Mixed-Integer Programming for Stochastic Optimization

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Acknowledgments

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- Yongjia Song
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Agenda

In the next two days, we will discuss

- Two-stage stochastic mixed-integer programs (MIPs):
  - Large-scale MIPs
  - How to decompose?
  - Desirable algorithmic properties: Finite convergence, scalability
- Other stochastic (continuous) optimization problems
  - Risk measures/distributional ambiguity modeled as MIPs
  - Exploit combinatorial structure for improved formulations
- Theory, algorithm design, computations, and (some) applications.

Outline

1. Two-Stage Stochastic Integer Programming
   - Two-Stage Stochastic Linear Programming
   - Classification Scheme for Two-Stage Stochastic Mixed-Integer Programming
   - Two-Stage Stochastic Pure Integer Programming
   - Two-Stage Stochastic Mixed-Integer Programming

2. Chance-Constrained Programming
   - Static Joint Chance-Constrained Programming
   - Two-stage (Dynamic) Chance-Constrained Programming
   - Distributionally Robust Chance-Constrained Programming

Outline

Motivation and Scope

Motivation:

- Large capital investment decisions must hedge against uncertain future
- First stage: Strategic decisions (Warehouse/data center/power generator locations)
- Second stage: Operational decisions (Shipments/routing/distribution)
- Applications: Energy, telecommunications, healthcare, supply chain, finance ...

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Scope:

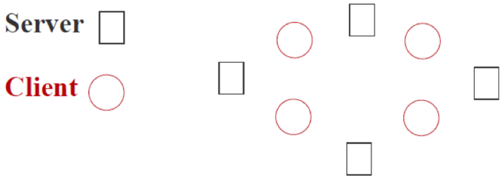
- Focus on Benders type methods
- Will not cover other methods such as Lagrangian relaxation, column generation, etc.

An Example: Stochastic Server Location and Sizing (SSLS)

Applications:

- Preparation and execution of disaster plans
- Location and sizing of data centers in cloud computing
- Supply chain planning with disruptions
- Battery charging infrastructure for electric vehicles

Planning Locations to Hedge Against Demand Uncertainty



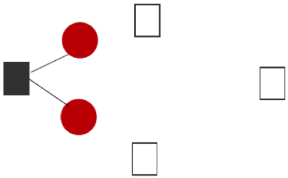
There are two sets of decisions:

- First stage: Determine data center locations (binary) and number of servers to locate (general integer)
- Second stage (once random demand is realized): Allocate servers to customers
- Constraints: capacity, demand satisfaction, etc.

Deterministic Server Location Problem

● Observed demand nodes, ■ Optimal server location

Scenario 1:



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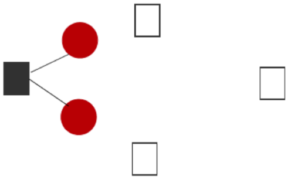
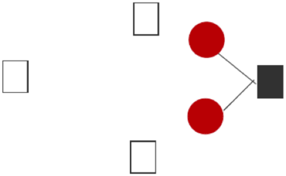


Scenario 2:



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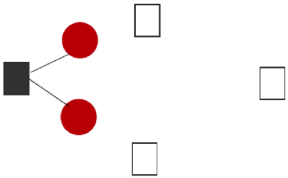

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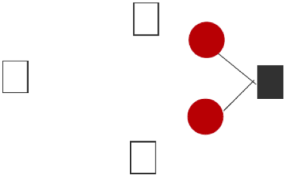
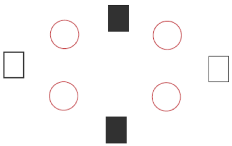


Suppose each scenario is equally likely? What is the optimal server location plan?

Stochastic Server Location Problem

Hedged Optimal Solution



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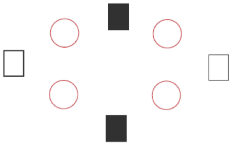

Dynamic Response to Demands/Threats

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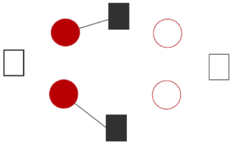

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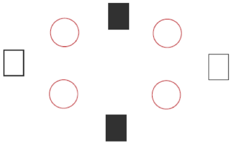

Dynamic Response to Demands/Threats

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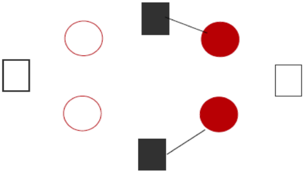

Outline

Standard (Risk-Neutral) Stochastic Programming Formulation

- $\tilde{\omega}$: a random vector with support Ω
- Order of events:

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- $x \in X := \{x \in \mathbb{R}_+^{n-n_1} \times \mathbb{Z}_+^{n_1} : Ax \geq b\}$: first-stage decision vector
- $y(\omega) \in \mathbb{R}_+^{n_2}$: second-stage decision vector for each ω
- $\mathcal{X}, \mathcal{Y}$: integer, continuous and sign restrictions on x, y, resp.

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  A two-stage stochastic program:

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\begin{aligned}
\min \quad & c^\top x + \mathbb{E}_{\tilde{\omega}}(h(x, \tilde{\omega})) \\
s.t. \quad & Ax \geq b, \\
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\end{aligned}
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where

$$\begin{aligned} h(x, \omega) = \min \quad & y_0 \\ & y_0 - g(\omega)^\top y = 0 \\ & W(\omega) y \geq r(\omega) - T(\omega) x \\ & y \in \mathcal{Y}. \end{aligned}$$

- All second stage data can be random $(T(\omega), W(\omega), r(\omega), g(\omega))$

Finite sample space assumption

- We consider the setting where Ω is a finite sample space:

$$\Omega = \{\omega^1, \ldots, \omega^N\}$$

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- Often, N is very large.

- Let $p_i \in [0,1]$: probability of scenario $\omega^i \in \Omega$, where $\sum_{i\in[N]} p_i = 1$.

Deterministic Equivalent Formulation

$$
\begin{array}{llllll}
\min & c^\top x & +p_1 g^\top(\omega^1)y(\omega^1) & +p_2 g^\top(\omega^2)y(\omega^2) + \cdots & +p_N g^\top(\omega^N)y(\omega^N) & \\
\text{s.t} & Ax & & & & \geq b \\
& T(\omega^1)x & +W(\omega^1)y(\omega^1) & & & \geq r(\omega^1) \\
& T(\omega^2)x & & +W(\omega^2)y(\omega^2) & & \geq r(\omega^2) \\
& \vdots & & \ddots & & \vdots \\
& T(\omega^N)x & & & +W(\omega^N)y(\omega^N) & \geq r(\omega^N) \\
& x \in \mathcal{X}, & y(\omega^i) \in \mathcal{Y}, i \in [N]. & & &
\end{array}
$$

It's HUGE!!!

Review of Benders Decomposition Algorithm

Algorithms for two-stage stochastic program with continuous second-stage variables:

Benders' decomposition [Benders, 1962], L-shaped method [van Slyke and Wets, 1969]

Master Problem MP^k at iteration $k = 0, 1, \ldots,$

$$\begin{aligned} \mathrm{MP}^k: \quad \min \quad & c^\top x + \sum_{\omega^i \in \Omega} p_i \eta_{\omega^i} \\ \mathrm{s.t} \quad & A^k(x, \eta) \geq b^k, \\ & x \in \mathcal{X} \end{aligned}$$

where η_j approximates the second-stage value function of scenario j.

- $A^k(x, \eta) \geq b^k$ includes:
  - $Ax \geq b$
  - Optimality cuts generated from the subproblems in iterations $j = 1, \ldots, k-1$
  - Feasibility cuts generated from the subproblems in iterations $j = 1, \ldots, k-1$

Subproblems

Subproblem $\mathrm{SP}^k(x,\omega)$, $\omega \in \Omega$ at iteration $k = 0, 1, \ldots,$

Given (x, η), the solution of the master problem at iteration k, solve for each ω:

$$\begin{aligned} \mathrm{SP}^k(x,\omega): \quad h^k(x,\omega) := \min \quad & g(\omega)^\top y(\omega) \\ \text{s.t} \quad & W(\omega)y(\omega) \geq r(\omega) - T(\omega)x, \\ & y(\omega) \in \mathbb{R}_+^{n_2}, \end{aligned}$$

Let ψ_ω^k be the dual vector of the subproblem $\mathrm{SP}^k(x,\omega)$.

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Let ψ_ω^k be the dual vector of the subproblem $\mathrm{SP}^k(x, \omega)$.

- If $\mathrm{SP}^k(x, \omega)$ is feasible, but $\eta_\omega < h^k(x, \omega)$, then add the optimality cut

$$\eta_\omega \geq {\psi_\omega^k}^\top (r(\omega) - T(\omega)x)$$

- If $\mathrm{SP}^k(x, \omega)$ is infeasible, then its dual is unbounded, so using the corresponding dual ray ψ_ω^k, add the feasibility cut

$$0 \geq {\psi_\omega^k}^\top (r(\omega) - T(\omega)x)$$

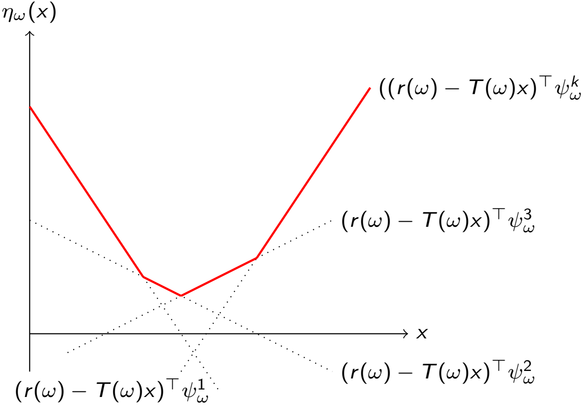

Figure 1: Piecewise-linear function, $\eta_\omega(x)$, for continuous recourse

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Classification Scheme For Stochastic MIPs

$$
\begin{aligned}
B &= \text{Stages with Binary decision variables} \\
C &= \text{Stages with Continuous decision variables} \\
D &= \text{Stages with Discrete (general integer) decision variables.}
\end{aligned}
$$

For example, two-stage stochastic MIP with continuous recourse has: $B = D = \{1\}, C = \{1, 2\}$.

Literature Overview

	First-stage	Second-stage
Laporte and Louveaux (1993) Sen and Sherali (2006)	Binary	Mixed-integer
Carøe and Tind (1997) Sherali and Zhu (2007)	Mixed-binary	Mixed-binary
Carøe and Tind (1998)	Mixed-integer	Integer
Schultz et al. (1998)	Continuous	Integer
Ahmed et al. (2004)	Mixed-binary	Integer
Sherali and Fraticelli (2002) Sen and Higle (2005) Ntaimo and Sen (2005, 2008) Ntaimo (2009)	Binary	Mixed-binary
Gade, K., Sen (2012)	Binary	Integer
Kong et al. (2006) Trapp et al. (2013) Zhang and K. (2014)	Integer	Integer
Qi and Sen (2017, 2021+)	Mixed-Integer	Mixed-Integer

Outline

A Two-Stage Stochastic Integer Program

Consider binary first stage and general integer second stage variables (i.e., B=$\{1,2\}$, D=$\{2\}$, C=$\emptyset$)

$$\begin{aligned} \min \quad & c^\top x + \mathbb{E}\left[h(x, \tilde{\omega})\right] \\ \text{s.t.} \quad & Ax \geq b \\ & x \in \mathbb{B}^n, \end{aligned}$$

where for a particular realization (scenario) ω of $\tilde{\omega}$, $h(x, \omega)$ is defined as

$$\begin{aligned} h(x, \omega) = \min \quad & y_0 \\ \text{s.t.} \quad & y_0 - g(\omega)^\top y = 0 \\ & W(\omega) y \geq r(\omega) - T(\omega) x \\ & y_0 \in \mathbb{Z}, y \in \mathbb{Z}_+^{n_2} \end{aligned}$$

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- $\tilde{\omega}$ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and has finite support
- $Y(x, \omega) := \{y_0 \in \mathbb{Z}, y \in \mathbb{Z}_+^{n_2} : y_0 - g(\omega)^\top y = 0, W(\omega)y \geq r(\omega) - T(\omega)x\}$.

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- Relatively complete recourse

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- Relatively complete recourse
- SIP has a finite optimum

Problem Structure

Deterministic Equivalent of SIP

$$
\begin{array}{llllll}
\min & c^\top x & +p_1 g(\omega^1)^\top y(\omega^1) + p_2 g(\omega^2)^\top y(\omega^2) & + \cdots + & p_N g(\omega^N)^\top y(\omega^N) & \\
& Ax & & & & \geq b \\
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$$x \in \mathbb{B}^n, y(\omega) \in \mathbb{Z}^{n_2}, \omega \in \Omega.$$

- Large-scale integer program
- For a fixed $x \in X$, SIP decomposes by scenario

Value Function Reformulation and Challenges

- Recall $X \cap \mathcal{X} = \{x \in \mathbb{B}^n : Ax \geq b\}$.
- Standard approach in L-shaped decomposition is the value function reformulation of SIP:

$$\min_{x \in X \cap \mathcal{X}} \{c^\top x + \eta : \eta \geq \mathcal{Q}(x)\}, \qquad \mathcal{Q}(x) := \mathbb{E}(h(x, \tilde{\omega}))$$

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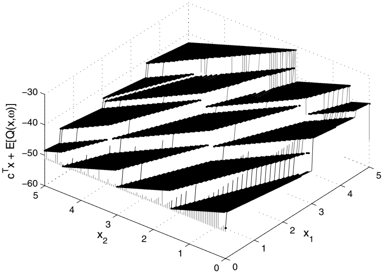
- If second stage is a linear program $\rightarrow h(\cdot, \omega)$, $\omega \in \Omega$: value function of an LP. It is piecewise linear and convex. Benders' decomposition and L-Shaped decomposition exploit this property.

Challenge for SIP

If second stage is an integer program, then $h(\cdot, \omega)$: value function of an integer program [Blair and Jeroslow, 1982]. It is non-linear & non-convex.



From [Ahmed et al., 2004]

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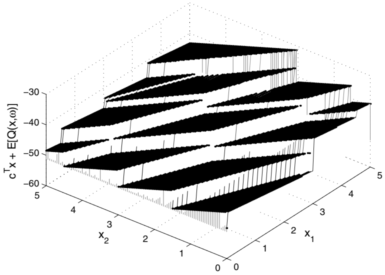

From [Ahmed et al., 2004]

How to create "good" lower bounding approximations practically?

