A Benders-Type Approach For Robust Kidney Exchanges under Recourse

Conference on Integer Programming and Combinatorial Optimization, IPCO XXII, May 19-21, 2021

Danny Blom*, Christopher Hojny, Bart Smeulders

Department of Mathematics and Computer Science, Eindhoven University of Technology

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Introduction

Kidney exchange programs (KEPs) are a recent innovation in organ donation. Participants of KEPs are

- incompatible *donor-recipient pairs* (“pairs”)
- *non-directed donors* (“NDDs”), i.e. donors not associated with recipient

Idea: recipients exchange donors (creating a cycle), NDDs can start chain of transplants

Given: a *compatibility graph* $G = (V, A)$, where

- $V = P \cup N$: set of pairs and NDDs
- arc $(i, j) \in A$ if i’s donor compatible with j’s recipient

Find: a packing of cycles and chains of maximum total length.

Extra constraint: cycle and chain lengths bounded by K and L respectively ($\mathcal{C}_K$ and $\mathcal{D}_L$ ($\mathcal{C}_K^j$ and $\mathcal{D}_L^j$)): sets of cycles of length $\leq K$ and chains of length $\leq L$ (involving j)

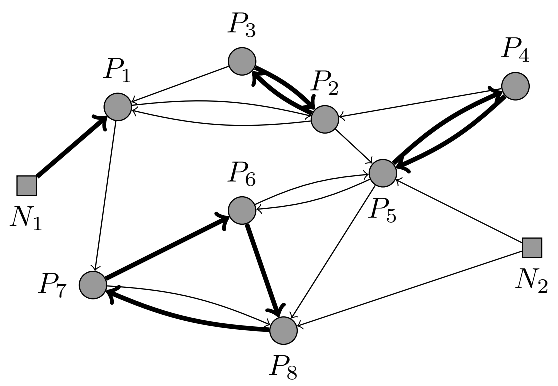

Figure 1: KEP with 2 NDDs and 8 pairs, with a solution for $K = 3$ and $L = 1$

Robust Kidney Exchange

Problem: donors can decide to withdraw, thus breaking cycles or chains.

Stage 1: identify a cycle / chain packing (“initial solution” $x \in \{0,1\}^{\mathcal{C}_K \cup \mathcal{D}_L}$).

Stage 2: $\leq B$ donors leave the KEP (“attack” $u \in \mathcal{U} = \{\chi^S : S \subseteq V, |S| \leq B\}$).

Stage 3: reconsider packing based on u (“recourse solution” $x^u \in \{0,1\}^{\mathcal{C}_K \cup \mathcal{D}_L}$).

Objective: maximize guaranteed number of pairs involved in initial **and** recourse solution under worst-case attack

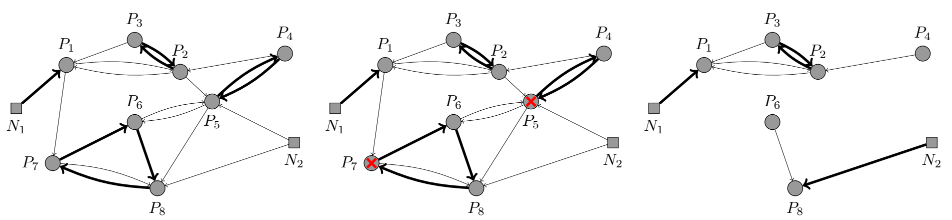

Figure 2: Robust kidney exchange, with $B = 2$, has objective value 4.

IP formulation (cycle-chain enumeration)

Variables:

$y_j^u \in \{0,1\}$: is $j \in P$ contained in initial & best recourse solution given $u \in \mathcal{U}$?

$x_c, x_c^u \in \{0,1\}$: is the cycle or chain $c \in \mathcal{C}_K \cup \mathcal{D}_L$ used both in initial & best recourse solution given $u \in \mathcal{U}$

$$
\begin{aligned}
z_{FR}(\mathcal{U}) = \max\ & Z, && && \text{(1a)}\\
\text{s.t.}\ & Z - \sum_{j \in P} y_j^u \leq 0, && u \in \mathcal{U} && \text{(1b)}\\
& y_j^u - \min\{\sum_{c \in \mathcal{C}_K^j \cup \mathcal{D}_L^j} x_c, \sum_{c \in \mathcal{C}_K^j \cup \mathcal{D}_L^j} x_c^u\} \leq 0, && u \in \mathcal{U}, j \in P && \text{(1c)}\\
& \sum_{c \in \mathcal{C}_K^j \cup \mathcal{D}_L^j} x_c \leq 1, && j \in P \cup N && \text{(1d)}\\
& \sum_{c \in \mathcal{C}_K^j \cup \mathcal{D}_L^j} x_c^u \leq 1 - u_j, && u \in \mathcal{U}, j \in P \cup N && \text{(1e)}
\end{aligned}
$$

Variants: position-indexed cycle edge formulation (PICEF): split up chain variables into variables indexed by arc and its position in the chain.

Iterative approach

The number of variables and constraints is exponential. To overcome this, consider small subset $\bar{\mathcal{U}} \subseteq \mathcal{U}$ initially and add new attacks if necessary.

