Fixed Parameter Tractable Exact Algorithms for Lot-Sizing Problems with Multiple Capacities and Concave Costs

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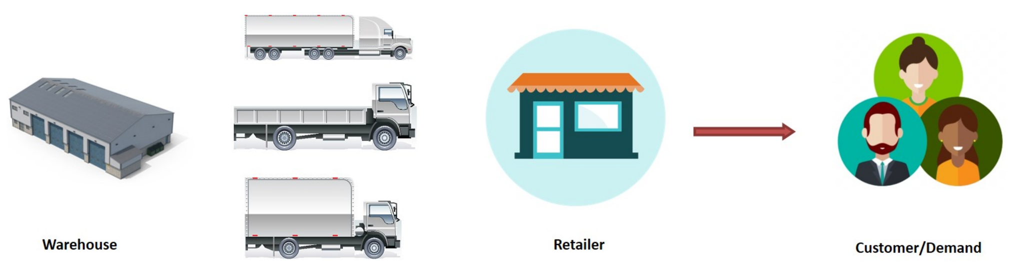
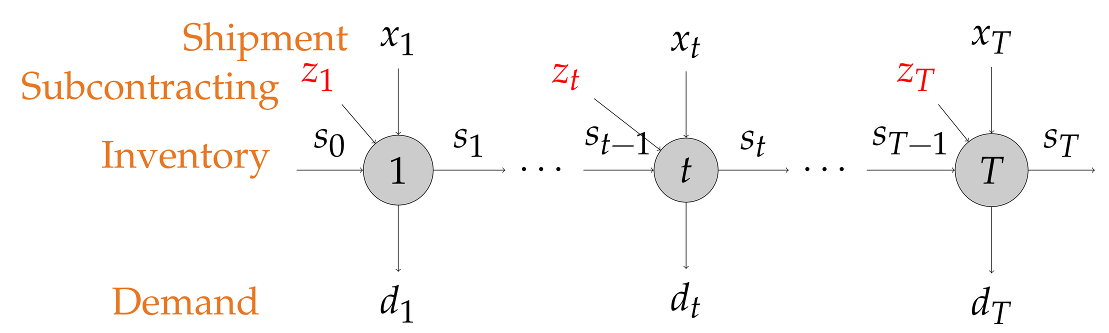
Multi-Capacitated Lot Sizing with Subcontracting (MCLS-S)



- *Data:* Demand is known over a planning horizon
- *Decision:* How much should the retailer order in each time period?
- *Objective:* Overall ordering costs and holding costs are minimized
- *Constraint:* Multiple trucks of different capacities $C_1, C_2, \ldots, C_n$ available for ordering in each time period

Network Flow Representation



MINLP Formulation for MCLS-S

$$
\begin{aligned}
\text{minimize} \quad & \sum_{t=1}^{T} (p_t(x_t) + h_t(s_t) + g_t(z_t)) + \sum_{t=1}^{T}\sum_{i=1}^{n} q_t^i y_t^i \\
\text{subject to} \quad & s_{t-1} + x_t + z_t = d_t + s_t, && t = 1, \ldots, T, \\
& x_t \leqslant \sum_{i=1}^{n} C_i y_t^i, && t = 1, \ldots, T \\
& y_t^i \in \{0,1\}, && t = 1, \ldots, T \text{ and } i = 1, \ldots, n, \\
& x_t, s_t, z_t \geqslant 0, && t = 1, \ldots, T.
\end{aligned}
$$

Notations:

- $p_t(.)$, $g_t(.)$ and $h_t(.)$ are concave transportation, subcontracting and holding cost functions respectively.
- $C_1, C_2, \ldots, C_n$ are capacities of n vehicles
- $q_t^1, q_t^2, \ldots, q_t^n$ ← setup cost of n vehicles in time period t
- x_t is quantity to be transported in time period t
- z_t is quantity to be subcontracted in time period t
- y_t^i is binary variable which is 1 iff vehicle i is setup in period t
- s_t is the inventory at the end of time period t

Open Question:

- Atamtürk and Hochbaum[2001]: MCLS-S with $n = 1 \rightarrow \mathcal{O}(T^5)$
- Does there exist a polynomial time exact algorithm for solving MCLS-S with fixed $n \geqslant 2$? (Note that the aforementioned MINLP formulation has nT binary variables.)