L-Shaped Algorithms for 2-Stage SMIP - Literature

- Integer L-shaped method [Laporte and Louveaux, 1993]: Binary first stage, mixed-integer second stage - First stage B&B and linear optimality cuts. Solve second stage MIPs to optimality. Improved in [Angulo et al., 2016]

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- Computations: e.g., [Laporte et al., 2002], [Ntaimo and Sen, 2005, 2008], [Yuan and Sen, 2009], [Ntaimo and Tanner, 2008].
- Global Optimization and other approaches for pure integer second stage: e.g., [Ahmed et al., 2004], [Kong et al., 2006], [Schultz et al., 1998], [Schultz and Hemmecke, 2003],[Klein, 2020]
- Gomory cuts for SMIP: [Carøe and Tind, 1998]

Gomory Fractional Cuts (GFC) for Deterministic Pure IPs

- Given first-stage vector $\bar{x}$, solve the LP relaxation of the second-stage IP with simplex.
- Let $\mathcal{B}, \mathcal{N}$ - Basic and nonbasic column index sets of LP.

Gomory Fractional Cuts (GFC) for Deterministic Pure IPs

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- Re-write **source** row, with $\nu_i \notin \mathbb{Z}$, as

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- A pure cutting plane algorithm using GFC is finitely convergent if one chooses the source row as the variable with the smallest index and use lexicographic dual simplex [Gomory, 1963]

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Continuous first stage, pure integer second stage.

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Research Question: Can we use Gomory cuts to develop a computationally amenable L-shaped algorithm for SIP?

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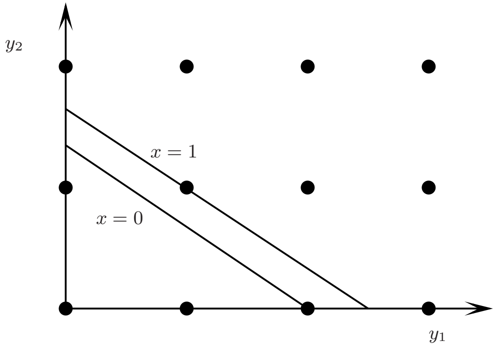
  - For mixed binary second stage, and disjunctive cuts, $\pi_0(\cdot, \omega)$ is piecewise linear concave [Sen and Higle, 2005]
  - What about general integers and Gomory cuts?

Lifting Gomory Cuts for Second Stage

$$\min\{-x + h(x) : x \in \{0,1\}\}$$

$$h(x) = \min\{-y_1 : 2y_1 + 3y_2 = 4 + x, y_1, y_2 \in \mathbb{Z}_+\}$$



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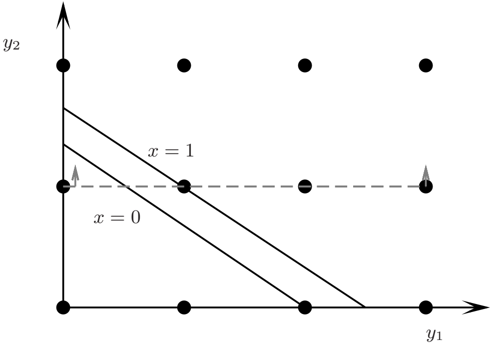

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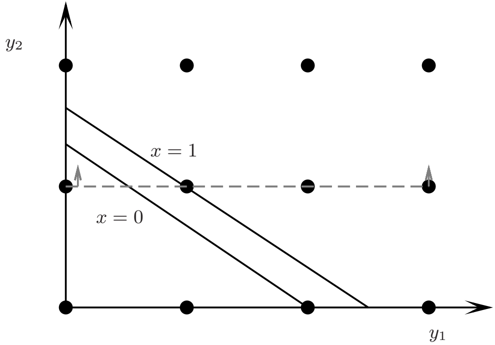

- Carøe and Tind approach: $\frac{1}{2}y_2 \geq \lceil \frac{x}{2} \rceil - \frac{x}{2}$ (Nonlinear)

Desiderata

- A second-stage cut that is valid for all x.
- A first-stage cut that is affine in x.
- Finite convergence

Lifting Gomory Cuts for Second Stage

Want the cut to be valid for all x. Let $x' := 1 - x$. Write source row as:

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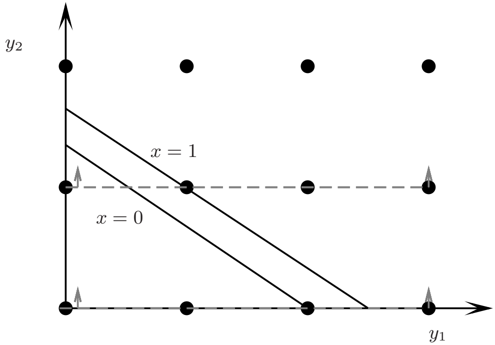

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- Assume w.l.o.g (by complementation, if necessary) that $\bar{x}_j = 0, \forall j = 1, \ldots, n_1$.

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- Furthermore, $\pi(\bar{\omega})^\top y \geq \pi_0(x, \bar{\omega})$, $\pi_0(\cdot, \omega)$ is affine.

Gomory Driven Decomposition Algorithm - Notation

- Second-stage linear approximations at the beginning of iteration k

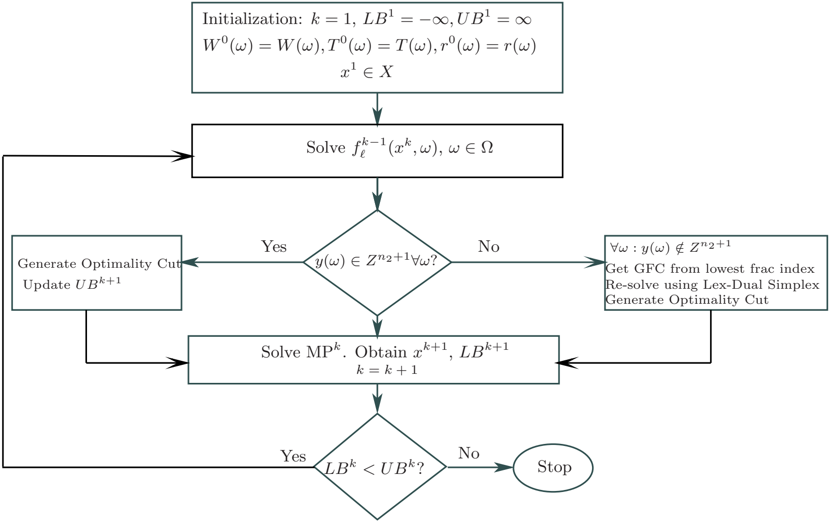
$$
\begin{aligned}
h_\ell^{k-1}(x,\omega) = \min\ & y_0 \\
& y_0 - g(\omega)^\top y = 0 \\
& W^{k-1}(\omega) y \geq r^{k-1}(\omega) - T^{k-1}(\omega) x \\
& y_0 \in \mathbb{R}, y \in \mathbb{R}_+^{n_2}.
\end{aligned}
$$

- $\psi^k(\omega)$: Dual multipliers of second-stage LP at iteration k
- $y^k(x,\omega)$: Lex-smallest solution to second-stage LP at iteration k, given x,ω
- Lower bounding Master Problem MP^k

$$
\begin{aligned}
& \min c^\top x + \eta \\
& Ax \geq b \\
& \eta \geq \sum_{\omega\in\Omega} p_\omega (\psi_\omega^t)^\top (r^t(\omega) - T^t(\omega)x), t = 1,\dots,k \\
& x \in \mathbb{B}^{n_1}, \eta \in \mathbb{R}.
\end{aligned}
$$

- LB^k, UB^k Lower and upper bounds on the SIP optimal solution

Gomory Driven Decomposition Algorithm is finitely convergent [Gade, , and Sen, 2014]



Proof of Convergence - Sketch

- Let $x^k = \bar{x}$ and $x^t = \bar{x}$, $t > k$
- Let $\alpha_k(\bar{x}, \omega) := \left(y_0^{k-1}(\bar{x}, \omega), y_1^{k-1}(\bar{x}, \omega), \ldots, y_{i_k-1}^{k-1}(\bar{x}, \omega), \lceil y_{i_k}^{k-1}(\bar{x}, \omega) \rceil, 0, \ldots, 0 \right)^\top$.

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- So $y^{t-1}(\bar{x},\omega) \succeq y^k(\bar{x},\omega) \succeq \alpha_k(\bar{x},\omega)$.

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- Hence $\alpha_t(\bar{x}, \omega) \succ \alpha_k(\bar{x}, \omega)$.

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- Let $\alpha_k(\bar{x}, \omega) := \left(y_0^{k-1}(\bar{x}, \omega), y_1^{k-1}(\bar{x}, \omega), \ldots, y_{i_k-1}^{k-1}(\bar{x}, \omega), \lceil y_{i_k}^{k-1}(\bar{x}, \omega) \rceil, 0, \ldots, 0\right)^\top$.
- Gomory cut for first fractional y_{i_k} and lex-dual simplex gives $y^k(\bar{x}, \omega) \succeq \alpha_k(\bar{x}, \omega)$
- Gomory cuts added during iterations $k+1, \ldots, t-1$ are all valid for $Y(\bar{x}, \omega)$.
- So $y^{t-1}(\bar{x}, \omega) \succeq y^k(\bar{x}, \omega) \succeq \alpha_k(\bar{x}, \omega)$.
- $\alpha_t(\bar{x}, \omega) \succ y^{t-1}(\bar{x}, \omega)$ by definition.
- Hence $\alpha_t(\bar{x}, \omega) \succ \alpha_k(\bar{x}, \omega)$.
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- Then the dual polyhedra of sub-problems remain fixed. Obtain full reformulation of SIP in (x,η).

Example from Literature

Variations of this example appear in [Schultz et al., 1998], [Sen et al., 2003], [Ahmed et al., 2004]

$$\begin{aligned} \min \quad & -1.5x_1 - 4x_2 + \mathbb{E}[f(x, \tilde{\omega})] \\ \text{s.t.} \quad & x \in \{0,1\}^2 \end{aligned}$$

where

$$\begin{aligned} f(x,\omega) = \min \quad & y_0 \\ \text{s.t.} \quad & y_0 + 16y_1 + 19y_2 + 23y_3 + 28y_4 - 100R = 0 \\ & 2y_1 + 3y_2 + 4y_3 + 5y_4 - R \leq r_1(\omega) - x_1 \\ & 6y_1 + 1y_2 + 3y_3 + 2y_4 - R \leq r_2(\omega) - x_2 \\ & y_0 \in \mathbb{Z}, y_i \in \{0, \ldots, 5\}, i = 1, \ldots, 4, R \in \mathbb{Z}_+, \end{aligned}$$

$\Omega = \{1, 2\}, p_1 = p_2 = 0.5.$

$(r_1(1), r_2(1)) = (10, 4), (r_1(2), r_2(2)) = (13, 8).$

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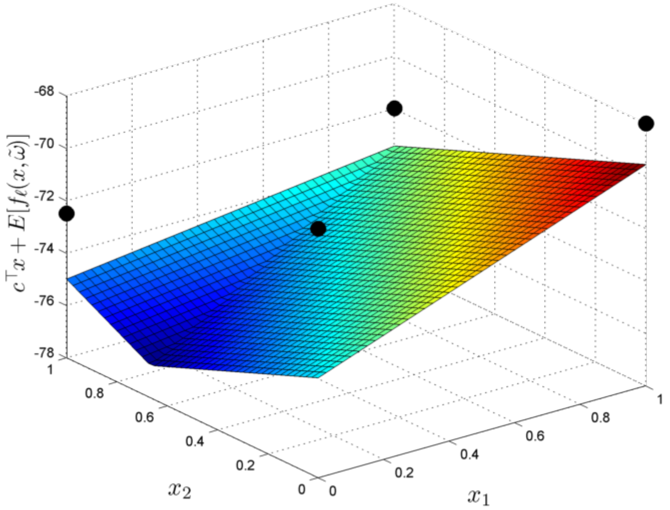
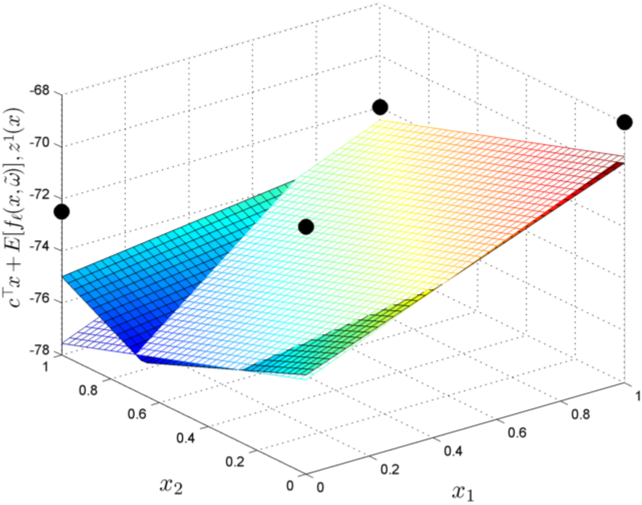
$$z^k(x) := c^\top x + \max_{t=1,\dots,k} \left\{ \sum_{\omega\in\Omega} p_\omega (\psi_\omega^t)^\top (r^t(\omega) - T^t(\omega)x) \right\}.$$