$R(x, u)$: problem of finding the best recourse solution given initial solution x and attack u (max. overlap)

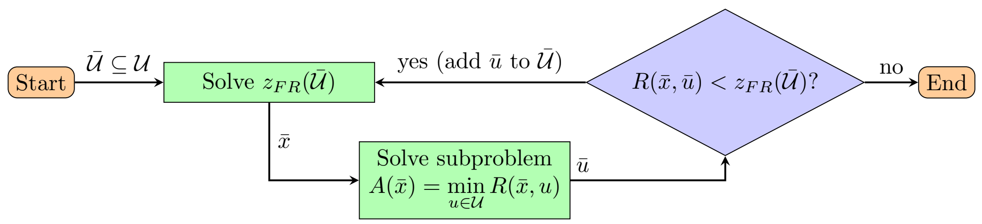

Figure 3: Framework for solving $z_{FR}(\mathcal{U})$

Main contribution (solving $A(\bar{x})$)

Reference method: branch-and-bound algorithm [1]

Drawback: can only solve instances with $V \leq 50$, $B \leq 4$ within one hour.

We consider a Benders-type approach instead. ($\mathcal{F}_G$: feasible KEP solutions)

$$
\begin{aligned}
z(\mathcal{F}_G, \bar{x}) = \min\ & Z && && \text{(2a)}\\
\text{s.t.}\ & Z \geq \sum_{c \in S} w_c(\bar{x}) x_c, && \forall S \in \mathcal{F}_G && \text{(2b)}\\
& x_c \geq \sum_{v \in V(c)} (1 - u_v) - |V(c)| + 1, && \forall c \in \mathcal{C}_K \cup \mathcal{D}_L && \text{(2c)}\\
& x \geq 0, && && \text{(2d)}\\
& u \in \mathcal{U}. && && \text{(2e)}
\end{aligned}
$$

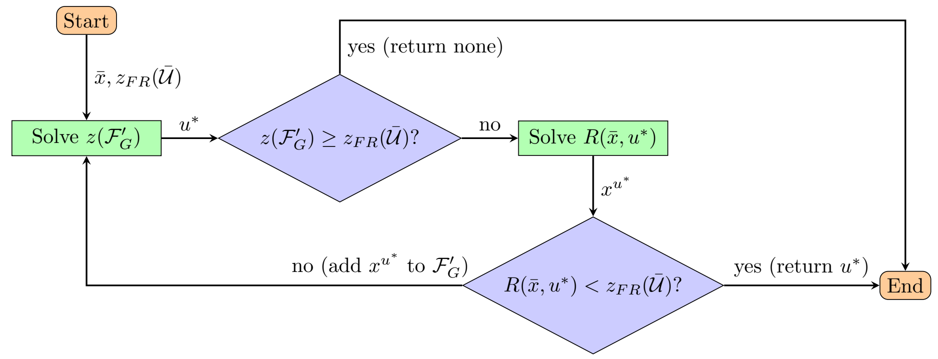

Figure 4: Benders-type framework for solving $A(\bar{x})$

Lifting

The recourse solution x^{u^*} only use cycles / chains not attacked by u^*.

Fact: we can “lift” best recourse solutions to the entire graph, while maintaining optimality on the weight of chosen nonattacked cycles / chains

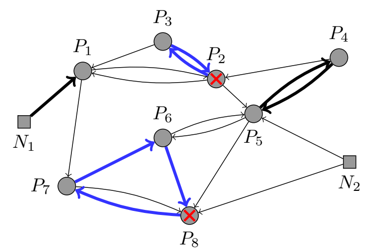

Figure 5: Black solution can be lifted to blue solution (stronger Benders cut)

Numerical results

Software: `SCIP 7.0.2` with `SoPlex 5.0.2` as LP solver

90 benchmark instances [1] with $|V| \in \{20, 50, 100\}$ (30 each), time limit 3600s

	B&B		CC		CC+lifting		PICEF		PICEF+lifting	
					$K=3, L=2$					
B	#opt	time	#opt	time	#opt	time	#opt	time	#opt	time
1	90	17.8	90	14.5	90	13.5	90	22.2	90	25.1
2	87	178.2	90	55.7	90	27.1	90	60.4	90	39.9
3	68	941.2	88	119.1	90	65.1	84	223.3	90	95.4
4	58	1335.4	82	285.4	90	81.0	77	599.4	89	118.3
					$K=3, L=3$					
1	87	242.3	87	190.9	88	185.5	90	63.0	90	39.3
2	68	929.3	81	442.0	84	358.5	87	229.2	90	85.5
3	59	1259.9	75	683.8	79	588.1	78	511.3	89	156.5
4	47	1600.9	68	908.0	76	686.2	66	909.2	88	225.1
					$K=3, L=4$					
1	63	1092.1	63	1119.2	63	1072.6	90	172.4	90	64.7
2	55	1328.4	62	1205.8	63	1228.1	80	540.4	90	147.5
3	47	1572.4	60	1252.3	62	1193.0	72	940.1	88	273.2
4	38	1956.3	58	1271.5	61	1221.9	61	1153.7	86	384.0

References

[1] M. Carvalho, X. Klimentova, K. Glorie, A. Viana, M. Constantino (2020). Robust models for the kidney exchange problem. *INFORMS Journal on Computing*.