parallel lines
no solutions

Three Variable Case

The equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane. The **solution set** to a system of **three equations** is the set of points were all planes intersect.

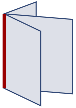
How many different ways can three planes intersect?

planes intersect at a point



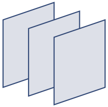
unique solution

planes intersect on a line



infinite number of solutions

parallel planes



no solution

Number of Solutions

Theorem: the Number of Solutions to a Linear System

The solution set to a system of linear equations can only have

- exactly one point (there is a unique solution), or
- infinitely many points (there are many solutions), or
- no points (there are no solutions)

Later in this course we will see why these are the only three possibilities.

Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?

We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one equation to another.
2. (Interchange) Interchange two equations.
3. (Scaling) Multiply an equation by a non-zero scalar.

When we apply these operations to a linear system we do not change the solution set. Let's use these operations to solve a system of equations.

Example: Solving a Linear System

Identify the solution set of the linear system.

$$\begin{aligned} x_1 \quad\quad\quad\quad & -7x_3 = 8 \\ 2x_2 & -8x_3 = 8 \\ 2x_1 \quad\quad\quad\quad & -2x_3 = 4 \end{aligned}$$

Summary

We explored the following concepts in this video.

- systems of linear equations
- elementary row operations
- applying elementary row operations to solve a linear system

Linear Algebra

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Consistent Systems

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- augmented matrices
- fundamental questions of existence and uniqueness of solutions
- row equivalence

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- express a set of linear equations as an augmented matrix
- characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent

Augmented Matrices

It is redundant to write $x_1, x_2, \ldots$ again and again. So we rewrite systems using matrices. For example,

$$\begin{aligned} x_1 \quad -2x_2 \quad +x_3 &= 0 \\ 2x_2 \quad -8x_3 &= 7 \end{aligned}$$

can be written as the **augmented matrix**,

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 7 \end{array}\right)$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3. Row operations can be applied to rows of augmented matrices as though they were coefficients in a system.

Consistent Systems and Row Equivalence

Definition: Consistent

A linear system is **consistent** if it has at least one solution.

Definition: Row Equivalence

Two matrices are **row equivalent** if a sequence of row operations transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then the systems have the same solution set.

Example for Consistent Systems and Row Equivalence

Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

1. Are A and B row equivalent? Are A and C row equivalent?

Example for Consistent Systems and Row Equivalence

Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

2. Do the augmented matrices $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right)$ and $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$ correspond to consistent systems?

Summary: Fundamental Questions

In this video we explored the following concepts.

- Augmented matrices, row equivalence, and consistent systems.
- Fundamental questions that we revisit many times throughout our course:
  1. Does a given linear system have a solution? In other words, is it consistent?
  2. If it is consistent, is the solution unique?

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Echelon Form and RREF

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- echelon form and row reduced echelon form

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- identify whether a matrix is in echelon form or in row reduced echelon form (RREF)
- give examples of matrices in echelon form or in RREF

Motivation: Identifying a Solution to a Linear System

This matrix below in a form referred to as **row reduced echelon form**.

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 7 \end{array}\right)$$

By inspection, what is the solution to the linear system?

Definition: Echelon Form

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. All entries below a leading entry (if any) are zero.

Examples

Matrix A is in echelon form. B is not in echelon form.

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Definition: Echelon Form

A matrix in echelon form is in **row reduced echelon form** (RREF) if

1. All leading entries, if any, are equal to 1.
2. Leading entries are the only nonzero entry in their respective column.

Examples

Matrix A is in RREF. B is not in RREF.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 6 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example of a Matrix in Echelon Form

$\blacksquare$ = non-zero number, $\quad *$ = any number

$$\begin{pmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example

Which of the following are in RREF?

$a)\quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$b)\quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$c)\quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$d)\quad \begin{pmatrix} 0 & 6 & 3 & 0 \end{pmatrix}$

$e)\quad \begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Summary: Echelon and RREF

In this video we explored the following concepts.

