Supplemental Material for "Nonlinear focusing in dynamic crack fronts and the micro-branching transition"

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REGARDING THE DERIVATION OF EQS. (1) AND (2)

Our derivation follows closely [1]. Our starting point is the wave equation for ϕ,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} - c^{-2}\phi_{tt} = 0\,, \tag{S1}$$

which is complemented by the dynamic boundary condition,

$$\mu\partial_y\phi(x, 0^{\pm}, z; t) = p(x - vt);\;\; x < vt + f(z, t)\,, \tag{S2}$$

where μ is the elastic modulus and $c \propto \sqrt{\mu}$. We rescale ϕ and the coordinates, and write the wave equation in a frame of reference moving at velocity v. $x \to (x - vt)/l, y \to y/l, z \to z/l, ct/l \to t, v \to v/c, p \to p/\mu$ Eq. (**??**) and (S2) become

$$\alpha^2\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{tt} + 2v\phi_{xt} = 0\,, \tag{S3}$$

$$\partial_y\phi = p(x)x < f(z, t)\,. \tag{S4}$$

We define $\epsilon = max_{z,t}|f(z, t)|/l$ and the "inner" variables $X = \epsilon x, Y = \epsilon y$. Writing $\Phi(X, Y, z; t) = \phi(x, y, z; t)$, the eqution for the "inner" problem is

$$\alpha^2\Phi_{XX} + \Phi_{YY} + 2\epsilon v\Phi_{Xt} + \epsilon^2(\Phi_{zz} - \Phi_{tt}) = 0\,. \tag{S5}$$

This equation may be solved order-by-order in the small parameter ϵ by subtituting the expansion $\Phi = \epsilon^{1/2}\Phi^{(1/2)} + \epsilon^{3/2}\Phi^{(3/2)} + \epsilon^{5/2}\Phi^{(5/2)} + \mathcal{O}(\epsilon^{7/2})$. Its integration yields a number of constants that are to be determined by matching to the outer problem.

To solve the outer problem, we recover the original coordinates and solve the wave equation for the unperturbed problem,

$$\alpha^2\phi_{xx} + \phi_{yy} + \phi_{zz} - \phi_{tt} + 2v\phi_{xt} = 0\,, \tag{S6}$$

$$\partial_y\phi = p(x) \quad x < 0\,. \tag{S7}$$

The solution proceeds by developing ϕ in powers of ϵ and assuming a harmonic variation in z and t, namely, $\phi = \phi^{(0)} + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + ... = \phi^{(0)} + P_0\,\mathrm{Im}\{(\epsilon q^{(1)}(x, y) + \epsilon^2 q^{(2)}(x, y))e^{i(kz-\omega t)}\}$.

The functions $q^i(x, y)$ satisfy the equation

$$\alpha^2 q_{xx} + q_{yy} + (\omega^2 - k^2)q - 2iv\omega q_x = 0 \tag{S8}$$

and the boundary conditions

$$\begin{aligned} q(x, 0) &= 0;\; x > 0 \\ q_y(x, 0) &= 0;\; x < 0\,. \end{aligned} \tag{S9}$$

This problem may be solved by applying a Fourier transform along the x coordinate, and using a Wiener-Hopf decomposition to resolve the mixed boundary conditions on the fracture plane.

EXPLICIT 2$^{\mathrm{ND}}$ ORDER EQUATION OF MOTION FOR SCALAR CRACK FRONTS

Here we derive the explicit equation of motion used to propagate the crack front in our simulations. In the following, we define the scalar wave speed $c = 1$ and $\alpha = \sqrt{1 - v^2}$. Combining Eq. (1) and (3) of the main text, local energy balance reads

$$G_0 g(v_\perp)(1 + H[f]) = \tilde{\Gamma}_0(1 + a v_\perp)(1 + \delta A)\,, \tag{S10}$$

where the instantaneous position of the crack front is given by $x = vt + f(z,t)$. Approximating $v_\perp \simeq v + f_t - \frac{v}{2} f_z^2$, where subscripts denote derivatives, and keeping only terms up to the 2$^{\mathrm{nd}}$ order in f and δA, Eq. (S10) becomes

$$\left(1 - \frac{1}{\alpha^2}\left(f_t - \frac{v}{2} f_z^2\right) + \frac{1 - 2v}{2\alpha^4} f_t^2\right)(1 + H[f]) = \left(1 + \frac{a}{1 + av}\left(f_t - \frac{v}{2} f_z^2\right)\right)(1 + \delta A)\,, \tag{S11}$$