Contributions

- For a given $n \in \mathbb{Z}_+$, we develop fixed parameter tractable (polynomial) dynamic programming (DP) algorithm to solve MCLS-S to optimality in $O(T^{2n+3})$ time.
- MCLS-S can be reformulated as Lot-sizing problem with piecewise concave production costs (LS-PC-S) with $2^n - 1$ breakpoints
- LS-PC-S is equivalent to MCLS-S with $n = 0$ and piecewise concave production costs.
- Koca, Yaman and Aktürk[2014]: Reformulation of MCLS-S $\rightarrow \mathcal{O}(T^{(2^{n+1})+1})$

No. of vehicles (n)	Complexity using LS-PC-S reformulation	Our Contribution
2	$\mathcal{O}(T^9)$	$\mathcal{O}(T^7)$
4	$\mathcal{O}(T^{33})$	$\mathcal{O}(T^{11})$
10	$\mathcal{O}(T^{2049})$	$\mathcal{O}(T^{23})$
n	$\mathcal{O}(T^{(2^{n+1})+1})$	$\mathcal{O}(T^{2n+3})$

- Developed a new DP algorithm for LS-PC-S that takes $O(T^{2m+3})$ time where m is the number of breakpoints; worst case complexity same as Koca et. al (2014), but our DP is computationally 16 times faster than the latter.
- Developed an algorithm for Discrete MCLS i.e. capacity constraint in MCLS-S formulation is replaced by $x_t = \sum_{i=1}^{n} C_i y_t^i$; generalized a result from Vyve (2007) for $n = 1$.
- Provided an extended formulation for MCLS with $n = 2$ and Wagner-Whitin (or non-speculative) cost structure; generalized a result from Pochet and Wolsey (1994) for MCLS with $n = 1$

Outline of the algorithm for MCLS-S

- Compute the optimal cost α_{kl} for each interval $[k, l]$ with $1 \leqslant k \leqslant l \leqslant T$
- Construct a directed graph $G = (V, A)$ for $V = (1, \ldots, T+1)$ and $A = \{k, l+1 : 1 \leqslant k \leqslant l \leqslant T\}$ and assign the cost α_{kl} on arc $(k, l+1)$
- Find the least cost route from node 1 to $T+1$ using shortest path algorithm

DP Algorithm for MCLS-S

- **In this algorithm, we compute α_{kl} for a given interval $[k, l]$**

Definitions:

Regeneration Interval $[k, l]$: $s_{k-1} = s_l = 0$, but $s_t > 0$ for $t = k, \ldots, l-1$.
Semi-Regeneration Interval $[k, T]$: $s_{k-1} = 0$, and $s_t > 0$ for $t = k, \ldots, T$.
Fractional Period for MCLS-S: For MCLS-S, a period t is a fractional period if either $x_t \notin \left\{ \sum_{i \in R} C_i \text{ for all } R \subseteq \{1, \ldots, n\} \right\}$ or $z_t > 0$.

Theorem 1: There exists an optimal solution to MCLS-S which comprises of: (a) a series of regeneration intervals, each having at most one fractional period, that span the interval $[1, T']$ for some $0 \leqslant T' \leqslant T$, and (b) a semi-regeneration interval $[T'+1, T]$ with $s_t > 0$ for all $T'+1 \leqslant t \leqslant T$ which has no fractional period.

Theorem 2: The DP algorithm below solves MCLS-S in $\mathcal{O}(T^{2n+3})$ time where T is the number of time periods in the planning horizon and n is the number of capacity modules.