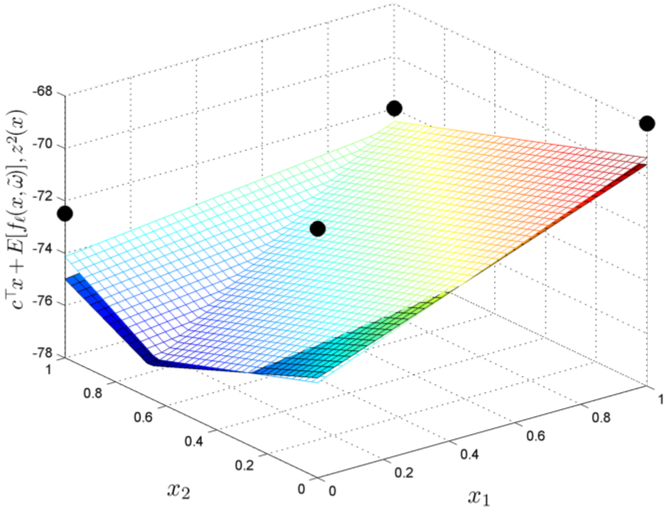
Best LP Approximation



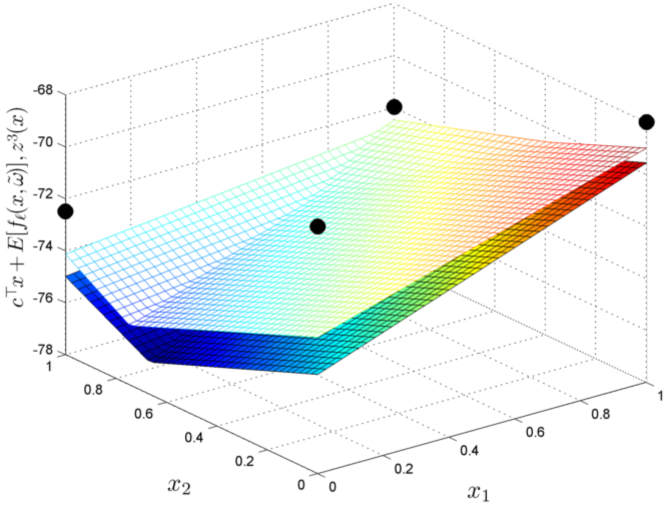
Approximation at $k = 1$



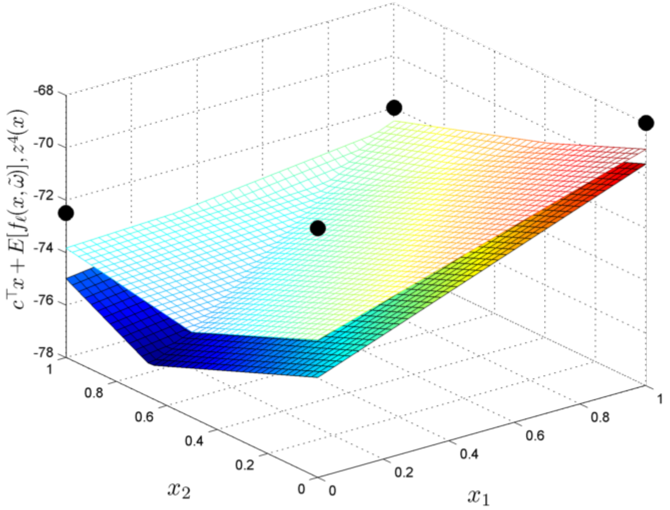
Approximation at $k = 2$



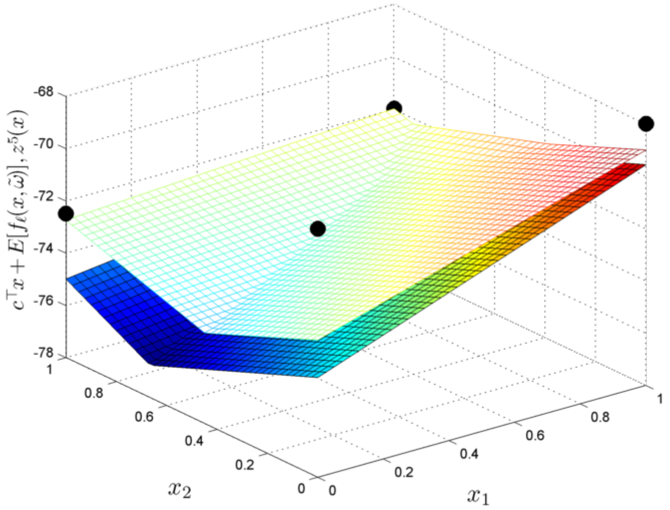
Approximation at $k = 3$



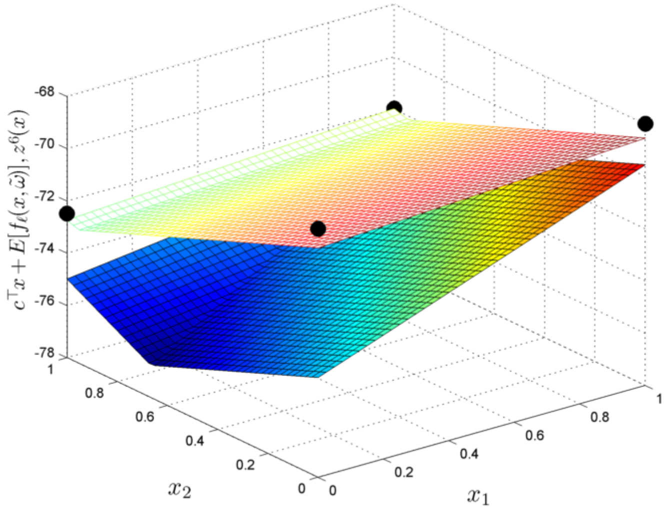
Approximation at $k = 4$



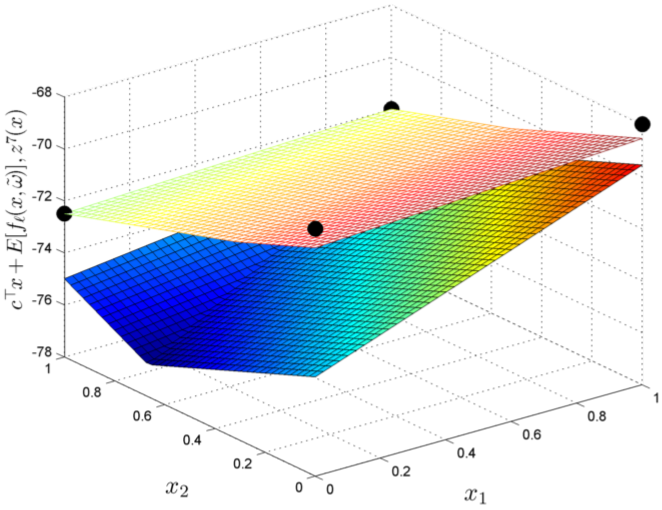
Approximation at $k = 5$



Approximation at $k = 6$



Approximation at $k = 7$



Deterministic Equivalent Comparison - SSLP Instances

Instances	DEF		Gomory	
	Time	Gap	Time	Gap
SSLP_5_25_50	2.03	0.00	0.18	0.00
SSLP_5_25_100	1.72	0.00	0.22	0.00
SSLP_5_50_50	1.06	0.00	0.27	0.00
SSLP_5_50_100	3.56	0.00	0.48	0.00
SSLP_5_50_1000	212.64	0.00	2.88	0.00
SSLP_5_50_2000	1020.54	0.00	5.73	0.00
SSLP_10_50_50	801.49	0.01	109.2	0.02
SSLP_10_50_100	*	0.10	218.42	0.02
SSLP_10_50_500	*	0.38	740.38	0.03
SSLP_10_50_1000	*	3.56	1615.42	0.02
SSLP_10_50_2000	*	18.59	2729.61	0.02

* 3600 second time limit

Alternative Implementations

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- Partial branch-and-cut for binary second-stage variables

Summary - First Stage Binary, Second Stage Integer

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- All alternative implementations with lex-dual simplex are finite
- One can now integrate alternative classes of cuts: Disjunctive, Gomory, structural

First and Second Stages Integer [Zhang and K., 2014]

$B = D = \{1, 2\}, C = \emptyset$

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How about mixed-integer variables? Gomory (or Gomory Mixed-Integer) pure cutting plane method is no longer finitely convergent...

Outline

1. Two-Stage Stochastic Integer Programming
   - Two-Stage Stochastic Linear Programming
   - Classification Scheme for Two-Stage Stochastic Mixed-Integer Programming
   - Two-Stage Stochastic Pure Integer Programming
   - Two-Stage Stochastic Mixed-Integer Programming

2. Chance-Constrained Programming
   - Static Joint Chance-Constrained Programming
   - Two-stage (Dynamic) Chance-Constrained Programming
   - Distributionally Robust Chance-Constrained Programming

Background: Deterministic 0-1 Mixed-Integer Linear Program (MILP)

$\min_{x \in X}\{c^T x | X = \{Ax \geq b, x \in \{0,1\}^{n_1} \times \mathbb{R}_+^{n-n_1}\}\}$.

- Let X_L be the LP relaxation of X.
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Theorem (Sequential convexification of 0-1 MILP [Balas, 1979])

$\text{clconv}(X) = \mathcal{H}_{n_1}(\mathcal{H}_{n_1-1}(\cdots(\mathcal{H}_1(X_L))\cdots))$.

Other finite characterizations: RLT [Sherali and Adams, 1990, 1994], SDP [Lovász and Schrijver, 1991], ...

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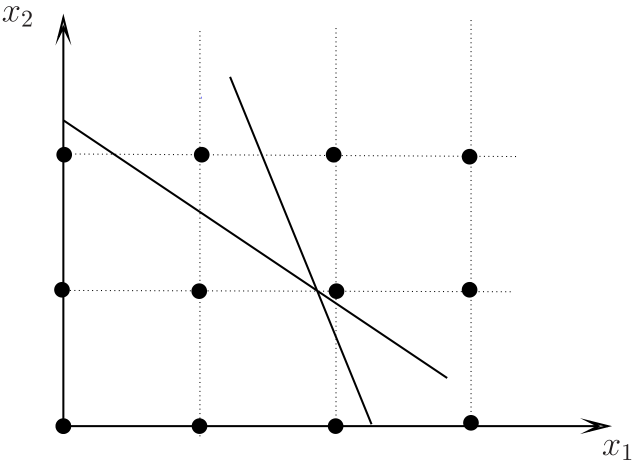
[Carøe and Tind, 1998] and [Sen and Higle, 2005] adapt this convexification scheme for two-stage stochastic mixed-binary optimization.

How about general MILP?

Example of [Owen and Mehrotra, 2001]



Convexification w.r.t x_1

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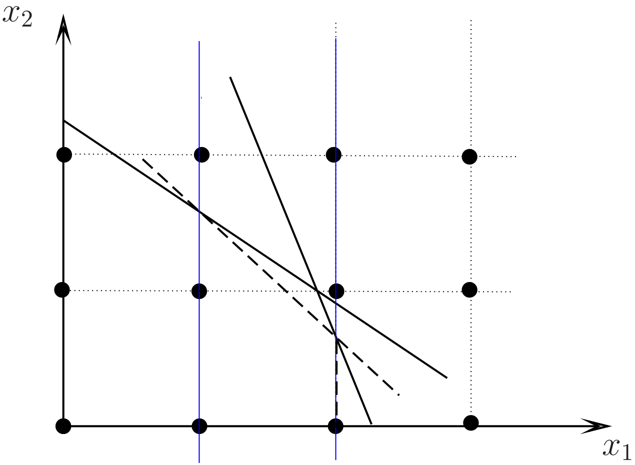

Convexification w.r.t first x_1, then $x_2 \neq \text{conv}(X)$!

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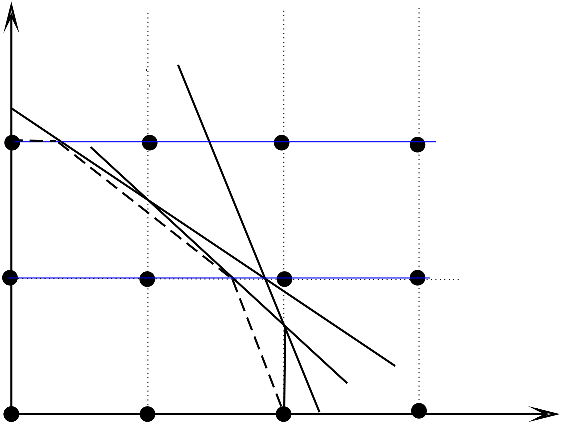

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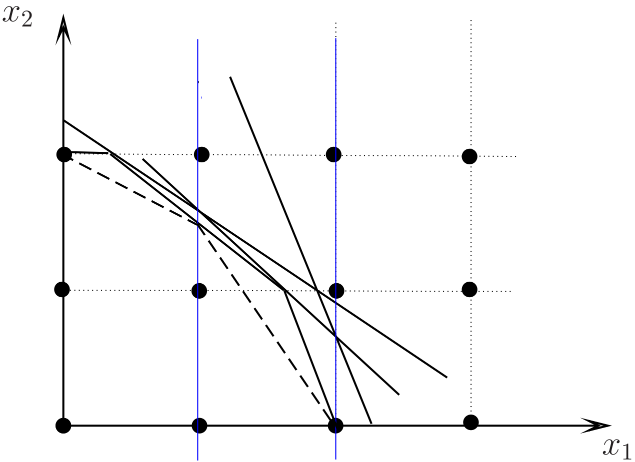

Ad infinitum

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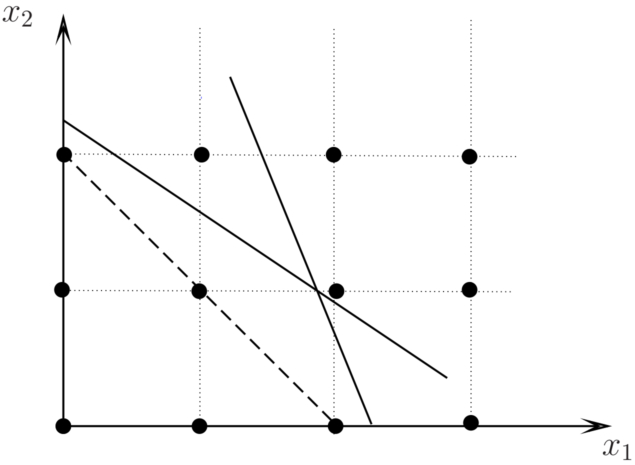

General MILP with bounded integer variables

$\min_{x \in X}\{c^T x | X = \{Ax \geq b, x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n-n_1}\}\}$.

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- Binary expansion of bounded integer variables may not be effective in practice [Owen and Mehrotra, 2002]
- [Adams and Sherali, 2005] give a finite RLT characterization using Lagrange interpolation polynomials

Questions

- Is there a finite disjunctive characterization of the convex hull of MILP solutions in the original space of general integer variables?

- Is there a finitely convergent cutting plane algorithm for a general MILP (with no assumptions on the integrality of the optimal objective)?

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- A **unit partition**, $\mathcal{P}^*$, of all integer points is a partition for which $u_{\kappa_j j} - \ell_{\kappa_j j} \leq 1$, for all $\kappa_j = 1, \ldots, t_j$, and all $j = 1, \ldots, n_1$.