- echelon and row reduced echelon forms
- identifying whether a matrix is in echelon or in RREF

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The Row Reduction Algorithm

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- row reduction algorithm
- pivots and pivot columns

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a linear system in terms of the number of leading entries, pivots, pivot columns, pivot positions
- apply the row reduction algorithm to reduce a linear system to echelon form, or to RREF

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the row reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

Example: Express the matrix in RREF and identify the pivot columns.

$$\begin{pmatrix} 0 & -3 & -6 & 9 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{pmatrix}$$

Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

Step 1: Swap the first row with a lower one so the leftmost nonzero entry is in the first row

Step 2: Scale the 1st row so that its leading entry is equal to 1

Step 3: Use row replacement so all entries above and below this leading entry (if any) are equal to zero

Then repeat these steps for row 2, then for row 3, and so on, for the remaining rows of the matrix.

Notes on the Row Reduction Algorithm

- There are many algorithms for reducing a matrix to echelon form, or to RREF.
- If we only need to count pivots, we do not need RREF. Echelon form is sufficient.

Summary: Fundamental Questions

In this video we explored the following concepts.

- pivot, pivot columns, pivot positions
- the row reduction algorithm

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Existence and Uniqueness

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- consistency, existence, uniqueness
- pivots, and basic and free variables

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- determine whether a linear system is consistent from its echelon form
- apply the row reduction algorithm to compute the coefficients of a polynomial

Basic and Free Variables

Consider the augmented matrix

$$\left(A \mid \vec{b}\right) = \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right)$$

The leading one's are in first, third, and fifth columns.

- The pivot columns of A are the first, third, and fifth columns
- The corresponding variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3, and x_5. Variables that correspond to a pivot are **basic variables**.
- Variables that are not basic are **free variables**. They can take any value.
- The free variables are x_2 and x_4. Any choice of the free variables leads to a solution of the system.

Notes on Basic and Free Variables

- Note that a matrix, on its own, does not have basic variables or free variables. Systems have variables.
- If A has n columns, then the linear system

$$\left(A \mid \vec{b}\right)$$

must have n variables. One variable for each column of the matrix.

- There are two types of variables: basic and free. And a variable cannot be both free and basic at the same time.

$$\begin{aligned} n &= \text{number of columns of } A \\ &= (\text{number of basic variables}) + (\text{number of free variables}) \end{aligned}$$

Existence and Uniqueness

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Theorem

A linear system is consistent if and only if (exactly when) the last column of the **augmented** matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no free variables, and
2. infinitely many solutions that are parameterized by free variables.

Example: Existence and Uniqueness

If possible, determine the coefficients of the polynomial $y(t) = a_0 t + a_1 t^2$ that passes through the points that are given in the form (t, y).

a) $L(-1, 0)$ and $M(1, 1)$

b) $P(2, 0)$, $Q(1, 1)$, and $R(0, 2)$

Summary: Fundamental Questions

In this video we explored the following concepts.

- augmented matrices and consistent systems
- pivots, and basic and free variables
- fundamental questions that we will revisit throughout the course regarding consistency, existence, uniqueness

Linear Algebra

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Vectors in $\mathbb{R}^n$

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- vectors in $\mathbb{R}^n$, and their basic properties

Learning Objectives

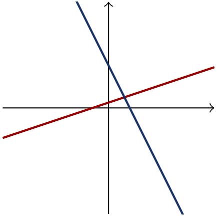
Students should be able to do the following after watching this video and completing the assigned homework.

- apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$$x - 3y = -3$$
$$2x + y = 8$$



This other perspective:

- gives us deeper insight into the properties of systems and their solutions
- requires that we introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

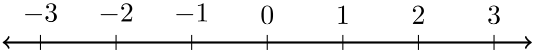
Definition of $\mathbb{R}^n$

$\mathbb{R}$ denotes the collection of all real numbers.

Let n be a positive whole number. We define

$$\mathbb{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.

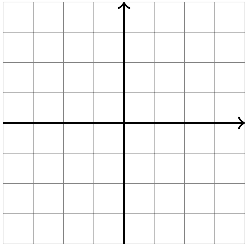


Definition of $\mathbb{R}^2$

Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

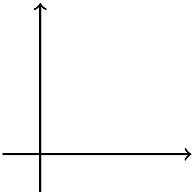
Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



Vectors as Points in $\mathbb{R}^n$

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.



For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column. For example, suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Vectors have the following properties.

1. **Scalar Multiples**:

$$c\vec{u} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$$

2. **Vector Addition**:

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

Note that vectors in higher dimensions have the same properties.