τ_i → number of times module i runs at full capacity from k through t
τ → a vector of τ_i's for $i \in \{1, \ldots, n\}$
S_j → $\binom{\{1, 2, \ldots, n\}}{j}$; denotes all j length combinations on $\{1, 2, \ldots, n\}$
d_{kl} → total demand in the interval $[k, l]$

Calculating All Fractional Production Levels

F → Set of fractional production levels;

$$
\Gamma := \left\{ \tau \in \mathbb{Z}_+^n : 0 < d_{kl} - \sum_{i=1}^{n} \tau_i C_i < C_n, \text{ and } d_{kl} - \sum_{i=1}^{n} \tau_i C_i \notin \{C_1, \ldots, C_{n-1}\} \right\}
$$

$$
F := \left\{ f^v : f^v = d_{kl} - \sum_{i=1}^{n} \tau_i^v C_i \text{ for all } \tau^v \in \Gamma \right\}
$$

Minimum Cost with Fractional Period

$G_k^u(t, \tau, 1)$ → value of minimum cost solution of producing $\tau_i C_i + f^v$ units from k to t

$$
G_k^u(t,\tau,1) = \begin{cases}
\infty, & \text{if } \tau_i > t-k+1 \text{ for any } i \in \{1,\ldots,n\} \text{ or } \sum_{i=1}^{n} \tau_i + 1 \geqslant n(t-k+1) \\
& \text{or } \sum_{i=1}^{n} \tau_i C_i + f^v \leqslant d_{kt} \text{ for } t \leqslant l-1 \ \sum_{i=1}^{n} \tau_i C_i + f^v \neq d_{kl} \text{ for } t = l \\
G_k(t,\tau,0), & \text{if } d_{kl} - \sum_{i=1}^{n} \tau_i C_i = 0 \\
\min \begin{cases}
G_k^u(t-1,\tau,1) + h_t\left(\sum_{i=1}^{n} \tau_i C_i + f^v - d_{kt}\right) \\
\min\limits_{\substack{j \in \{1,\ldots,n\} \\ S \in \mathcal{S}_j}} \left\{ G_k^u(t-1, \tau - \sum_{i \in S} e_i, 1) + \sum_{i \in S} \left(p_t(C_i) + q_t^i\right) \right. \\
\qquad \left. + h_t\left(\sum_{i=1}^{n} \tau_i C_i + f^v - d_{kt}\right) \right. \\
\min\limits_{\substack{j \in \{1,\ldots,n-1\} \\ S \in \mathcal{S}_j : v_{f^v} \in S \\ f^v < C_n}} \left\{ G_k(t-1, \tau - \sum_{i \in S \setminus \{v_{f^v}\}} e_i, 0) \right. \\
\qquad \left. + \sum_{i \in S \setminus \{v_{f^v}\}} \left(p_t(C_i + f^v) + q_t^i + q_t^{v_{f^v}}\right) + h_t\left(\sum_{i=1}^{n} \tau_i C_i + f^v - d_{kt}\right) \right. \\
\min\limits_{\substack{j \in \{1,\ldots,n\} \\ S \in \mathcal{S}_j}} \left\{ G_k(t-1, \tau - \sum_{i \in S} e_i, 0) + \sum_{i \in S} \left(p_t(C_i) + q_t^i\right) \right. \\
\qquad \left. + g_t(f^v) + h_t\left(\sum_{i=1}^{n} \tau_i C_i + f^v - d_{kt}\right) \right.
\end{cases} & \text{, otherwise.}
\end{cases}
$$

Minimum Cost without Fractional Period

$G_k(t, \tau, 0)$ → value of minimum cost solution of producing $\tau_i C_i$ units from time period k to t (computed in a similar manner as $G_k^v(t, \tau, 1)$)

Overall Optimal Solution of a given interval $[k, l]$

$$
\alpha_{kl} = \begin{cases}
\min\limits_{\tau^u \in \Gamma} \{G_k^u(l, \tau^u, 1)\} & \text{for } 1 \leqslant k \leqslant l < T, \\
\min \left\{ \min\limits_{\tau^u \in \Gamma} \{G_k^u(l, \tau^u, 1)\}, \min\limits_{\tau \in \{0,\ldots,T\}^n} \{G_k(l, \tau, 0)\} \right\} & \text{for } 1 \leqslant k \leqslant l = T.
\end{cases}
$$