A Finite Disjunctive Characterization for General MILP

For a given vector $\kappa \in K(\mathcal{P}^*)$, an index j, and a polyhedron $\bar{X}$, let

$$P^-(\kappa, j, \bar{X}) := \{x \in \bar{X} | \ell_{\kappa_i i} \leq x_i \leq u_{\kappa_i i}, i = 1, \dots, n_1; x_j \leq \ell_{\kappa_j j}\},$$

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Theorem (Sequential convexification of General MILP [Chen, K., and Sen, 2011])

Given a set $X = \{x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n-n_1} | Ax \ge b\}$, $X \neq \emptyset$, with bounded integer variables, for any unit partition $\mathcal{P}^*$,

$$\text{clconv}(X) = \text{clconv}\{\cup_{\kappa \in K(\mathcal{P}^*)}[\mathcal{H}_{n_1}^\kappa(\mathcal{H}_{n_1-1}^\kappa(\cdots(\mathcal{H}_1^\kappa(X_L))\cdots)) \setminus \emptyset]\}.$$

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For a given vector $\kappa \in K(\mathcal{P}^*)$, an index j, and a polyhedron $\bar{X}$, let

$$P^-(\kappa, j, \bar{X}) := \{x \in \bar{X} | \ell_{\kappa_i i} \leq x_i \leq u_{\kappa_i i}, i = 1, \dots, n_1; x_j \leq \ell_{\kappa_j j}\},$$

$$P^+(\kappa, j, \bar{X}) := \{x \in \bar{X} | \ell_{\kappa_i i} \leq x_i \leq u_{\kappa_i i}, i = 1, \dots, n_1; x_j \geq u_{\kappa_j j}\}.$$

Also let $\mathcal{H}_j^\kappa(\bar{X}) := \text{clconv}(P^-(\kappa, j, \bar{X}) \cup P^+(\kappa, j, \bar{X}) \setminus \emptyset)$

Theorem (Sequential convexification of General MILP [Chen, K., and Sen, 2011])

Given a set $X = \{x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n-n_1} | Ax \geq b\}$, $X \neq \emptyset$, with bounded integer variables, for any unit partition $\mathcal{P}^*$,

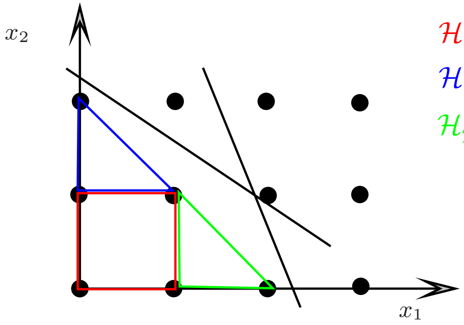
$$\text{clconv}(X) = \text{clconv}\{\cup_{\kappa \in K(\mathcal{P}^*)}[\mathcal{H}_{n_1}^\kappa(\mathcal{H}_{n_1-1}^\kappa(\cdots(\mathcal{H}_1^\kappa(X_L))\cdots)) \setminus \emptyset]\}.$$

Proof idea. The set $K(\mathcal{P}^*)$ decomposes the problem into boxes of at most unit size, each of which can be sequentially convexified.

Example (cont.)

A unit partition $\mathcal{P}^*$ is given by $x_j \in \{[0,1],[1,2],[2,3]\}$ for $j = 1,2$, $t_j = 3$ and $\kappa_j \in \{1,2,3\}$ for $j = 1,2$.

$$K(\mathcal{P}^*) = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$



$\mathcal{H}_2^{(1,1)}(\mathcal{H}_1^{(1,1)}(X_L))$
$\mathcal{H}_2^{(1,2)}(\mathcal{H}_1^{(1,2)}(X_L))$
$\mathcal{H}_2^{(2,1)}(\mathcal{H}_1^{(2,1)}(X_L))$

How can we make this practical?

Unit partition contains exponentially many pieces.

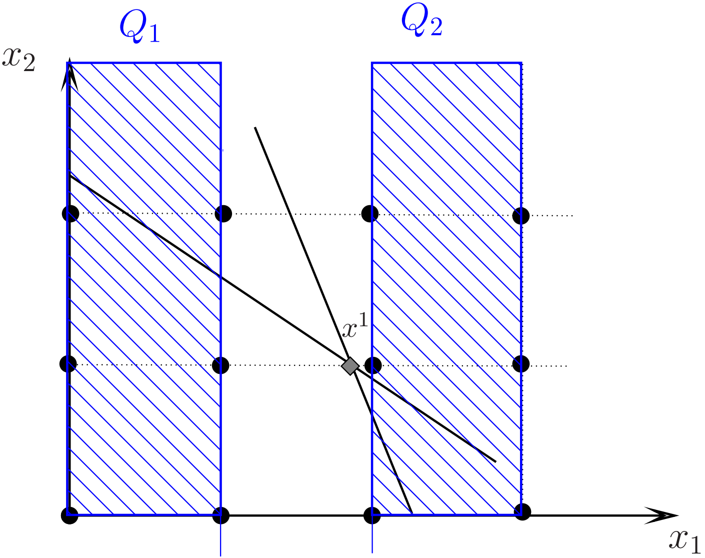
Overview of the Cutting plane tree (CPT) algorithm

Given a fractional point x, find and add a violated disjunctive cut, re-solve LP.

- Add one valid cut at a time from "box" disjunctions (Q_t's), using a cut generation LP (CGLP)
- Obtain Q_t's **on-the-fly** using a **cutting plane tree**
- CPT provides the memory needed for finite convergence.

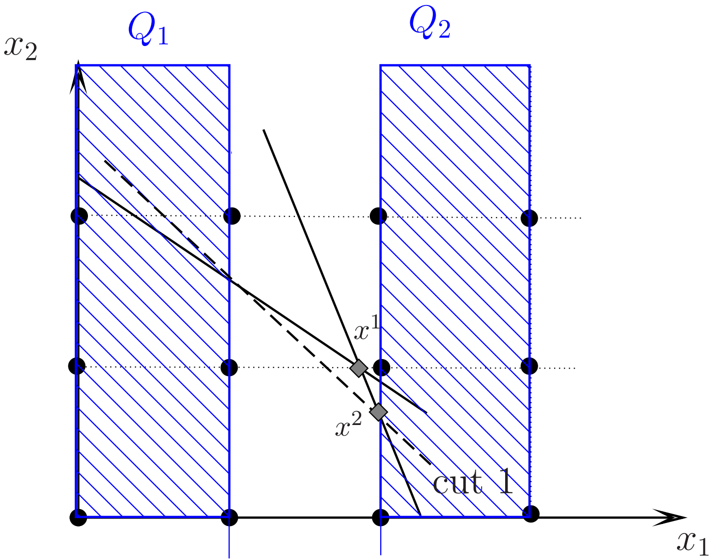
Example (cont.)

Cutting plane tree algorithm



Example (cont.)

Cutting plane tree algorithm



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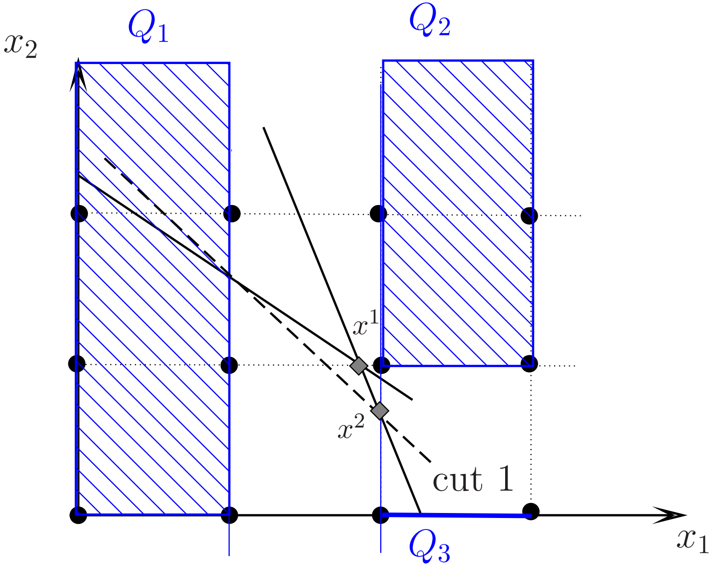

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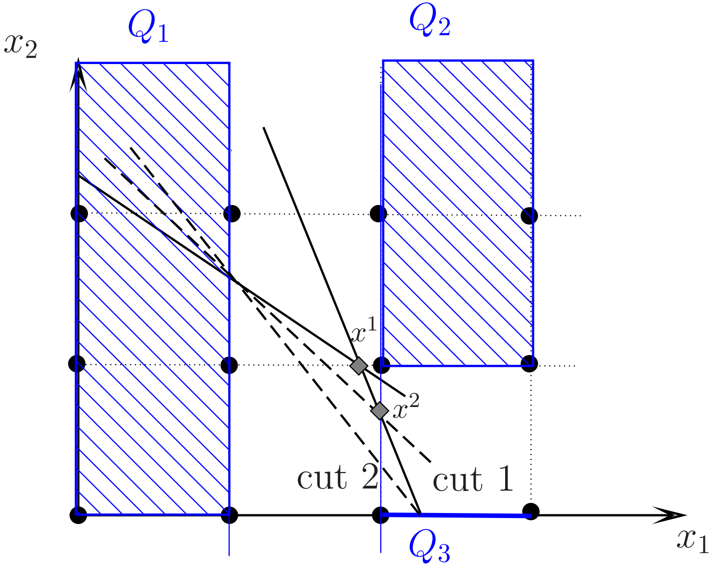


Example (cont.)

CPT algorithm

Iteration 1.

- Solve LP relaxation: $x^1 = (15/8, 1)$.

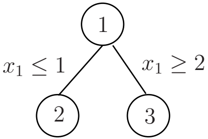
Example (cont.)

CPT algorithm

Iteration 1 (cont.)

- Create two branches in CPT: $x_1 \leq 1$ and $x_1 \geq 2$
- Solve the CGLP based on the two disjunctions (nodes 2&3) to generate a violated cut:

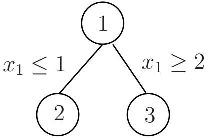
$$\frac{11}{12}x_1 + x_2 \leq \frac{5}{2}$$



Example (cont.)
CPT algorithm

Iteration 2.

- Solve LP relaxation: $x^2 = (2, 2/3)$.
- Search the current CPT to find where x^2 falls. (Node 3)



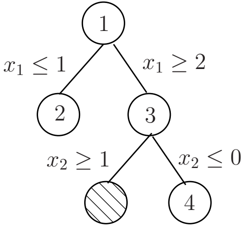
Example (cont.)

CPT algorithm

Iteration 2 (cont.)

- Create 2 branches for node 3: $x_2 \leq 0$ and $x_2 \geq 1$, remove infeasible nodes (crossed).
- Solve the CGLP based on the 2 disjunctions (nodes 2&4) to generate a violated cut:

$$x_1 + \frac{15}{19}x_2 \leq \frac{9}{4}$$

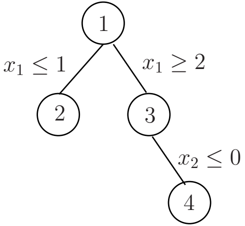


Example (cont.)

CPT algorithm

Iteration 3.

- Solve LP relaxation: $x^3 = (1, 19/12)$.
- Search the current CPT to find where x^3 falls. (Node 2)



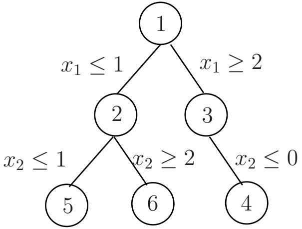
Example (cont.)

CPT algorithm

Iteration 3 (cont.)

- Create 2 branches for node 2: $x_2 \leq 1$ and $x_2 \geq 2$.
- Solve the CGLP based on the 3 disjunctions (nodes 4,5&6) to generate a violated cut:

$$x_1 + \frac{15}{16}x_2 \leq \frac{9}{4}$$

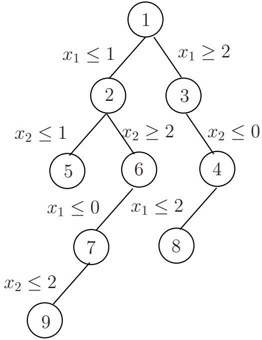


Example (cont.)

CPT algorithm

Iteration 7.

- Solve LP relaxation: $x^7 = (2, 0)$.



Finite convergence of CPT

Theorem ([Chen, K., and Sen, 2011])

For a general MILP with bounded integer variables, the cutting plane tree algorithm converges to an optimal solution in finitely many iterations.

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Proof sketch.

- The number of possible leaf nodes is finite. In the worst case, we reach a unit partition, $\mathcal{P}^*$.
- There are finitely many extreme points of the CGLP for clconv$\{\cup_{Q_t \in \mathcal{P}^*}(Q_t \cap X_{m_\sigma})\}$
- A node σ is visited finitely many times.
- The unique path from the root node to each leaf node defines a $\kappa \in K(\mathcal{P}^*)$.
- Now use General MILP Sequential Convexification Theorem.

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[Chen, K., Sen, 2012] tests CPT algorithm on (deterministic) MIPLIB instances
[Qi and Sen, 2017, 2021+] leverage the CPT algorithm for two-stage stochastic MIPs

Discussion

- Successful adaptation of Benders-type approaches require
  - finite convexification in second stage,
  - tractable lifting of first-stage variables
- Extended formulations in second stage, e.g., [Kim and Mehrotra, 2015], [Bansal et al., 2018]
- Convex approximations, e.g., [Romeijnders et al., 2016], [van der Laan and Romeijnders, 2020+]
- Multi-stage stochastic MIP: SDDiP (JuMP) [Zou et al., 2019]
- Progressive hedging (Py-SP), e.g., [Rockafellar and Wets, 2004], [Watson et al., 2012], [Gade et al., 2016]
- Two-stage stochastic mixed-integer nonlinear programs, e.g., [Mehrotra and Özevin, 2009], [Li and Grossmann, 2018, 2019]

Outline

1. Two-Stage Stochastic Integer Programming
   - Two-Stage Stochastic Linear Programming
   - Classification Scheme for Two-Stage Stochastic Mixed-Integer Programming
   - Two-Stage Stochastic Pure Integer Programming
   - Two-Stage Stochastic Mixed-Integer Programming

2. Chance-Constrained Programming
   - Static Joint Chance-Constrained Programming
   - Two-stage (Dynamic) Chance-Constrained Programming
   - Distributionally Robust Chance-Constrained Programming

Risk-Averse Optimization

Modeling risk/reliability/quality-of-service restrictions

- Rare events with dire consequences
- Not every realization of uncertain data may lead to a feasible solution
- Using risk-neutral models (expectations) do not capture the risk involved with low probability events
- There exist multiple correlated risk criteria
- Supply chain disruptions, natural disasters, pandemic, etc.

Risk Models and Challenges

- Quantitative risk models
    - Models with (multivariate) conditional-value-at-risk (CVaR)
    - Stochastic multi-objective optimization: Efficient frontier stochastic
- Qualitative risk models
    - Models with joint chance-constraints
    - Feasible region highly non-convex
- A large number of samples (scenarios) needed to represent uncertainty

Preliminaries: Value-at-Risk (VaR)

Definition

For a univariate random variable X, with cumulative distribution function F_X, the *value-at-risk* (VaR) at confidence level $(1-\epsilon)$, also known as $(1-\epsilon)$-quantile, is given by:

$$\text{VaR}_{1-\epsilon}(X) = \min\{\eta \ : \ F_X(\eta) \geq 1-\epsilon\}. \tag{1}$$

- From (1), for any $x \in \mathbb{R}$, the inequalities $\text{VaR}_{1-\epsilon}(X) \leq \tau$ and $\mathbb{P}(X \leq \tau) \geq 1-\epsilon$ are equivalent.
- In optimization context, the r.v. X is dependent on the decision vector x and uncertain parameters ω
- In this context, a chance constraint on random variable X can be equivalently represented as a constraint on its VaR.
- Here, larger values of X are considered risky (e.g., losses).