Parallelogram Rule for Vector Addition

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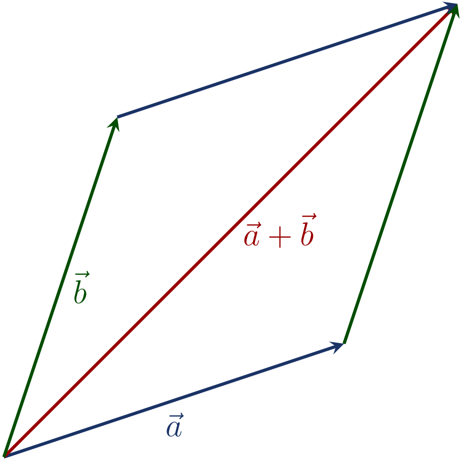

Summary

We explored the following concepts in this video.

- geometric and algebraic properties of vectors in $\mathbb{R}^n$
- vector algebra: compute vector additions and scalar multiplications

Linear Algebra

Linear Equations

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Linear Combinations

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- linear combinations of vectors

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a set of vectors in terms of **linear combinations**

Linear Combinations Definition

Definition

Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \dots, c_p$, the vector $\vec{y}$, where

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

is called a **linear combination** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ with weights $c_1, c_2, \dots, c_p$.

Linear Combinations Example

Can $\vec{y}$ be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{y} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Solution

If $\vec{y}$ can be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, we can find c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$. The vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$ is

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Can we represent this vector equation as a system of equations?

Linear Combinations Example

Our vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$ is

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This can be written as

$$\begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

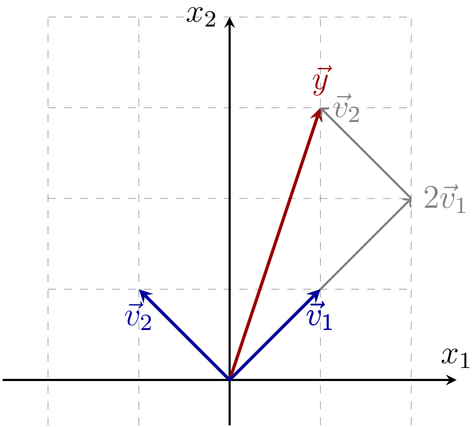
Thus, we have the linear system

$$c_1 - c_2 = 1$$
$$c_1 + c_2 = 3$$

There is a solution to this system, $c_1 = 2$, $c_2 = 1$. Therefore, $\vec{y}$ can be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

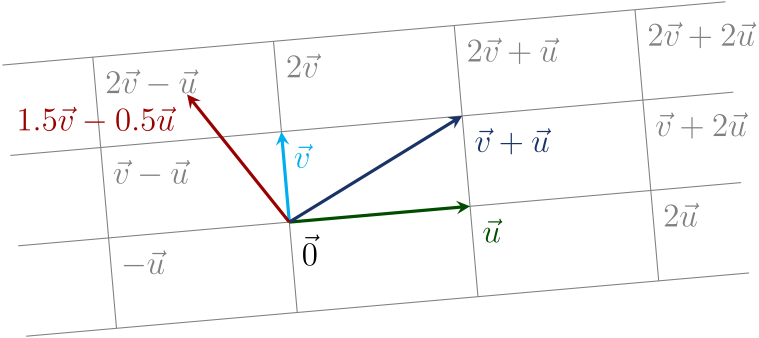
Linear Combinations Example

We found that $2\vec{v}_1 + \vec{v}_2 = \vec{y}$.



Geometric Interpretation of Linear Combinations

Any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors in $\mathbb{R}^2$ that are not multiples of each other.



Linear Combinations Example in $\mathbb{R}^3$

Can $\vec{y}$ be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{y} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Solution

If $\vec{y}$ can be represented as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, we can find c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$. The vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$ is

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Linear Combinations Example in $\mathbb{R}^3$

Expressing this as a linear system, we obtain

$$\begin{aligned} c_1 - c_2 &= 1 \\ c_1 + c_2 &= 3 \\ 0c_1 + 0c_2 &= 1 \end{aligned}$$

Thus, the system is inconsistent.

- There is no solution to this system.
- There are no values of c_1 and c_2 so that $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{y}$
- $\vec{y}$ cannot be expressed as a linear combination of the other two vectors.

Summary

We explored the following concepts in this video.