where we assumed that for the unperturbed crack, $G_0 g(v) = \tilde{\Gamma}_0(1 + av)$. To obtain an explicit expression for f_t, we substitute $H[f]$ with the right-hand-side of Eq. (2) and expand the parentheses in (S11). The resulting equation is quadratic in f_t, which we solve, retaining only terms that are up to 2$^{\mathrm{nd}}$ order in f. The result is an explicit equation of motion for f:

$$\begin{aligned} f_t =& \frac{v}{2} f_z^2 + \frac{1}{1+\chi}\left(-\Psi[f] - \frac{1 + 2\chi - \chi^2 + 4v}{4\alpha^2(1+\chi)^2}\Psi[f]^2 + \frac{1}{2\alpha^2}\Psi[f\Psi[f]] - \frac{1 - 2v}{4\alpha^2}\Psi_2[f^2] - \frac{1 + 2v}{2\alpha^2} f \Psi_2[f]\right) \\ &+ \frac{1}{1+\chi}\left(-\alpha^2 \delta A + \frac{\chi^2 - 2v}{(1+\chi)^2}\Psi[f]\delta A + \frac{2\chi(1-\chi) + 1 - 2v}{2(1+\chi)^2}\alpha^2 \delta A^2\right)\,. \end{aligned} \tag{S12}$$

$\Psi_2[f]$ is defined before Eq. (2) in the main text.

DETAILS OF THE NUMERICAL SOLUTION

We discretize Eq. (S12) in both z and t, assuming periodic boundary conditions. Since the problem does not contain an intrinsic length-scale we conveniently set the periodic z interval length to $Z = 2\pi$. The crack front is propagated in a straightforward Euler scheme. For a z-mesh of N points, the time step is defined as $\Delta = \frac{0.2}{N\alpha}$. This time step is sufficiently small to finely sample the oscillations of the Bessel functions appearing in the history functionals Ψ and Ψ_2 for the highest wavenumber $k_{max} = N/2$. The history functionals are evaluated in k-space by trapezoidal integration with the same time step Δ used to propagate the crack front, and then transformed back to z-space with a Fast Fourier Transform. This integration scheme results in an error of $\sim 0.1\%$. To prevent negative crack front velocities, we equated $f_t = -v$ locally when Eq. (S12) gives $v + f_t < 0$.

Disk-shaped obstacle — Fig. 1(d) and Fig. 2. For the crack front interaction with an obstacle, we assume that the front is straight and initially positioned at $x = 0$. The z interval is discretized over $N = 512$ mesh points. We insert an disk-shaped obstacle of diameter $d = 0.05\pi$ centered at $x = d/2 + 2\epsilon$ where $\epsilon = 20\pi/N$. Inside the obstacle diameter $\delta A = D = const.$ and outside $\delta A = D\exp\left(-2(r - d/2)^2/\epsilon^2\right)$, where r is the radial distance from the center of the obstacle.

Step-lines — Fig. 4. To speed up simulations while maintaining numerical accuracy, the crack front was discretized on a $N = 4096$ mesh which was made coarser at predetermined times as δA grew wider; we doubled the mesh size at $t_N = (10\pi/2^N)^2/\xi v - \xi/v$ which roughly corresponds to the times when w increased past 5 times the mesh size. Both crack front velocity and curvature evolved smoothly during the simulation, ensuring that re-meshing did not affect the overall dynamics.

FOCUSING FOR $\delta A = D\cos(z)$ AND FOR THE DISK OBSTACLE

Eq. (S12) may be solved analytically for the time-independent $\delta A = D\cos(z)$. Assuming that the spatial variation of the crack front is also time-independent, $\Psi[f] \to \alpha\mathcal{H}[f_z]$, $\Psi_2[f] \to -\alpha^2 f_{zz}/2$, defining the Hilbert transform $\mathcal{H}[g] = \pi^{-1}\int \frac{dz'}{z - z'} g(z')$. Then, Eq. (S12) reduces to

$$\begin{aligned} (1+\chi) f_t + \alpha\mathcal{H}[f_z] + \alpha^2\delta A =& -\frac{1 + 2\chi - \chi^2 + 4v}{4(1+\chi)^2}\mathcal{H}[f_z]^2 + \frac{1}{2}\mathcal{H}[\partial_z(f\mathcal{H}[f_z])] + \frac{1 + 2\chi v}{4} f_z^2 + \frac{1}{4} f f_{zz} \\ &+ \frac{\chi^2 - 2v}{(1+\chi)^2}\alpha\mathcal{H}[f_z]\delta A + \frac{2\chi(1-\chi) + 1 - 2v}{2(1+\chi)^2}\alpha^2\delta A^2\,. \end{aligned} \tag{S13}$$

Substituting $f = Df_1\cos(z) + D^2 v_2 t + D^2 f_2 \cos(2z)$ in Eq. (S13) and solving for order-by-order we obtain the coefficients

$$f_1 = -\alpha;\ v_2 = \alpha^2 \frac{\chi\left(-4\chi + v(1+\chi)^2\right)}{4(1+\chi)^3};\ f_2 = \alpha\frac{1+(2-v)\chi - (3+2v)\chi^2 - v\chi^3}{8(1+\chi)^2}\,. \tag{S14}$$

In Fig. S1 we compare the maximum curvature obtained by this solution with that observed during the dynamic interaction with the disk obstacle. The curvatures in the two cases show similar dependencies on a and v, as well as a transition from nonlinear defocusing to focusing when $\sqrt{1-v^2}a/(1+av) \sim 1$.