Variants of MCLS-S

Lot-sizing with piecewise concave production costs (LS-PC-S)

- Present a new algorithm that solves LS-PC-S
- The complexity of our algorithm is $\mathcal{O}(T^{2m+3})$ where m is the number of breakpoints in the piecewise cost function (as good as Koca et al.[2014])
- Utilize the fact that fractional production levels are dependent only on the total demand and the value of breakpoints; enables parallel computing
- Hence, despite same complexity, our algorithm is computationally 16 times faster

Discrete MCLS with and without Backlogging

- "All-or-nothing" production in each time period
- Capacity constraint in the MCLS formulation is replaced by $x_t = \sum_{i=1}^{n} C_i y_t^i$
- Consider two cases: with and without backlogging
- Develop a fixed parameter tractable (polynomial) algorithm that takes $\mathcal{O}(T^{n+1})$ time for a fixed $n \geqslant 2$.
- Generalize the algorithm of Vyve (2007) for Discrete MCLS with $n = 1$.

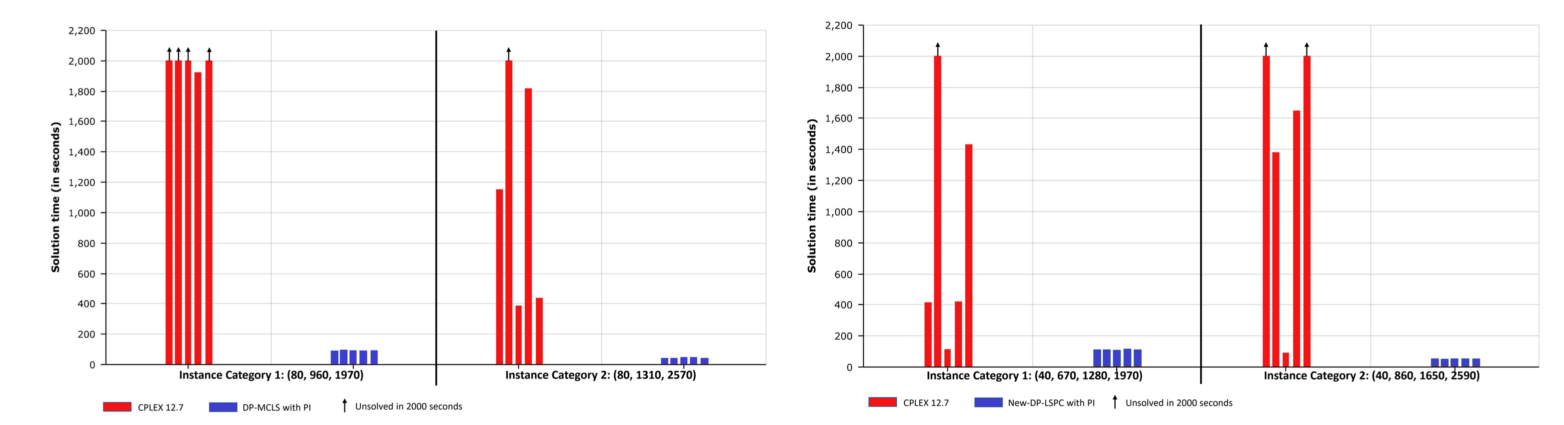
Computational Results

- Compared the running time of our algorithm with Koca et al.[2014] and CPLEX 12.70
- `Old-DP-LSPC` denotes the algorithm for LS-PC presented by Koca et al.[2014]
- `New-DP-LSPC` denotes the algorithm for LS-PC presented in this work
- `DP-MCLS` denotes the algorithm for MCLS presented in this work
- No-PI and With-PI denote without and with parallel computing, respectively.
- T denotes number of periods in the planning horizon
- Avg. denotes average solution time (in seconds) over five instances of each category;
- S.Dev. denotes standard deviation of solution times (in seconds);
- #USI denotes number of instances unsolved using corresponding methodology.
- '-' denotes none of the five instances were solved within 2000 seconds.