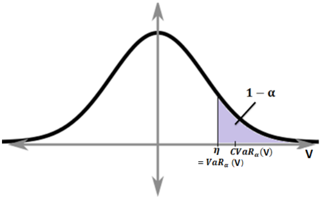
Preliminaries: Conditional Value-at-Risk (CVaR)

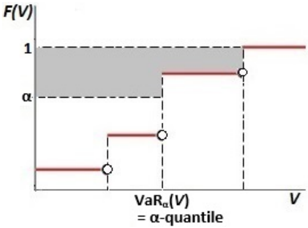
Definition ([Rockafellar and Uryasev, 2000,2002])

The *conditional value-at-risk* (CVaR) at confidence level $(1-\epsilon) \in (0,1]$ is given by

$$\mathrm{CVaR}_{1-\epsilon}(X) = \min\left\{\eta + \frac{1}{\epsilon}\mathbb{E}\left([X-\eta]_+\right) \ : \ \eta \in \mathbb{R}\right\}, \tag{2}$$

where $(a)_+ := \max\{0, a\}$.





Here $\alpha = 1 - \epsilon$.

Preliminaries: Alternative Representations of CVaR

- Suppose X is a r.v. with realizations $X_1, \ldots, X_N$ and probabilities $p_1, \ldots, p_N$.
- The optimization problem in (2) can equivalently be formulated as the linear program (LP):

$$\min \left\{ \eta + \frac{1}{\epsilon} \sum_{i \in [N]} p_i w_i \ : \ w_i \geq X_i - \eta, \ \forall \ i \in [N], \quad w \in \mathbb{R}_+^N \right\}. \tag{3}$$

- Let ρ denote an ordering of the realizations such that $X_{\rho_1} \leq X_{\rho_2} \leq \cdots \leq X_{\rho_N}$. Then, for a given confidence level $\epsilon \in (0,1]$ we have

$$\mathrm{VaR}_{1-\epsilon}(X) = X_{\rho_q}, \text{ where } q = \min \left\{ j \in [N] : \sum_{i \in [j]} p_{\rho_i} \geq 1 - \epsilon \right\}. \tag{4}$$

- CVaR provides a tractable approximation to an individual VaR constraint. (Replace $\mathrm{VaR}_{1-\epsilon}(X) \leq \tau$ with $\mathrm{CVaR}_{1-\epsilon}(X) \leq \tau$.)
- How about the multivariate case? [Prékopa, 1990], [K. and Noyan, 2016], [Meraklı and K., 2018]

Outline

Static Joint chance-constrained program (CCP)

- A linear joint chance-constrained program (CCP) with right-hand-side uncertainty is an optimization problem of the following form:

$$\min \left\{ c^\top x : \ \mathbb{P}\left[Ax \geq b(\omega) \right] \geq 1 - \epsilon, \ x \in X \right\} \tag{CCP}$$

  where
  - $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space,
  - X is a (polyhedral) domain,
  - $\epsilon \in (0, 1)$ is a risk level, and
  - $b(\omega)$ is the random right-hand-side vector that depends on the random variable $\omega \in \Omega$.

- Dates back to [Charnes et al., 1958], [Charnes and Cooper, 1959, 1963] (*individual* chance constraints), and [Miller and Wagner, 1965], [Prékopa,1973] (*joint* chance constraints)

- Why can't we handle $\mathbb{P}[f(x, \xi) \geq 0] \geq 1 - \epsilon$ directly?
  - Non-convex unless certain restrictive assumptions, e.g., [Prékopa, 1990], [Sen, 1992], [Dentcheva et al., 2000]
  - Evaluating $\mathbb{P}[f(x, \xi) \geq 0]$ is difficult (multidimensional integration).
  - In practice, $\mathbb{P}$ is often unknown. (We'll address this later.)

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- Dates back to [Charnes et al., 1958], [Charnes and Cooper, 1959, 1963] (*individual* chance constraints), and [Miller and Wagner, 1965], [Prékopa,1973] (*joint* chance constraints)
- Used in modeling problems with "random supplies/demands".
- Why can't we handle $\mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon$ directly?
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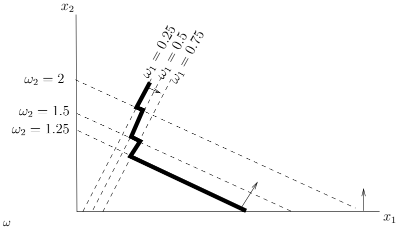
Non-convex feasible region example adapted from [Sen, 1992]

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & \mathbb{P}\left\{ \begin{array}{ccc} 2x_1 - x_2 & \geq & \omega_1 \\ x_1 + 2x_2 & \geq & \omega_2 \end{array} \right\} \geq 0.6 \\
& x \geq 0,
\end{aligned}
$$

with joint probability density function of ω

Scenario	1	2	3	4	5	6	7	8	9
ω_1	0.75	0.5	0.5	0.25	0.25	0.25	0	0	0
ω_2	1.25	1.5	1.25	1.75	1.5	1.25	2	1.5	1.25
Probability	0.2	0.14	0.06	0.06	0.06	0.3	0.04	0.04	0.1



Finite sample space assumption

- We consider the setting where Ω is a finite sample space:

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- Assuming that $\mathbb{P}\left[\omega = \omega^i\right] = p_i$ for $i \in [N]$,

$$\min\left\{c^\top x : \ \mathbb{P}\left[Ax \geq b(\omega)\right] \geq 1 - \epsilon, \ x \in X\right\} \tag{CCP}$$

can be rewritten as

$$\min\left\{c^\top x : \sum_{i\in[N]} p_i \mathbb{1}\left[Ax \geq b(\omega^i)\right] \geq 1 - \varepsilon, \ x \in X\right\}.$$

  - Also known as (ML) empirical risk, (stats) Monte Carlo.

Reformulation

- There is a deterministic reformulation: the problem can be reformulated as the following mixed-integer program [Ruszczyński, 2001],

$$
\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax = y, \\
& y \geq b(\omega^i)(1 - z_i), \quad \forall i \in [N], \\
& \sum_{i \in [N]} p_i (1 - z_i) \geq 1 - \epsilon, \\
& x \in X, \; y \in \mathbb{R}^k_+, \; z \in \{0, 1\}^N,
\end{aligned}
$$

where

- we assume that $Ax \geq \mathbf{0}$ holds for all $x \in X$,
- $b(\omega^i) \geq \mathbf{0}$ for all i, i.e., $Ax \geq \mathbf{0}$ is satisfied for all $x \in X$,
- $1 - z_i \simeq \mathbb{1}\left[Ax \geq b(\omega^i)\right]$:

$$Ax \geq b(\omega^i) \text{ if } z_i = 0 \quad \text{and} \quad Ax \geq \mathbf{0} \text{ if } z_i = 1.$$

Big-M Reformulation

The problem can be reformulated as the following mixed-integer program:

$$
\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax = y, \\
& y_j \geq w_{ij}(1 - z_i), \quad \forall i \in [N], \forall j \in [k], && \text{(big-M)} \\
& \sum_{i \in [N]} p_i z_i \leq \epsilon, && \text{(knapsack)} \\
& x \in X, \ y \in \mathbb{R}^k_+, \ z \in \{0,1\}^N,
\end{aligned}
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where $W = \{w_{ij}\} \in \mathbb{R}^{N \times k}_+$ is a nonnegative matrix.

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$$

- We will strengthen the formulation by integer programming techniques.

Known substructures

- We refer to the set

$$\left\{(y,z) \in \mathbb{R}^k_+ \times \{0,1\}^N : \; y_j \geq w_{ij}(1-z_i), \; \forall i \in [N], \forall j \in [k]\right\} \tag{Mix}$$

  as a (joint) mixing set (term coined by [Günlük and Pochet, 2001] for related set with general integer variables).

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- Random technology matrix and right-hand-side extensions [Tanner and Ntaimo, 2010], [Luedtke, 2014]

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- One can obtain the convex hull of (Mix) by adding the so-called mixing (or star) inequalities [Atamtürk, Nemhauser, Savelsbergh, 2000].
- We call the set

$$\left\{(y, z) \in \text{(Mix)} : \ \sum_{i \in [N]} p_i z_i \leq \epsilon\right\} \qquad \text{(Mix-knapsack)}$$

  as a (joint) mixing set with a knapsack constraint.
- Valid inequalities for (Mix-knapsack) are given in [Luedtke, Ahmed, Nemhauser, 2010], [K., 2012], [Abdi and Fukasawa, 2016], [Zhao, Huang, Zeng, 2017].
- Random technology matrix and right-hand-side extensions [Tanner and Ntaimo, 2010], [Luedtke, 2014]
- It is harder to convexify (Mix-knapsack) due to the knapsack structure.

Binary mixing (star) inequalities

- The basic mixing set for given $j \in [k]$:

$$\left\{(y_j, z) \in \mathbb{R} \times \{0,1\}^N : \; y_j \geq w_{ij}(1 - z_i), \; \forall i \in [N]\right\}$$

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- The mixing inequality for a given subset $\Pi_j = \{j_1, \ldots, j_\tau\}$ with $w_{j_1 j} \geq \cdots \geq w_{j_\tau j}$ is:

$$y_j + \sum_{s \in [\tau]} (w_{j_s j} - w_{j_{s+1} j}) z_{j_s} \geq w_{j_1 j}$$

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- For example, the convex hull of

$$\left\{ (y_1, z) \in \mathbb{R}_+ \times \{0,1\}^3 \quad : \quad \begin{array}{l} y_1 \geq 8(1 - z_1) \\ y_1 \geq 6(1 - z_2) \\ y_1 \geq 13(1 - z_3) \end{array} \right\}$$

  is

$$\left\{ (y_1, z) \in \mathbb{R}_+ \times [0,1]^3 \quad : \quad \begin{array}{l} y_1 \geq 13 - 6z_2 - 7z_3 \\ y_1 \geq 13 - 13z_3 \\ y_1 \geq 13 - 8z_1 - 5z_3 \\ y_1 \geq 13 - 2z_1 - 6z_2 - 5z_3 \end{array} \right\}$$

$$= \left\{(y_1, z) \in \mathbb{R}_+ \times [0,1]^3 : \ \text{the mixing inequalities for } y_1\right\}.$$

How about the knapsack constraint?

- Typically, $p_i = \frac{1}{N}$ due to i.i.d. sampling
- In this case, the knapsack constraint is a cardinality constraint:

$$\sum_{i \in [N]} z_i \leq \lfloor N\epsilon \rfloor =: q$$

- Suppose $w_{1j} \geq \cdots \geq w_{Nj}$, then we must have

$$y_j \geq w_{(q+1)j}$$

- Use this to strengthen the formulation as

$$\left\{ (y_j, z) \in \mathbb{R} \times \{0,1\}^N : \; y_j + (w_{ij} - w_{(q+1)j}) z_i \geq w_{ij}, \; \forall i \in [q], \sum_{i \in [N]} z_i \leq q \right\}$$

- Apply mixing inequalities to the strengthened formulation [Luedtke et al., 2010].

Quantile cuts

- We can exploit the knapsack structure "indirectly" by the quantile cuts [Luedtke, 2014], [Xie and Ahmed, 2018].

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- A quantile cut is of the following form: for some $h \in \mathbb{R}_+^k$,

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- Quantile cuts are valid for (Mix-knapsack), and thus, for the formulation.
- We replace/relax the knapsack constraint by the quantile cut

$$y_1 + \cdots + y_k \geq \varepsilon$$

Mixing set with lower bounds

- Consider the set

$$\left\{ (y,z) : \begin{array}{ll} y_j \geq w_{ij}(1 - z_i), & \forall i \in [N], \forall j \in [k], \\ y_1 + \cdots + y_k \geq \varepsilon, & \\ y \in \mathbb{R}^k_+, \ z \in \{0,1\}^N & \end{array} \right\} \tag{Mix-lb}$$

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- Our goal is to understand the polyhedral structure of (Mix-lb) to generate strong valid inequalities.

Example 1

- The convex hull of

$$\left\{ (y,z) \in \mathbb{R}^2_+ \times \{0,1\}^3 \;:\; \begin{array}{l} y_1 \geq 8(1-z_1) \\ y_1 \geq 6(1-z_2) \\ y_1 \geq 13(1-z_3) \end{array} \;,\; \begin{array}{l} y_2 \geq 3(1-z_1) \\ y_2 \geq 4(1-z_2) \\ y_2 \geq 2(1-z_3) \end{array} \right\}$$

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$$= \left\{ (y,z) \in \mathbb{R}^2_+ \times [0,1]^3 : \text{the mixing inequalities for } y_1, y_2 \right\}.$$

- This was shown by [Atamtürk, Nemhauser, Savelsbergh '00].

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Example 2

The convex hull of

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Are the mixing and the aggregated mixing inequalities enough to describe the convex hull of (Mix-lb)?