- characterizing a set of vectors in terms of **linear combinations**
- determining whether a given vector can be represented by a linear combination of a set of vectors

Linear Algebra

Linear Equations

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Span

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- the span of a set of vectors

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a set of vectors in terms of **linear combinations** and their **span**, and how they are related to each other geometrically

Span

Definition

Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Span Example

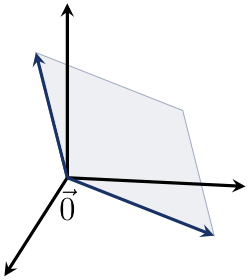
Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.



Summary

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We explored the following concepts in this video.

- characterizing a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically

Linear Algebra

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The Matrix-Vector Product

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- matrix notation for systems of equations
- the matrix product $A\vec{x}$

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- compute matrix-vector products
- express linear systems as vector equations and matrix equations

Multiple Representations

"Mathematics is the art of giving the same name to different things."
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

Notation for Dimensions of Vectors and Matrices

symbol	meaning
$\in$	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m \times n}$	the set of real-valued matrices with m rows and n columns

Example

The notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.

Matrix-Vector Product as a Linear Combination

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Definition

If $A \in \mathbb{R}^{m \times n}$ has columns $\vec{a}_1, \ldots, \vec{a}_n$ and $\vec{x} \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of A.

$$A\vec{x} = \begin{pmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Note that $A\vec{x}$ is in the span of the columns of A.

Linear Combination Examples

Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

1. The following product can be written as a linear combination of vectors:

   $$A\vec{x} =$$

2. Is $\vec{b} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$ in the span of the columns of A?

Summary

We explored the following concepts in this video.

- computing matrix-vector products
- expressing linear systems as vector equations and matrix equations

Linear Algebra

Linear Equations

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Existence of Solutions

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- solution sets

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- express linear systems as vector equations and matrix equations
- characterize solution sets of linear systems using the concepts of span, linear combinations

Equivalent Solution Sets

Note that if A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and $\vec{x} \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$A\vec{x} = \vec{b}$$

has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

which as the same set of solutions as the set of linear equations with the augmented matrix

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Linear Combinations and the Existence of Solutions

Theorem

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

This follows directly from our definition of $A\vec{x}$ being a linear combination of the columns of A.

Using Linear Combinations to Characterize a System

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

Multiple Representations of Linear Systems

We now have four **equivalent** ways of representing a linear system.

1. A list of equations: $2x_1 + 3x_2 = 7, \quad x_1 - x_2 = 5$
2. An augmented matrix: $\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$
3. A vector equation: $x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$
4. A matrix equation: $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$

Each representation gives us a different way to think about linear systems.

Summary

We explored the following concepts in this video.

- computing matrix-vector products
- expressing linear systems as vector equations and matrix equations
- characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots

Linear Algebra

Linear Equations

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Homogeneous Systems

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- homogeneous systems
- parametric **vector** forms of solutions to linear systems

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms

Homogeneous Systems

Definition

Linear systems of the form $A\vec{x} = \vec{0}$ are **homogeneous**.

Linear systems of the form $A\vec{x} = \vec{b},\ \vec{b} \neq \vec{0}$, are **inhomogeneous**.

Because homogeneous systems always have the **trivial solution**, $\vec{x} = \vec{0}$, the interesting question is whether they have **non-trivial** solutions.

Homogeneous Systems

Observation

$A\vec{x} = \vec{0}$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

Summary

We explored the following concepts in this video.

- characterizing homogeneous and inhomogeneous systems
- relationships between free variables, pivots, and solutions
- identifying free variables of homogeneous systems

Linear Algebra

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Parametric Vector Forms

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- homogeneous systems
- parametric **vector** forms of solutions to linear systems

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- express the solution set of a linear system in parametric vector form

Recall: Homogeneous Systems

Definition

Linear systems of the form $A\vec{x} = \vec{0}$ are **homogeneous**.

Linear systems of the form $A\vec{x} = \vec{b}$, $\vec{b} \neq \vec{0}$, are **inhomogeneous**.

These systems are related to each other in a way that is easier to see with **parametric vector form**.

Parametric Vector form of the Solution of a Non-homogeneous System

Write the solution as a sum of vectors. Give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 - x_2 - 5x_3 &= 1 \\ x_1 - 2x_3 &= 1 \end{aligned}$$

Note that the left-hand side is the same as a previous example.

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Parametric Forms, Homogeneous Case

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.

Summary

We explored the following concepts in this video.