Table: Summary of Results for MCLS and LS-PC

n	T	$(C_1, .., C_n)$	CPLEX 12.7			`Old-DP-LSPC`		`New-DP-LSPC` No-PI		`New-DP-LSPC` With-PI		`DP-MCLS` No-PI		`DP-MCLS` With-PI	
			Avg.	S.Dev.	#USI	Avg.	S.Dev.	Avg.	S.Dev.	Avg.	S.Dev.	Avg.	S.Dev.	Avg.	S.Dev.
2	40	(670, 1280)	184.6	127.5	0	759.4	1.5	226.5	2.3	36.6	0.7	28.6	1.1	4.7	0.3
		(850, 1590)	423.3	408.9	2	405.8	2.1	101.0	1.7	17.0	0.7	16.8	0.5	3.7	0.1
	60	(960, 1970)	475.0	366.5	1	-	-	781.7	1.7	129.4	1.2	96.3	0.7	15.7	0.1
		(870, 1590)	688.0	348.1	2	-	-	1429.9	6.1	253.1	3.3	145.3	1.9	23.7	0.3
	80	(960, 1970)	1922.5	0.0	4	-	-	-	-	-	-	527.9	2.1	86.0	1.3
		(1310, 2570)	945.5	584.9	1	-	-	-	-	-	-	250.4	1.7	41.4	0.6
3	40	(970, 1950,2810)	-	-	5	-	-	-	-	-	-	174.4	1.9	35.4	1.0
		(790, 1650, 2410)	864.7	680.3	2	-	-	-	-	-	-	333.8	2.2	51.2	1.3
	40	(670, 1280,1970)	557.6	433.6	2	-	-	-	-	-	-	725.5	3.7	156.4	2.2
		(860, 1650, 2590)	249.6	274.3	2	-	-	-	-	-	-	327.1	2.5	49.4	1.7

m	T	$(b_1, .., b_m)$	CPLEX 12.7			`Old-DP-LSPC`		`New-DP-LSPC` No-PI		`New-DP-LSPC` With-PI	
			Avg.	S.Dev.	#USI	Avg.	S.Dev.	Avg.	S.Dev.	Avg.	S.Dev.
2	50	(800, 1600)	76.9	50.4	0	162.1	3.3	38.2	1.0	7.5	0.4
		(1310, 2570)	98.2	58.6	0	83.0	2.6	25.8	1.0	6.0	0.2
	75	(1310, 2570)	811.2	597.6	0	701.6	7.3	199.3	1.9	39.0	1.0
		(1790, 3470)	239.8	212.7	0	375.2	3.5	102.1	0.9	20.0	0.9
3	40	(970, 1950, 2810)	573.3	413.0	2	377.3	1.1	143.8	1.6	28.8	0.9
		(790, 1650, 2410)	1038.8	678.6	2	702.1	1.9	245.2	1.8	49.1	1.6
	40	(670, 1280, 1970)	593.2	497.9	1	1323.9	5.3	543.0	1.8	106.7	1.2
		(860, 1650, 2590)	98.6	0.0	4	482.6	2.1	186.2	0.9	36.8	1.5



(a) CPLEX vs. DP-MCLS with $n = 2$ (b) CPLEX vs. New-DP-LSPC with $n = 3$

Summary of Results:

- Our algorithm for LS-PC in comparison to Koca et al.[2014] is consistently
  - 18 times faster in case of 2 breakpoints - 13 times faster in case of 3 breakpoints
- Our algorithm for MCLS in comparison to CPLEX 12.70 is on an average
  - 20 times faster in case of 2 machines - 9 times faster in case of 3 machines
- Our algorithm for MCLS in comparison to Koca et al.[2014] is consistently
  - 150 times faster for 2 machines and 40 time periods

Computational Results for Discrete MCLS:

- Compared running time of our algorithms for Discrete MCLS with and without backlogging with time taken by CPLEX 12.7
- CPLEX unable to solve 81 out of 240 instances within 2000 seconds;
- For the remaining 159 instances, average solution time of CPLEX is 810 seconds; average solution time of our algorithms is 83 seconds.

Major References

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