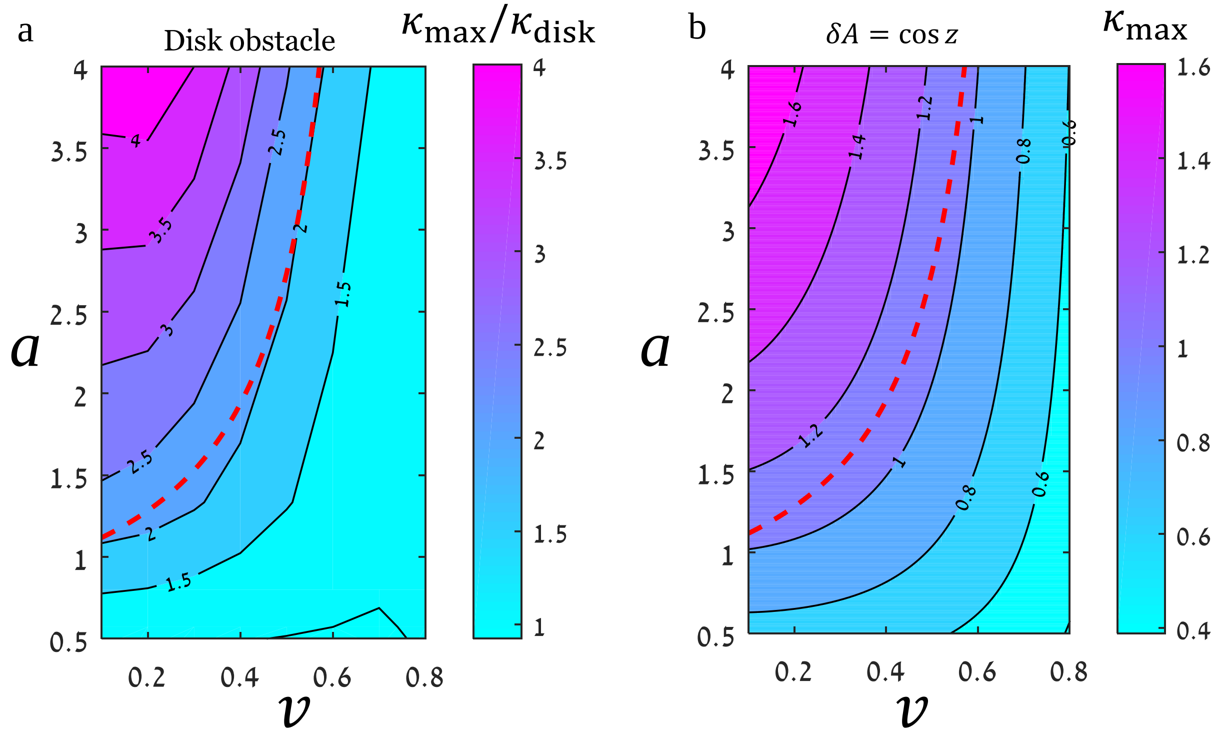

FIG. S1. Comparison of maximum curvatures in (a) interaction of a crack front with the disk obstacle and (b) the time independent $\delta A = D\cos(z)$. In both cases $D = 1$. The disk curvature is $\kappa_{disk} = 2/(0.05\pi)$. The red dashed line corresponds to $\sqrt{1-v^2}a/(1+av) = 1$ and indicates the transition from defocusing to focusing.

SUPPLEMENTAL MOVIE FILES

fig1b_movie.avi — The movie shows the propagation of crack fronts depicted in Fig. 1(b) and contains 190 frames originally taken at 15000 fps (total duration: 12.7ms), displayed here at 20fps.

fig4left_movie.avi, fig4right_movie.avi — The two movie files show the crack front dynamics depicted in Fig. 4 of the main text. The duration of the first movie is $T = 14.09$ and the duration of the second movie is $T = 14.82$, where we take $Z = 2\pi$ and $c = 1$.

[1] A. N. Norris and I. D. Abrahams, Journal of Engineering Mathematics **59**, 399 (2007).