- expressing the solution set of a linear system in parametric vector form
- the geometric relationship between the solution to $A\vec{x} = \vec{b}$ and $A\vec{x} = \vec{0}$

Linear Algebra

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A Definition of Linear Independence

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- linear independence
- geometric interpretation of linearly independent vectors

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a set of vectors and linear systems using the concept of linear independence

A Motivating Question

What is the smallest number of vectors needed in a parametric solution to a linear system?

Linear Independence

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the **trivial** solution. It is **linearly dependent** otherwise.

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

How to Establish Linear Independence

Georgia Tech

Consider the vectors:

$$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$$

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = V\vec{c} \overset{??}{=} \vec{0}$$

Linear independence: there is **NO** non-zero solution $\vec{c}$

Linear dependence: there is a non-zero solution $\vec{c}$.

Example: Determine Whether Set is Independent

For what values of h, if any, is the set of vectors linearly independent?

$$\begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}, \begin{pmatrix} 1 \\ h \\ 1 \end{pmatrix}, \begin{pmatrix} h \\ 1 \\ 1 \end{pmatrix}$$

Summary

We explored the following concepts in this video.

- characterizing a set of vectors using the concept of linear independence

Linear Algebra

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Linear Independence Theorems

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- linear independence
- geometric interpretation of linearly independent vectors

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a set of vectors and linear systems using the concept of linear independence
- construct dependence relations between linearly dependent vectors

A Motivating Question

What is the smallest number of vectors needed in a parametric solution to a linear system?

Recall: Linear Independence

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the **trivial** solution. It is **linearly dependent** otherwise.

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Example: Two Dependent Vectors

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent? Provide a geometric interpretation.

Solution

From our definition of linear dependence, if $\vec{v}_1, \vec{v}_2$ are dependent, then there exists a c_1 and a c_2, not **both** zero, so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

Example: Two Dependent Vectors

We consider two cases:

1) If $\vec{v}_1$ and/or $\vec{v}_2$ is the zero vector, then the vectors are dependent. If for example $\vec{v}_1 = \vec{0}$, then $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ is satisfied for $c_2 = 0$ and any c_1.
2) If $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$, then $\vec{v}_2 = -\frac{c_1}{c_2}\vec{v}_1$, so $\vec{v}_1$ and $\vec{v}_2$ are multiples of each other. The vectors are parallel (one vector is in the span of the other).

Example: Two Dependent Vectors (continued)

Thus, two vectors in $\mathbb{R}^n$ are dependent when either or both of the following occur.

- One or both vectors are the zero vector.
- One vector is a multiple of the other.

Linear Independence Theorems

1) **More Vectors Than Elements:** Suppose $\vec{v}_1, \dots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly dependent.

Wny? Every column of the matrix

$$A = (\vec{v}_1, \dots, \vec{v}_k)$$

would have to be pivotal for the vectors to be independent. But A has **more columns than rows**, so every column cannot be pivotal. The vectors must be linearly dependent.

Linear Independence Theorems

2) **Set Contains Zero Vector:** If any one or more of $\vec{v}_1, \ldots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

Wny? Every column of the matrix

$$A = (\vec{v}_1, \ldots, \vec{v}_k)$$

would have to be pivotal for the vectors to be independent. But A **has a zero column**, so every column cannot be pivotal. The vectors must be linearly dependent.

Application of our Linear Independence Theorems

By inspection, which matrices have linearly independent columns?

1. $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$ *zero column* $\Rightarrow$ *dependent*
2. $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ *more columns than rows* $\Rightarrow$ *dependent*
3. $C = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$ *last column is the sum of the first two* $\Rightarrow$ *dependent*
4. $D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *every column is pivotal* $\Rightarrow$ *linearly independent*

Summary

We explored the following concepts in this video.

- characterizing a set of vectors and linear systems using the concept of linear independence
- constructing dependence relations between linearly dependent vectors

Linear Algebra

Linear Equations

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Domain, Codomain, Range

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- the definition of a linear transformation
- domain, codomain, image, and range
- the interpretation of matrix multiplication as a linear transformation

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize linear transforms using the concepts of domain, codomain, image, and range

From Matrices to Functions

Let A be an $m \times n$ matrix. We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{x}$$

This is called a **matrix transformation**.

