Lower Bounds on the Size of General Branch-and-Bound Trees by Santanu S. Dey, *Yatharth Dubey*, Marco Molinaro

The Problem and Main Result

$$\max_{x \in P \cap \{0,1\}^n} c^T x$$

Does there exist a small branch-and-bound tree solving this problem?

NO for Packing, Set Cover, Cross-polytope, Cross-polytope + noise, TSP

$$2^{\Omega(n)}$$

Abstract Branch-and-Bound Trees

- Consider a branch-and-bound tree as an operator applied to a polytope
  - Let $\mathcal{T}$ be a branch-and-bound tree formed using disjunctions of the form $\pi^T x \leq \pi_0 \vee \pi^T x \geq \pi_0 + 1$
  - Denote $\mathcal{T}(P)$ the union of the atoms of the leaves of the tree $\mathcal{T}$ applied to P
  - $\mathcal{T}$ solves $\max_{x \in P \cap \{0,1\}^n} c^T x$ if, for all leaf nodes v, at least one of the following conditions hold:
    - The atom of v is empty
    - The atom of v has an integral optimal solution
    - The atom of v has optimal value worse than the best integer point in P

$$\text{BBhardness(P)} = \max_{c \in \mathbb{R}^n} \left(\min\{|\mathcal{T}| : \mathcal{T} \text{ solves } \max_{x \in P \cap \{0,1\}^n} c^T x\}\right)$$

From Packing to Set Cover

$$P_{\text{PA}} = \left\{x \in [0,1]^n : \sum_{i \in S} x_i \leq \frac{n}{2} \text{ for all } S \subseteq [n], |S| = \frac{n}{2} + 1\right\}$$

$$f(x) = 1 - x$$

$$T_{\text{SC}} = \left\{y \in [0,1]^n : \sum_{i \in S} y_i \geq 1 \text{ for all } S \subseteq [n], |S| = \frac{n}{2} + 1\right\}$$

Turns out BBrank is conserved through this kind of reduction!

If f is a one-to-one integral affine function such that $f(P) \subseteq T$ and $T \cap \mathbb{Z}^m \subseteq f(P \cap \mathbb{Z}^n)$ then $\text{BBrank}(T) \geq \text{BBrank}(P)$

Previous Work

- (Jeroslow 1974) and (Chvatal 1980) show simple instances that have exponential lower bounds for *variable* branching
  - Can be solved with polynomial size trees using *general* branching
- (Cook et al. 1990) show an exponential lower bound for TSP instances, still using *variable* branching
- (Basu et al. 2020) show that CG proofs are at least as strong as *variable* BB proofs for 0-1 problems
- (Basu et al. 2020) show that the sparsity of the disjunctions used can have a large impact on the size of the tree
- (Beame et al. 2018) ask as an open question whether there are superpolynomial lower bounds for *general* BB proofs
- (Dadush et al. 2020) answer this question in the affirmative by showing the Cross Polytope requires a *general* BB proof of infeasibility of size $2^n/n$
- (Fleming et al. 2021) present a relationship between *general* BB proofs and CG proofs

Notions of Hardness

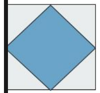
$$\text{BBhardness(P)} = \max_{c \in \mathbb{R}^n} \left(\min\{|\mathcal{T}| : \mathcal{T} \text{ solves } \max_{x \in P \cap \{0,1\}^n} c^T x\}\right)$$

$$\text{BBdepth}(x, P) = \text{size of the smallest } \mathcal{T} \text{ such that } x \notin \text{conv}(\mathcal{T}(P))$$

$$\text{BBrank}(P) = \max_{x \in P \setminus P_I} \text{BBdepth}(x, P)$$

$$\text{BBhardness}(P) \geq \text{BBrank}(P)$$

Cross Polytope (Cropped Cube)



$$P_n = \left\{x \in [0,1]^n : \sum_{i \in J} x_i + \sum_{i \notin J} (1 - x_i) \geq \frac{1}{2} \text{ for all } J \subseteq [n]\right\}$$

$$\text{BBhardness}(P_n) = 2^{n+1} - 1 \qquad \text{BBdepth}(\tfrac{1}{2}\mathbf{1}, P_n) \geq 2^{n/2}$$

$$Q_n = \left\{x \in [0,1]^n : \sum_{i \in J} \gamma x_i + \sum_{i \notin J} (1 - \gamma x_i) \geq \frac{1.6n}{20} \text{ for all } J \subseteq [n]\right\}$$

where $\gamma \sim 1 + N\left(0, \frac{1}{20^2}\right)$

$$\text{BBhardness}(Q_n) = 2^{\Omega(n)} \text{ (w.h.p.)}$$

Comparison with Previous Work

(Fleming et al. 2021) give the following result:

Let $P \subseteq [0,1]^n$ be an integer infeasible polytope such that any CG proof of infeasibility of P has length at least L. Then, any general BB proof of infeasibility of P with maximum coefficient c has size at least $L^{\frac{1}{1+\log(cn)}}$.

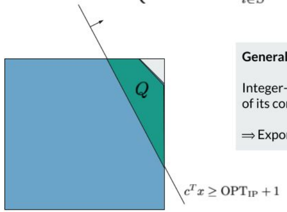
1) **New problems with exponential lower bounds**
2) **Improved quality of bounds**

   (Fleming et al. 2021): $2^{\Omega(\frac{n}{\log cn})}$

   (Dey, Dubey, Molinaro 2021): $2^{\Omega(n)}$
3) **Removing dependence on maximum coefficient size of disjunctions**

A BB Hard Packing Polytope

$$P_{\text{PA}} = \left\{x \in [0,1]^n : \sum_{i \in S} x_i \leq \frac{n}{2} \text{ for all } S \subseteq [n], |S| = \frac{n}{2} + 1\right\}$$



Generalized Dadush-Tiwari Lemma:

Integer-infeasible polytope that requires an exponential number of its constraints to remain integer-infeasible

$\Rightarrow$ Exponential size BB proof of infeasibility

$$\text{BBhardness}(Q) \geq 2^{\Omega(n)} \implies \text{BBrank}(P_{\text{PA}}) \geq 2^{\Omega(n)}$$

(using a technical lemma of (Dash, Gunluk, Molinaro 2015))

From Cross Polytope to TSP

$$T_{\text{TSP}_n} = \{x \in [0,1]^{n^2} : x(\delta(v)) = 2 \quad \forall v \in V, \; x(\delta(W)) \geq 2 \quad \forall W \subset V, W \neq \emptyset\}$$

there exists an integral affine function f such that $f(P_{\lfloor n/8 \rfloor}) \subseteq T_{\text{TSP}_n}$ (Cross Polytope)

and $f(\frac{1}{2}\mathbf{1})$ is not in the integer hull of T_{TSP_n}

(by a reduction from (Cook, Dash 2001))

$$\text{BBrank}(T_{\text{TSP}_n}) \geq \text{BBdepth}(f(\tfrac{1}{2}\mathbf{1}), T_{\text{TSP}_n}) \geq \text{BBdepth}(\tfrac{1}{2}\mathbf{1}, P_{\lfloor n/8 \rfloor}) \geq 2^{\lfloor n/16 \rfloor}$$

By Defn — We show this as part of the reduction framework — By Cross Polytope result