Example 3

- The convex hull of

$$\left\{ \begin{array}{c} (y,z) \in \\ \mathbb{R}_+^2 \times \{0,1\}^3 \end{array} : \begin{array}{l} y_1 \geq 8(1-z_1) \\ y_1 \geq 6(1-z_2) \\ y_1 \geq 13(1-z_3) \\ y_1 \geq (1-z_4) \\ y_1 \geq 4(1-z_5) \end{array} , \begin{array}{l} y_2 \geq 3(1-z_1) \\ y_2 \geq 4(1-z_2) \\ y_2 \geq 2(1-z_3) \\ y_2 \geq 2(1-z_4) \\ y_2 \geq (1-z_5) \end{array} , y_1 + y_2 \geq 9 \right\}$$

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$$\left\{ \begin{array}{c} (y,z) \in \\ \mathbb{R}^2_+ \times [0,1]^3 \end{array} : \begin{array}{l} \text{the mixing inequalities for } y_1, y_2 \\ \text{the aggregated mixing inequalities for } y_1 + y_2 \\ 7y_1 + 6y_2 \geq 115 - 12z_2 - 49z_3 \\ 6y_1 + 5y_2 \geq 98 - 10z_2 - 42z_3 - z_4 \\ 3y_1 + 2y_2 \geq 47 - 4z_2 - 21z_3 - z_4 - 3z_5 \\ 3y_1 + 2y_2 \geq 47 - 4z_2 - 21z_3 - 4z_5 \\ 2y_1 + 3y_2 \geq 38 - 6z_2 - 14z_3 \\ y_1 + 2y_2 \geq 21 - 4z_2 - 7z_3 - z_5 \end{array} \right\}$$

Example 4

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Example 4

- The convex hull of

$$\left\{ \begin{array}{l}(y,z)\in \\ \mathbb{R}^2_+\times\{0,1\}^3\end{array} : \begin{array}{l} y_1 \geq 8(1-z_1) \\ y_1 \geq 6(1-z_2) \\ y_1 \geq 13(1-z_3) \\ y_1 \geq (1-z_4) \\ y_1 \geq 4(1-z_5) \end{array} , \begin{array}{l} y_2 \geq 3(1-z_1) \\ y_2 \geq 4(1-z_2) \\ y_2 \geq 2(1-z_3) \\ y_2 \geq 3(1-z_4) \\ y_2 \geq (1-z_5) \end{array} , y_1+y_2\geq 7 \right\}$$

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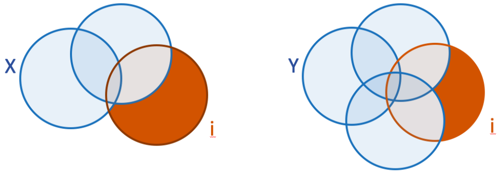
- We discover an underlying submodularity in (Mix-lb)!
- A function $f \in \{0,1\}^N \to \mathbb{R}$ is submodular if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad \forall A, B \subseteq [N].$$

- Alternatively, a function $f \in \{0,1\}^N \to \mathbb{R}$ is submodular if

$$f(X \cup \{i\}) - f(X) \geq f(Y \cup \{i\}) - f(Y) \quad \forall X \subset Y \subseteq [N], i \notin Y.$$



Submodularity in joint mixing sets

- (Mix) can be written as

$$\left\{(y,z):\ y_j \geq \max_{i\in[N]}\left\{w_{ij}(1-z_i)\right\},\ \forall j\in[k]\right\}$$
$$=\left\{(y,z):\ y_j \geq f_j(\mathbf{1}-z),\ \forall j\in[k]\right\}$$

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Remark

Each f_j is a submodular function:

$$\max_{i\in A}\left\{w_{ij}\right\}+\max_{i\in B}\left\{w_{ij}\right\}\geq \max_{i\in A\cup B}\left\{w_{ij}\right\}+\max_{i\in A\cap B}\left\{w_{ij}\right\}$$

for any $A, B\subseteq[N]$.

Submodularity and polymatroid inequalities

- Given a submodular (set) function $f : 2^{[N]} \to \mathbb{R}$, the extended polymatroid of f is

$$EP_f := \{\pi \in \mathbb{R}^n : \ \pi(V) \leq f(V), \ \forall V \subseteq [N]\} .$$

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Theorem [Lovász, 1983, Atamtürk and Narayanan 2008]

The convex hull of Q_f is given by
$$\left\{(y, z) \in \mathbb{R} \times [0,1]^N : \ y \ge \pi^\top(\mathbf{1} - z) + f(\emptyset), \ \forall \pi \in EP_{f - f(\emptyset)}\right\}.$$

Theorem [Edmonds, 1970]

Let $f : \{0,1\}^n \to \mathbb{R}$ be a submodular function. Then $\pi \in \mathbb{R}^n$ is an extreme point of EP_f if and only if there exists a permutation σ of $[N]$ such that $\pi_{\sigma(t)} = f(V_t) - f(V_{t-1})$, where $V_t = \{\sigma(1), \ldots, \sigma(t)\}$ for $t \in [N]$ and $V_0 = \emptyset$.

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Let $f : \{0,1\}^n \to \mathbb{R}$ be a submodular function. Then $\pi \in \mathbb{R}^n$ is an extreme point of EP_f if and only if there exists a permutation σ of $[N]$ such that $\pi_{\sigma(t)} = f(V_t) - f(V_{t-1})$, where $V_t = \{\sigma(1), \ldots, \sigma(t)\}$ for $t \in [N]$ and $V_0 = \emptyset$.

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- Separating the polymatroid inequalities can be done in $O(N \log N)$ time.

Example 1 (revisited)

- The convex hull of

$$\left\{ (y_1, z) \in \mathbb{R}_+ \times \{0,1\}^3 \quad : \quad \begin{array}{l} y_1 \geq 8(1 - z_1) \\ y_1 \geq 6(1 - z_2) \\ y_1 \geq 13(1 - z_3) \end{array} \right\},$$

is

$$\left\{ (y_1, z) \in \mathbb{R}_+ \times [0,1]^3 \quad : \quad \begin{array}{l} y_1 \geq 13 - 6z_2 - 7z_3 \\ y_1 \geq 13 - 13z_3 \\ y_1 \geq 13 - 8z_1 - 5z_3 \\ y_1 \geq 13 - 2z_1 - 6z_2 - 5z_3 \end{array} , \right\}$$

$$= \left\{ (y_1, z) \in \mathbb{R}_+ \times [0,1]^3 : \text{ the mixing inequalities for } y_1 \right\}.$$

- Consider $\sigma = \{2, 3, 1\}$.

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Joint mixing sets and mixing inequalities

- Recall the basic mixing set:

$$\left\{(y,z) \in \mathbb{R} \times \{0,1\}^N : \; y_j \geq f_j(\mathbf{1}-z), \; \forall j \in [k]\right\}$$

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- The mixing inequality from a subset $\Pi_j = \{j_1, \cdots, j_\tau\}$ with $w_{j_1 j} \geq \cdots \geq w_{j_\tau j}$ is:

$$y_j + \sum_{s \in [\tau]} (w_{j_s j} - w_{j_{s+1} j}) z_{j_s} \geq w_{j_1 j}$$

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Theorem [Kılınç-Karzan, Küçükyavuz, Lee, 2019+]

The polymatroid inequalities of f_j of the form

$$y_j \geq \pi^\top (\mathbf{1} - z) + f_j(\emptyset) \text{ for } \pi \in EP_{f_j - f_j(\emptyset)}$$

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Multiple submodular constraints

Theorem [Baumann et al., 2013]

Given submodular functions $f_1, \ldots, f_k : \{0,1\}^N \to \mathbb{R}$, the convex hull of

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Let $f_1, \dots, f_\ell : \{0,1\}^N \to \mathbb{R}$ be submodular. If $h_1, \dots, h_\ell \in \mathbb{R}^k$ are weakly independent, then

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- Now consider (Mix-lb):

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$$\left\{(y,z) : \; y_j \geq f_j(\mathbf{1} - z), \; \forall j \in [k], \quad y_1 + \cdots + y_k \geq g(\mathbf{1} - z)\right\}$$

where

$$f_j(z) = \max_{i \in [N]} \left\{w_{ij} z_i\right\}, \quad g(z) = \max\left\{\varepsilon, \sum_{j \in [k]} f_j(z)\right\} \quad \text{for } z \in \{0,1\}^N.$$

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- In contrast to f_j, the function g is not always submodular.

- Can we characterize when g is submodular?

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- Let $\bar{I}(\varepsilon) \subseteq [N]$ be a collection of scenarios defined as follows:

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- In Example 1, $\bar{I}(\varepsilon) = \{4, 5\}$.

$$\left\{ \begin{array}{l} (y,z) \in \\ \mathbb{R}^2_+ \times \{0,1\}^3 \end{array} : \begin{array}{l} y_1 \geq 8(1-z_1) \\ y_1 \geq 6(1-z_2) \\ y_1 \geq 13(1-z_3) \\ y_1 \geq (1-z_4) \\ y_1 \geq 4(1-z_5) \end{array}, \begin{array}{l} y_2 \geq 3(1-z_1) \\ y_2 \geq 4(1-z_2) \\ y_2 \geq 2(1-z_3) \\ y_2 \geq 2(1-z_4) \\ y_2 \geq (1-z_5) \end{array}, y_1 + y_2 \geq 7 \right\}$$

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  (1) $\sum_{j \in [k]} \max_{i \in \bar{I}(\varepsilon)} \{w_{ij}\} \leq \varepsilon$,
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g is submodular if and only if ε satisfies

1. $\bar{I}(\varepsilon)$ is ε-negligible,
2. $\varepsilon \leq L_W(\varepsilon) := \begin{cases} \min\limits_{p,q \in [N] \setminus \bar{I}(\varepsilon)} \left\{ \sum\limits_{j \in [k]} \min\{w_{pj}, \ w_{qj}\} \right\}, & \text{if } \ \bar{I}(\varepsilon) \neq [N], \\ +\infty, & \text{if } \ \bar{I}(\varepsilon) = [N] \end{cases}.$

- Now we know when (Mix-lb) has a submodularity structure.

Aggregated mixing inequalities

- (Mix-lb) can be written as

$$\left\{(y,z):\ y_j \geq f_j(\mathbf{1}-z),\ \forall j \in [k], \quad y_1 + \cdots + y_k \geq g(\mathbf{1}-z)\right\}$$

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The polymatroid inequalities of g of the form

$$y_1 + \cdots + y_k \geq \pi^\top(\mathbf{1}-z) + g(\emptyset) \text{ for } \pi \in EP_{g-g(\emptyset)}$$

are aggregated mixing inequalities. They can be separated in $O(kN \log N)$ time.

Example 2 (revisited)

The convex hull of

$$\left\{ \begin{array}{c} (y,z) \in \\ \mathbb{R}^2_+ \times \{0,1\}^3 \end{array} : \begin{array}{l} y_1 \geq 8(1-z_1) \\ y_1 \geq 6(1-z_2) \\ y_1 \geq 13(1-z_3) \\ y_1 \geq (1-z_4) \\ y_1 \geq 4(1-z_5) \end{array} , \begin{array}{l} y_2 \geq 3(1-z_1) \\ y_2 \geq 4(1-z_2) \\ y_2 \geq 2(1-z_3) \\ y_2 \geq 2(1-z_4) \\ y_2 \geq (1-z_5) \end{array} , y_1 + y_2 \geq 7 \right\}$$

is

$$\left\{ \begin{array}{c} (y,z) \in \\ \mathbb{R}^2_+ \times [0,1]^3 \end{array} : \begin{array}{l} \text{the mixing inequalities for } y_1, y_2 \\ y_1 + y_2 \geq 17 - z_1 - z_2 - 8z_3 \\ y_1 + y_2 \geq 17 - 2z_2 - 8z_3 \\ y_1 + y_2 \geq 17 - 3z_2 - 7z_3 \\ y_1 + y_2 \geq 17 - 2z_1 - 3z_2 - 5z_3 \\ y_1 + y_2 \geq 17 - 4z_1 - z_2 - 5z_3 \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (y,z) \in \\ \mathbb{R}^2_+ \times [0,1]^3 \end{array} : \begin{array}{l} \text{the mixing inequalities for } y_1, y_2 \\ \text{the ``aggregated'' mixing inequalities for ``} y_1 + y_2 \text{''} \end{array} \right\}$$

Consider $\sigma = \{2, 3, 1, 4, 5\}$.

Convex hull of (Mix-lb)

Theorem [Kılınç-Karzan, K., Lee, 2019+]

The following statements are equivalent:

(i) the convex hull of (Mix-lb) is obtained after adding the mixing and the aggregated mixing inequalities,

(ii) $f_1, \ldots, f_k, g$ are submodular.

(iii) ε satisfies the following 2 conditions:

1. $\bar{I}(\varepsilon)$ is ε-negligible,
2. $\varepsilon \leq L_W(\varepsilon) := \begin{cases} \min\limits_{p,q \in [N] \setminus \bar{I}(\varepsilon)} \left\{ \sum\limits_{j \in [k]} \min\{w_{pj},\ w_{qj}\} \right\}, & \text{if } \bar{I}(\varepsilon) \neq [N], \\ +\infty, & \text{if } \bar{I}(\varepsilon) = [N] \end{cases}.$

Outline

Two-stage (dynamic) chance-constrained problem (2CCP)

- Order of events:

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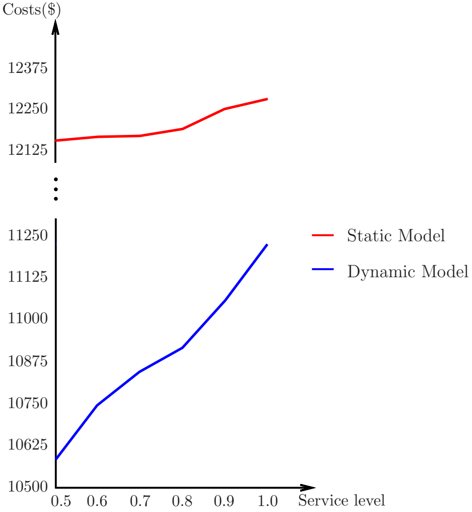
A two-stage chance-constrained program:

$$\begin{aligned} \min \quad & c^\top x + \mathbb{E}_\omega(g(\omega)^\top y(\omega)) \\ s.t. \quad & \mathbb{P}\{W(\omega)x + T(\omega)y(\omega) \geq r(\omega)\} \geq 1 - \epsilon \\ & x \in X \cap \mathcal{X}, y(\omega) \in \mathbb{R}_+^{n_2}, \omega \in \Omega. \end{aligned}$$

- Assume (wlog) i.i.d sample ($\mathbb{P}(\omega) = \frac{1}{N}$) and $g(\omega) \geq \mathbf{0}$.

Static vs Dynamic Decisions

Multi-stage inventory control problem with a service level constraint [Zhang, K., Goel, 2014]



- Significant cost savings by dynamic model.
- Higher service level gives rise to higher cost.
- Static model: limited flexibility

  Dynamic model: large cost savings with small decrease in service level

Deterministic Equivalent Formulation (DEF)

$$
\begin{array}{lllllll}
\min\limits_{x,y,z} & c^\top x & +\frac{1}{N}(g(\omega^1)^\top y(\omega^1) z_1 + g(\omega^2)^\top y(\omega^2) z_2 & \ldots + g(\omega^N)^\top y(\omega^N) z_N) & & & \\
 & T(\omega^1)x & +W(\omega^1)y(\omega^1) & & & +\bar{M}_1 z_1 & \geq r(\omega^1) \\
 & T(\omega^2)x & & +W(\omega^2)y(\omega^2) & & +\bar{M}_2 z_2 & \geq r(\omega^2) \\
 & \vdots & & & \ddots & & \vdots \\
 & T(\omega^N)x & & & +W(\omega^N)y(\omega^N) & +\bar{M}_N z_N & \geq r(\omega^N)
\end{array}
$$

$$
\sum_{k=1}^{N} z_k \leq \lfloor N\epsilon \rfloor = p; \quad x \in X \cap \mathcal{X},\ y(\omega) \in \mathbb{R}_+^{n_2}, \omega \in \Omega, z \in \mathbb{B}^N,
$$

where $\bar{M}_i$ is a vector of very large numbers, $\omega^i \in \Omega$, and

$$
z_i = \begin{cases} 0 & \text{if scenario } \omega^i \text{ is satisfied} \\ 1 & \text{otherwise.} \end{cases}
$$

Let $g(\omega^i) = g_i$, $T(\omega^i) = T_i$, $W(\omega^i) = W_i$, $r(\omega^1) = r_i$.