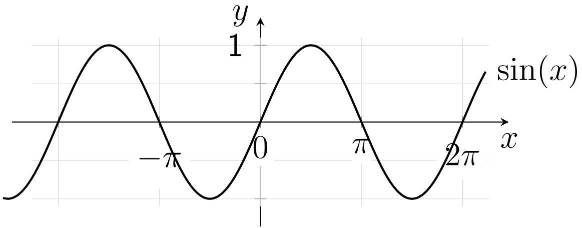
- The **domain** of T is $\mathbb{R}^n$.
- The **codomain** of T is $\mathbb{R}^m$.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T.
- The set of all possible images $T(\vec{x})$ is the **range**.

Functions from Calculus

Many of the functions we know have **domain** and **codomain** $\mathbb{R}$. We can express the **rule** that defines the function $\sin$ this way:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph. The horizontal axis is the **domain**, the vertical axis is the **codomain**.



Example: A Matrix Transformation

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad T(\vec{x}) = A\vec{x}$

a) What is the domain and codomain of T?

b) Compute the image of $\vec{u}$ under T.

c) What is the range of T?

From Matrices to Functions

The function

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{x}$$

gives us **another** interpretation of $A\vec{x} = \vec{b}$. We now have five ways of representing $A\vec{x} = \vec{b}$:

- set of linear equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Example: A Matrix Transformation as a System

Consider again the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, and associated transform $T(\vec{x}) = A\vec{x}$.

a) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$

b) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$.
or: Give a $\vec{c}$ that is not in the range of T.
or: Give a $\vec{c}$ that is not in the span of the columns of A.

Summary

We explored the following concepts in this video.

- Characterized linear transforms using the concepts of domain, codomain, image, and range.
- The interpretation of matrix multiplication as a linear transformation.

Linear Algebra

Linear Equations

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Geometric Interpretations of Linear Transforms

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- geometric interpretations of a linear transform

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear)

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$.

So if T is linear, then

$$T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$$

This is called the **principle of superposition**.

Fact: Every matrix transformation T_A is linear.

Geometric Interpretations of Transforms in $\mathbb{R}^2$

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

3) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ for $k \in \mathbb{R}$

Geometric Interpretations of Transforms in $\mathbb{R}^3$

Georgia Tech

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Constructing the Matrix of the Transformation

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}5\\-7\\2\end{pmatrix}, \qquad T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-3\\8\\0\end{pmatrix}$$

What is the matrix, A, so that $T = Ax$?

Summary

We explored the following concepts in this video.

- constructing linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ and geometric interpretations for them

We will need to go into more detail on linear transformations and their relationships to linear systems.

Linear Algebra

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The Standard Vectors

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- the **standard vectors** and the **standard matrix**

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- identify and construct linear transformations of a matrix

Definition: The Standard Vectors

The **standard vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \qquad \vec{e}_2 = \qquad \vec{e}_n =$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \qquad \vec{e}_2 = \qquad \vec{e}_3 =$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 =$$

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{pmatrix}$$

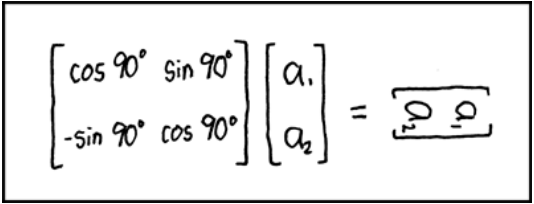
The matrix A is the **standard matrix** for a linear transformation.

Standard Matrix for a Counterclockwise Rotation

What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T(\vec{x}) = \vec{x} \textit{ rotated counterclockwise by angle } \theta?$$

Standard Matrix for a Clockwise Rotation



https://xkcd.com/184

Example: Constructing a Standard Matrix

Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

Is T one-to-one? Is T onto?

Summary

We explored the following concepts in this video.

- constructing linear transformations and standard matrices in $\mathbb{R}^2$
- constructing the standard matrix for a rotation matrix

The rotation matrix was just one of the standard matrices that are defined in the textbook. There are other standard matrices for transformations that we will explore.

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Standard Matrices of Linear Transforms

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- the **standard vectors** and the **standard matrix**
- two dimensional transformations in more detail

Learning Objectives

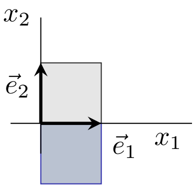
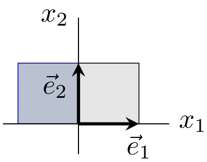
Students should be able to do the following after watching this video and completing the assigned homework.