Decomposition algorithm for 2CCP

If there are second stage costs, and only a subset of scenarios are satisfied, then the traditional Benders feasibility and optimality cuts are no longer valid.

Goal: Develop valid feasibility and optimality cuts to the master problem of 2CCP.

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Goal: Develop valid feasibility and optimality cuts to the master problem of 2CCP.

- First, the algorithm requires solving a master problem (MP):

$$\begin{aligned} \mathrm{MP}(C,B) = \min_{x,z,\eta} \quad & c^\top x + \frac{1}{N}\sum_{i\in[N]} \eta_i \\ \text{s.t.} \quad & \sum_{i\in[N]} z_i \le q \\ & z \in \mathbb{B}^N \\ & x \in X \cap \mathcal{X},\ \eta \in \mathbb{R}_+^N \\ & (x,z) \in \mathcal{F},\ (x,z,\eta) \in \mathcal{O}, \end{aligned}$$

- $\mathcal{F}$ represents the collection of feasibility cuts and
- $\mathcal{O}$ represents the collection of optimality cuts.
- Let $P_i = \{x \in X \cap \mathcal{X} | \exists y \ge \mathbf{0} : T_i x + W_i y \ge r_i\}$, $i \in [N]$.

Subproblem 1 (SP1): Optimality Cut Generation (Basic)

- SP1 is used to cut off a feasible solution $(\hat{x}, \hat{z})$ which has incorrect second stage value $\hat{\eta}$.
- If the solution $(\hat{x}, \hat{z})$ is feasible, then $\forall \hat{z}_i = 0$, we solve single scenario linear optimization problem (SP1$_i$):

$$Y_i = \min_{y \in \mathbb{R}_+^{n_2}} g_i^\top y$$
$$s.t. \quad W_i y \geq r_i - T_i \hat{x} \qquad (\psi_i)$$

where ψ_i is the vector of dual variables for kth scenario subproblem.

- If SP1$_i$ is feasible, then compare $\hat{\eta}_i$ with Y_i. If $\hat{\eta}_i < Y_i$, then add the modified Benders optimality cut to $\mathcal{O}$:

$$\eta_i + M_i z_\omega \geq \psi_i^\top (r_i - T_i x)$$

M_i : big-M

- If SP1$_i$ (or equivalently $(\hat{x}, \hat{z})$) is infeasible, then go to the second subproblem (feasibility cut generation). [Luedtke, 2014]

Computations

A call center staffing problem

Instances		DEF		Basic Decomposition	
(N, ϵ)	(n_1, d)	`Time (slvd)`	Gap(%)	`Time (slvd)`	Gap(%)
(300, 0.05)	(5,10)	55.8 (5)	0	**54.6 (5)**	0
	(10,20)	258.3 (4)	0.1	**134.2 (5)**	0
(300, 0.1)	(5,10)	**126.0 (5)**	0	258.3 (4)	0.1
	(10,20)	1294.7 (4)	1.3	**483.7 (3)**	0.3
(400, 0.05)	(5,10)	**83.6 (5)**	0	133.8 (5)	0
	(10,20)	781(3)	2.3	**233.2 (5)**	0
(400, 0.1)	(5,10)	243 (5)	0	**220 (3)**	0.0
	(10,20)	>3600 (0)	3.4	**909.8 (5)**	0
(500, 0.05)	(5,10)	**170.6 (5)**	0	221(5)	0
	(10,20)	>3600 (0)	2.9	**313.2(5)**	0
(500, 0.1)	(5,10)	730 (2)	1.3	**166 (3)**	0.3
	(10,20)	>3600 (0)	5.8	**142.7 (3)**	0.3
Avg (Sum)	(n, m)	916.2 (38)	3.2	**276.1 (51)**	0.2

n_1: number of first stage variables (servers); d: number of customers.

Improved optimality cuts [Liu, K., Luedtke, 2016]

- For a given $\alpha \in \mathbb{R}^{n_1}$ and each $i \in [N]$, let

$$v_i(\alpha) = \min\{\alpha^\top x : x \in P_i\}$$

- Note $v_i(\alpha) \leq \alpha^\top x$ for all feasible x

- Then an improved optimality cut with $\phi = \psi_i^\top T_i$ is:

$$\eta_i + \left(\psi_i^\top r_i - v_i(\phi)\right) z_i \geq \psi_i^\top (r_i - T_i x).$$

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For $z_i = 1$, we get $\underbrace{\eta_i}_{\geq 0} \geq \underbrace{v_i(\phi) - \phi x}_{\leq 0}$, so it is valid.

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- We also give another class of strong optimality cuts

Computational results with strong decomposition

Instances		DEF	Basic Decomp.	Strong Decomp.
(N, ϵ)	(n_1, d)	Time(slvd) / gap	Time / gap	Time(slvd) / gap
(2000, 0.05)	(5,10)	120	1.8%	133
	(10,20)	9.0%	1.8%	1012
	(15,30)	14.6%	3.8%	343
(2500, 0.05)	(5,10)	165(2) / 6.5%	3.0%	131
	(10,20)	9.5%	2.8%	1246
	(15,30)	–	3.3%	1246
(3000, 0.05)	(5,10)	262(1) / 5.9%	1.8%	273
	(10,20)	17.4%	2.2%	2030
	(15,30)	–	3.2%	1207(2) / 0.4%

- " -" : failed to find solution.
- If the algorithm hits the time or memory limit, we report the end gap, otherwise we report time.
- For DEP (3000,0.05) (5,10), CPLEX successfully solved 1 instance in 262 seconds, and failed to solve the other 2 instances, with 5.9% end gap.

Do we really know $\mathbb{P}$?

- So far we discussed two-stage stochastic MIPs and chance-constrained programs with a given (finite) $\mathbb{P}$.

- Do we really know $\mathbb{P}$?

Outline

Chance-constrained program (CCP)

Consider **chance-constrained programs** in the general form:

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & \mathbb{P}^*[f(x,\xi) \geq 0] \geq 1-\epsilon, \\ & x \in \mathcal{X}. \end{aligned} \qquad \text{(CCP)}$$

Often, we do not know $\mathbb{P}^*$ precisely.

Sample average approximation (SAA)

- **Sample average approximation**: draw i.i.d. samples $\{\xi_i\}_{i\in[N]}$ from $\mathbb{P}^*$.

$$\mathbb{P}^*[f(x,\xi) \geq 0] \approx \mathbb{P}_N[f(x,\xi) \geq 0] := \frac{1}{N}\sum_{i\in[N]} \mathbb{1}(f(x,\xi_i) \geq 0).$$

- Focus on constraint functions $f(x,\xi)$ in piecewise linear form

$$f(x,\xi) := \min_{p\in[P]} \left\{ (b_p - A^\top x)^\top \xi + (d_p - a_p^\top x) \right\}.$$

Sample average approximation (SAA)

Approximate , (CCP) by

$$
\begin{aligned}
\min_x \quad & c^\top x \\
\text{s.t.} \quad & \underbrace{\frac{1}{N}\sum_{i\in[N]} \mathbb{1}(f(x,\xi_i)\geq 0) \geq 1-\epsilon,}_{\text{MIP-representable}} \qquad \text{(SAA)} \\
& x \in \mathcal{X}.
\end{aligned}
$$

Essentially, we need to ensure that that at least $N(1-\epsilon)$ samples satisfy $f(x,\xi_i)\geq 0$.

Sample average approximation (SAA)

Approximate , (CCP) by

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & \underbrace{\frac{1}{N} \sum_{i \in [N]} \mathbb{1}(f(x, \xi_i) \geq 0) \geq 1 - \epsilon,}_{\text{MIP-representable}} \\ & x \in \mathcal{X}. \end{aligned} \tag{SAA}$$

Essentially, we need to ensure that that at least $N(1-\epsilon)$ samples satisfy $f(x, \xi_i) \geq 0$.

The out-of-sample performance of the solution from (SAA) is often poor, particularly for small N.

- Just because $\mathbb{P}_N[f(x, \xi) \geq 0] \geq 1 - \epsilon$ does **not** mean that $\mathbb{P}^*[f(x, \xi) \geq 0] \geq 1 - \epsilon$.
- The so-called "Optimizer's Curse" [Smith and Winkler, 2006].

Improving out-of-sample performance

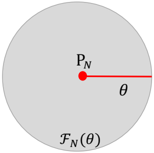
- Distributionally robust chance constrained program:

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & \mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon \;\; \forall\, \mathbb{P} \in \mathcal{F}_N(\theta), \\ & x \in \mathcal{X}, \end{aligned} \tag{DR-CCP}$$

where $\mathcal{F}_N(\theta)$: an ambiguity set of distributions on $\mathbb{R}^K$ that contains the empirical distribution $\mathbb{P}_N$:

$$\mathcal{F}_N(\theta) := \{\mathbb{P} : d(\mathbb{P}_N, \mathbb{P}) \leq \theta\}, \quad \text{w.h.p. } \mathbb{P}^* \in \mathcal{F}_N(\theta).$$

- **Intuition:** $\mathbb{P}_N$ will be (w.h.p.) close to $\mathbb{P}^*$, so make sure $\mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon$ **for all** $\mathbb{P}$ in a radius θ ball around $\mathbb{P}_N$.



- When N large, make the radius θ smaller.
- When N small, we are not as confident that $\mathbb{P}_N$ is close to $\mathbb{P}^*$, so make the radius θ larger.

Ambiguity set

Wasserstein ambiguity set with radius θ:

$$\mathcal{F}_N(\theta) := \{\mathbb{P} : d_W(\mathbb{P}_N, \mathbb{P}) \leq \theta\}$$

where

$$d_W(\mathbb{P}, \mathbb{P}') := \inf_{\Pi} \left\{\mathbb{E}_{(\xi,\xi')\sim\Pi}[\|\xi - \xi'\|] : \Pi \text{ has marginal distributions } \mathbb{P}, \mathbb{P}'\right\}.$$

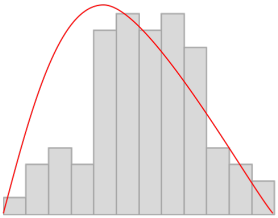


Figure 2: Wasserstein distance $d_W(\mathbb{P}_N, \mathbb{P})$: minimum distance required to transport grey bars to red curve.

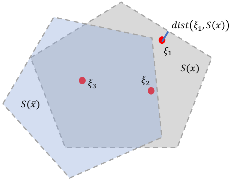
Has recently become very popular in optimization and machine learning [Mohajerin Esfahani and Kuhn, 2018].

Distance to violation

- For a given parameter ξ and decision x, define the **distance to violation**:

$$\mathrm{dist}(\xi, x) := \inf_{\Delta} \left\{ \|\Delta\| : f(x, \xi + \Delta) < 0 \right\}.$$

- Safe set $\mathcal{S}(x) = \{\xi : f(x, \xi) \geq 0\}$



Reformulation of (DR-CCP)

We now need to reformulate semi-infinite constraint $\mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon \ \ \forall\, \mathbb{P} \in \mathcal{F}_N(\theta)$.

- [Blanchet and Murthy, 2019], [Gao and Kleywegt, 2016], [Xie, 2019] show that for Wasserstein ambiguity

$$\mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon \ \ \forall\, \mathbb{P} \in \mathcal{F}_N(\theta) \iff \mathrm{CVaR}^{\mathbb{P}_N}_{1-\epsilon}(\mathrm{dist}(\xi,x)) \geq \frac{\theta}{\epsilon}$$

$$\begin{aligned}\mathrm{CVaR}^{\mathbb{P}_N}_{1-\epsilon}(\mathrm{dist}(\xi,x)) &:= \text{take the lowest } \epsilon N \text{ distances amongst } \{\mathrm{dist}(\xi_i,x)\}_{i\in[N]},\\ &\quad\ \text{then take their average}\\ &= \max_{t,r}\left\{ t - \frac{1}{\epsilon N}\sum_{i\in[N]} r_i : \begin{array}{l} r_i \geq 0,\ i \in [N] \\ t - r_i \leq \mathrm{dist}(\xi_i,x),\ i \in [N] \end{array}\right\}.\end{aligned}$$

Here larger distances are preferred, so distances are acceptability functionals rather than risk. CVaR definition is adapted accordingly.

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  Here larger distances are preferred, so distances are acceptability functionals rather than risk. CVaR definition is adapted accordingly.

- Usual SAA-CCP formulation implies $\mathrm{VaR}^{\mathbb{P}_N}_{1-\epsilon}(\mathrm{dist}(\xi,x)) \geq 0$. Its (conservative) CVaR approximation gives $\mathrm{CVaR}^{\mathbb{P}_N}_{1-\epsilon}(\mathrm{dist}(\xi,x)) \geq 0$. Compare with (DR-CCP).

Reformulation of (DR-CCP)

This implies that (DR-CCP) can be reformulated as

$$\begin{aligned} \min_{x,t,r} \quad & c^\top x \\ \text{s.t.} \quad & \epsilon t \geq \theta + \frac{1}{N} \sum_{i \in [N]} r_i, \\ & t - r_i \leq \text{dist}(\xi_i, x), \quad i \in [n] \\ & r_i \geq 0, \quad i \in [n] \\ & x \in \mathcal{X}. \end{aligned} \tag{DR-CCP-f}$$

The last step is to reformulate the constraint $t - r_i \leq \text{dist}(\xi_i, x)$.

- This depends on how we define $f(x, \xi)$.

Linear constraints

- For simple presentation, we focus on a single linear function with right-hand side uncertainty (no bilinear term):

$$f(x, \xi) := \xi + d - a^\top x,$$

  for given a, d.
- Distance to violation:

$$\mathrm{dist}(\xi, x) = \max\{0, \xi + d - a^\top x\} = \max\{0, f(x, \xi)\}.$$

- Our results extend to polyhedral structures of the form

$$f(x, \xi) := \min_{p \in [P]} \left\{ (b_p - A^\top x)^\top \xi + (d_p - a_p^\top x) \right\} \geq 0.$$

  - The *only* condition we impose is that the bilinear term $(A^\top x)^\top \xi$ is *the same for all* $p \in [P]$.

Reformulation of (DR-CCP)

However, $t - r_i \leq \text{dist}(\xi_i, x) = \max\{0, f(x, \xi_i)\}$

$$\iff t - r_i \leq 0 \quad \text{OR} \quad t - r_i \leq f(x, \xi_i).$$

is a non-convex constraint.

- We can model this with a binary variable and big-M constants:

$$\begin{aligned} & z_i \in \{0,1\}, \\ & t - r_i \leq f(x, \xi_i) + M_i z_i \\ & t - r_i \leq M_i(1 - z_i) \end{aligned}$$

  $z_i = 1$ indicates when $t - r_i \leq 0$, and $z_i = 0$ indicates when $t - r_i \leq f(x, \xi_i)$.

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  $z_i = 1$ indicates when $t - r_i \leq 0$, and $z_i = 0$ indicates when $t - r_i \leq f(x, \xi_i)$.

- M_i is a sufficiently large constant. For some fixed *optimal* decision x of (DR-CCP), we need

$$M_i \geq |f(x, \xi_i)| \quad \forall i \in [N].$$

  Choosing in this way requires understanding the structure of optimal solutions, which is not easy, and can still result in large values.

The basic MIP reformulation of (DR-CCP)

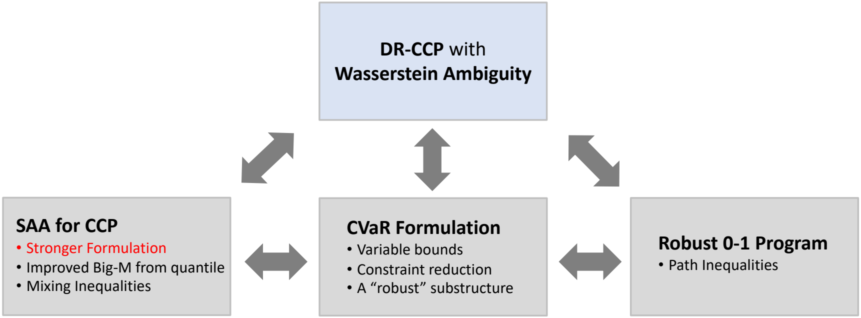
[Chen et al., 2018], [Xie, 2019] gave the following MIP reformulation for (DR-CCP):

$$\begin{aligned}
\min_{z,r,t,x} \quad & c^\top x \\
\text{s.t.} \quad & z \in \{0,1\}^N,\ t \geq 0,\ r \geq \mathbf{0},\ x \in \mathcal{X}, \\
& \epsilon\, t \geq \theta + \frac{1}{N} \sum_{i\in[N]} r_i, \\
& M_i(1-z_i) \geq t - r_i, \quad i \in [N], \\
& f(x,\xi_i) + M_i z_i \geq t - r_i, \quad i \in [N].
\end{aligned} \tag{DR-CCP-MIP}$$

Difficult to solve, especially for small θ even for $N = 100$.
In [Ho-Nguyen,Kılınç-Karzan, K., Lee, 2021a], we scale this up to $N = 1000 \sim 3000$.

Improvements to (DR-CCP-MIP) [Ho-Nguyen,Kılınç-Karzan, K., Lee, 2021a+]

Our key insight finds a link between (SAA) and (DR-CCP). This leads to a number of enhancements.



Connection to (SAA)

Denote the feasible regions of (SAA) and (DR-CCP) as

$$
\begin{aligned}
\mathcal{X}_{\mathrm{SAA}} &:= \{x \in \mathcal{X} : \mathbb{P}_N[f(x,\xi) \geq 0] \geq 1-\epsilon\}, \\
&= \left\{ x \in \mathcal{X} : \begin{array}{l} \frac{1}{N}\sum_{i \in [N]} w_i \leq \epsilon, \quad w \in \{0,1\}^N \\ f(x,\xi_i) + M_i w_i \geq 0, \quad i \in [N] \end{array} \right\} \\
\mathcal{X}_{\mathrm{DR}} &:= \left\{ x \in \mathcal{X} : \inf_{\mathbb{P} \in \mathcal{F}_N(\theta)} \mathbb{P}[f(x,\xi) \geq 0] \geq 1-\epsilon \right\} \\
&= \left\{ x \in \mathcal{X} : \begin{array}{l} \epsilon\, t \geq \theta + \frac{1}{N}\sum_{i \in [N]} r_i, \quad z \in \{0,1\}^N \\ M_i(1-z_i) \geq t - r_i, \quad i \in [N], \\ f(x,\xi_i) + M_i z_i \geq t - r_i, \quad i \in [N] \end{array} \right\}.
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\end{aligned}$$

Observation: in general $\mathcal{F}_N(0) = \{\mathbb{P}_N\} \subseteq \mathcal{F}_N(\theta)$ for any $\theta \geq 0$, so $\mathcal{X}_{\mathrm{DR}} \subseteq \mathcal{X}_{\mathrm{SAA}}$.

Naïvely, BLUE constraints are valid for $\mathcal{X}_{\mathrm{DR}}$, but require **different** binary variables (w vs. z).

Stronger formulation

Key result 1: for both RED and BLUE constraints, **the same binary variables** z **can be used.**

$$
\begin{aligned}
\min_{z,r,t,x} \quad & c^\top x \\
\text{s.t.} \quad & z \in \{0,1\}^N,\ t \geq 0,\ r \geq \mathbf{0},\ x \in \mathcal{X}, \\
& \epsilon\, t \geq \theta + \frac{1}{N} \sum_{i \in [N]} r_i, \\
& M_i(1 - z_i) \geq t - r_i, \quad i \in [N], \\
& f(x, \xi_i) + M_i z_i \geq t - r_i, \quad i \in [N], \\
& \frac{1}{N} \sum_{i \in [N]} z_i \leq \epsilon, \\
& f(x, \xi_i) + M_i z_i \geq 0, \quad i \in [N].
\end{aligned}
$$

Big-M reduction via the mixing procedure

Key result 2: we gain much more from the SAA constraints

$$\sum_{i\in[N]} z_i \le \epsilon N, \qquad f(x,\xi_i) + M_i z_i \ge 0,\ \forall i \in [N].$$

(Mixing procedure) [Luedtke et al., 2010] showed that we can **drastically** reduce M_i to

$$\sum_{i\in[N]} z_i \le \epsilon N, \qquad f(x,\xi_i) + m_i z_i \ge 0,\ \forall i \in [N].$$

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$$\sum_{i \in [N]} z_i \leq \epsilon N, \qquad f(x, \xi_i) + m_i z_i \geq 0, \ \forall i \in [N].$$

- For each $i \in [N]$, we have the inequalities

$$t - r_i \leq M_i(1 - z_i), \quad t - r_i \leq f(x, \xi_i) + M_i z_i$$
$$0 \leq f(x, \xi_i) + m_i z_i.$$

- It is easily checked that these imply

$$t - r_i \leq f(x, \xi_i) + m_i z_i.$$

- These can replace the inequalities $t - r_i \leq f(x, \xi_i) + M_i z_i$ in (DR-CCP-MIP).

Compact formulation of (DR-CCP-MIP) via CVaR interpretation

Key result 3: recall that the DR-CCP is

$$\mathrm{CVaR}_{1-\epsilon}^{\mathbb{P}_N}(\mathrm{dist}(\xi, x)) = \max_{t,r} \left\{ t - \frac{1}{\epsilon N} \sum_{i \in [N]} r_i : \begin{array}{l} r_i \geq 0,\ \ i \in [N] \\ t - r_i \leq \mathrm{dist}(\xi_i, x),\ \ i \in [N] \end{array} \right\} \geq \frac{\theta}{\epsilon}.$$

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- There always exists an optimal solution to the program such that

$$t = (\lfloor \epsilon N \rfloor + 1)\text{-th smallest value amongst } \left\{ \mathrm{dist}(\xi_i, x) = (\xi_i + d - a^\top x)_+ \right\}_{i \in [N]}$$

$$q = (\lfloor \epsilon N \rfloor + 1)\text{-th smallest value amongst } \{\xi_i\}_{i \in [N]}.$$

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$$q = (\lfloor \epsilon N \rfloor + 1)\text{-th smallest value amongst } \{\xi_i\}_{i\in[N]}.$$

- Suppose $\xi_i \geq q$. Then immediately $t \leq \mathrm{dist}(\xi_i,x)$. But then

$$t - r_i \leq \mathrm{dist}(\xi_i,x) \iff 0 \leq r_i + (\mathrm{dist}(\xi_i,x) - t).$$

Therefore when $\xi_i \geq q$, this constraint is **vacuous**, so we can **remove** $N - \lfloor \epsilon N \rfloor$ **constraints.**

Strengthened compact formulation of (DR-CCP-MIP)

$$
\begin{aligned}
\min_{z,r,t,x} \quad & c^\top x \\
\text{s.t.} \quad & z \in \{0,1\}^N,\ t \geq 0,\ r \geq \mathbf{0},\ x \in \mathcal{X}, \\
& \epsilon\, t \geq \theta + \frac{1}{N} \sum_{i \in [N]} r_i, \\
& M_i(1 - z_i) \geq t - r_i, \quad i \in [N], \\
& f(x, \xi_i) + (q - \xi_i) z_i \geq 0, \quad i \in [N], \\
& \frac{1}{N} \sum_{i \in [N]} z_i \leq \epsilon, \\
& f(x, q) - t \geq 0 \\
& f(x, \xi_i) + m_i z_i \geq t - r_i, \quad i \in [N] \text{ s.t. } q > \xi_i.
\end{aligned}
$$

Valid inequalities for (DR-CCP-MIP)

Key result 4: classes of valid inequalities can be derived by analysing different substructures in the formulation.

- Consider again the so-called mixing substructure from the (SAA) constraints:

$$\text{MIX} = \left\{ (x,z) : \begin{array}{ll} f(x,\xi_i) + m_i z_i \geq 0, & i \in [N] \\ z \in \{0,1\}^N & \end{array} \right\}$$

$$\text{conv}(\text{MIX}) = \text{MIX} \cap \{\text{mixing inequalities}\}.$$

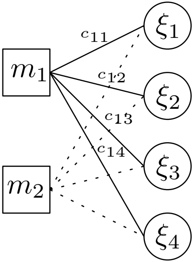
- There is also a substructure arising from robust 0-1 programming [Bertsimas and Sim, 2003]:

$$\text{ROB} = \left\{ (x,z,r,t) : \begin{array}{l} f(x,\xi_i) + m_i z_i \geq t - r_i, \ \ i \in [N] \text{ s.t } q > \xi_i \\ z \in \{0,1\}^N \end{array} \right\}$$

$$\text{conv}(\text{ROB}) = \text{ROB} \cap \{\text{path inequalities [Atamtürk, 2006]}\}.$$

Computational study

A distributionally robust chance-constrained transportation problem [Chen et al., 2018].



Given a set of factories $[F]$ with capacities $m_f, f \in [F]$, a set of distribution centers $[D]$ must meet the random demands ξ_d, $d \in [D]$ with high probability at minimum cost.

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & \mathbb{P}\left[\sum_{f\in[F]} x_{fd} \geq \xi_d, \quad \forall d \in [D]\right] \geq 1-\epsilon, \quad \mathbb{P} \in \mathcal{F}(\theta), \\ & \sum_{d\in[D]} x_{fd} \leq m_f, \quad f \in [F], \\ & x_{fd} \geq 0, \quad f \in [F],\ d \in [D]. \end{aligned}$$

$F = 5,\ D = 50,\ \epsilon = 0.1,\ \theta_1 = 0.001,\ \theta_j = \frac{j-1}{10}\theta_{\max}\ j = 2, \ldots, 10$

Performance analysis

We compare the following formulations (1 hour time limit)

- `Basic`: the basic formulation
- `Improved`: the strengthened compact formulation
- `Mixing+Path`: the strengthened compact formulation with both mixing and path inequalities.

Metrics:

- Time: recorded in seconds if instance is solved to optimality within one hour.
- Gap: if instance not solved in one hour, the final optimality gap as a percentage.

Summary of computational results

$N = 100$

	Basic	Improved	Mixing+Path	
	Time(Gap)$^{\text{Fnd}}$	Time	Time	M/P Cuts
θ_1	*(1.16)^{10}	4.29	8.40	41.7/274.6
θ_2	26.58(*)	0.04	0.06	0.3/88.2
θ_3	4.27(*)	0.04	0.05	0.0/73.8

$N = 3000$

	Basic	Improved	Mixing+Path	
	Time(Gap)$^{\text{Fnd}}$	Time(Gap)$^{\text{Fnd}}$	Time(Gap)$^{\text{Fnd}}$	M/P Cuts
θ_1	n/a^{0}	*(0.78)^{10}	*(0.48)^{10}	1470.3/4228.1
θ_2	*(69.56)^{5}	*(0.49)^{10}	*(0.41)^{10}	0.0/6102.2
θ_3	*(48.65)^{4}	17.89(*)	18.29(*)	0.0/200.8
θ_4	*(15.01)^{4}	13.74(*)	13.94(*)	0.0/94.1
θ_5	*(1.11)^{10}	12.75(*)	13.55(*)	0.0/88.3

Summary of computational results

$N = 3000$

	Basic		Improved		Mixing+Path	
	R.time	R.gap	R.time	R.gap	R.time	R.gap
θ_1	n/a	n/a	72.08	0.80	3601.05	0.48
θ_2	3144.09	70.41	134.46	0.55	3600.22	0.41
θ_3	2952.26	51.31	17.89	0.01	18.29	0.01
θ_4	2684.77	15.72	13.74	0.01	13.94	0.01
θ_5	3181.43	1.14	12.75	0.00	13.55	0.00
θ_6	3176.11	0.63	12.29	0.00	12.68	0.00
θ_7	2958.81	0.55	12.28	0.01	12.95	0.01
θ_8	2876.49	0.47	12.48	0.01	12.65	0.01
θ_9	2781.77	0.45	11.96	0.01	12.52	0.01
θ_{10}	2439.69	0.41	8.04	0.01	8.94	0.01

Discussion

- Strong reformulation of (DR-CCP) that exploits connections with various other models for uncertainty
  - nominal (SAA) relaxation
  - conditional value-at-risk (CVaR) interpretation
  - a substructure that arises in robust 0-1 programming.

  Using these connections we provided two classes of valid inequalities for (DR-CCP).
- Extended to more general polyhedral safety sets involving multiple linear constraints and left-hand side uncertainty. [Ho-Nguyen,Kılınç-Karzan, K., Lee, 2021b+]
- Left-hand side uncertainty case involves conic constraints in the form

$$\|Ax\|_p \leq t.$$

  - [Xie, 2019] use polymatroid inequalities to strengthen the formulation when x is a pure binary decision vector, using submodularity of $\|Ax\|_p$.
  - [Kılınç-Karzan, K., and Lee, 2020+] extend the polymatroid inequalities to obtain valid inequalities when x is mixed-binary. (MIP Workshop, May 25, 2021)
  - Submodularity can also be exploited for distributionally robust pure binary optimization problems under moment-based ambiguity sets, e.g., [Zhang et al., 2018].

Parting thoughts

- Stochastic optimization problems often give rise to large-scale MIPs
- Opportunities for theoretical, methodological, and computational MIP research
- Wide range of applications with broad impact (disaster logistics, energy, healthcare, and more).

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