- identify and construct linear transformations of a matrix

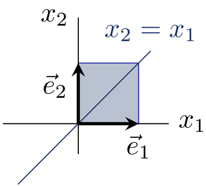
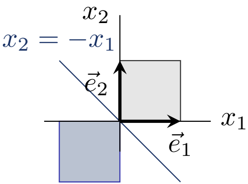
Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, ...).
- Please familiarize yourself with them: you are expected to memorize them, or be able to derive them.

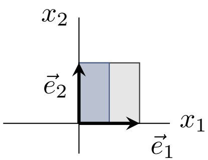
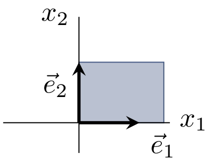
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1−axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection through x_2−axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

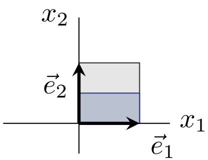
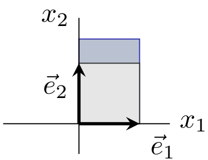
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through $x_2 = x_1$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

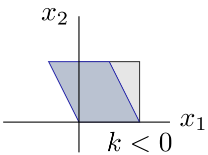
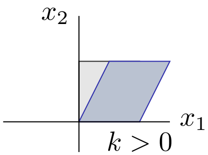
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
horizontal contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}. \; \lvert k\rvert < 1$
horizontal expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}, \; k > 1$

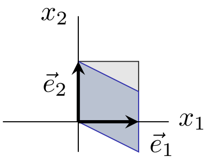
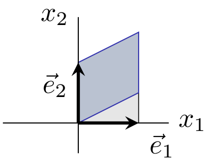
Two Dimensional Examples: Contractions and Expansions

transformation	image of unit square	standard matrix
vertical contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k\rvert < 1$
vertical expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

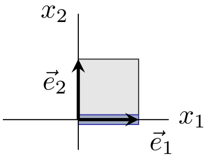
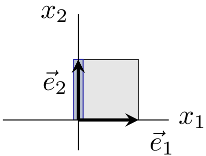
Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
horizontal shear (left)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k < 0$
horizontal shear (right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
vertical shear (down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$
vertical shear (up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Projections

transformation	image of unit square	standard matrix
projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Example: Composite Transform

Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.

Summary

We explored the following concepts in this video.

- constructing linear transformations in $\mathbb{R}^2$ and gave geometric interpretations for them
- constructing composite transform that involve two ore more linear transforms

Linear Algebra

Linear Equations

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Onto and One-to-One

Topics and Learning Objectives

Topics

We will explore the following concepts in this video.

- **onto** and **one-to-one** transformations

Learning Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize and construct linear transformations that are onto and/or one-to-one

Onto

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = A\vec{x} = \vec{b}$.

Implications

- Onto is an **existence property:** for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.
- T is onto if and only if its standard matrix has a pivot in every row.

One-to-One

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = A\vec{x} = \vec{b}$.

Implications

- One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.
- T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if every column of A is pivotal.

Example: Matrix Completion, One-to-one and Onto

Complete the matrices by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why**.

a) A is a 2×3 standard matrix for a one-to-one transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$

b) B is a 3×3 standard matrix for a transform that is one-to-one and onto.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}$$

Theorem for Onto Transforms

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A, these are equivalent statements.

1. T is onto.
2. A has columns that span $\mathbb{R}^m$.
3. Every row of A is pivotal.

Theorem for One-to-one Transforms

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A, these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. A has linearly independent columns.
4. Each column of A is pivotal.

Example: Constructing a Standard Matrix, One-to-one and Onto

Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

Construct the standard matrix for the transformation. Is T one-to-one? Is T onto?

Example: Linear Transform Review

Suppose A is an $m \times n$ standard matrix for transform T, and there are some vectors $\vec{b} \in \mathbb{R}^m$ that are not in the range of $T(\vec{x}) = A\vec{x}$.

True or false:

1. $A\vec{x} = \vec{b}$ could be inconsistent
2. there cannot be a pivot in every column of A
3. T could be one-to-one

Summary

We explored the following concepts in this video.

- constructing linear transformations of a matrix that are one-to-one and/or onto
- characterizing transforms that are one-to-one/onto