Strong Valid Inequalities for a Class of Concave Submodular Minimization Problems under Cardinality Constraints

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Motivation

Let $N = \{1,2,\ldots,n\}$ and $2^N = \{S : S \subseteq N\}$.

Definition. A set function $g : 2^N \to \mathbb{R}$ is *submodular* if for any $X, Y \in 2^N$, $g(X) + g(Y) \geq g(X \cap Y) + g(X \cup Y)$.

- For $a \in \mathbb{R}^n_+$, concave $f : \mathbb{R} \to \mathbb{R}$,
$$F(X) := f\left(\sum_{i\in X} a_i\right), \forall X \subseteq N \text{ is submodular.}$$
- W.L.O.G. $f(0) = 0$.
- Problem of consideration:
$$\min\left\{ f\left(\sum_{i=1}^n a_i x_i\right) : x \in \{0,1\}^n, \sum_{i=1}^n x_i \leq k \right\}$$
- Key structure:
$$\mathscr{P}^m_k = \left\{ (w,x) \in \mathbb{R} \times \{0,1\}^n : w \geq f\left(\sum_{i=1}^n a_i x_i\right), \sum_{i=1}^n x_i \leq k \right\}$$
  m: # of distinct weights in a
- Applications: mean-risk optimization, concave cost facility location, etc.
- This problem is polynomial-time solvable [1, 2].
- How to fully describe $\text{conv}(\mathscr{P}^m_k)$ when $m \geq 2$ is an open problem.

Main Contributions

- Propose three classes of strong valid linear inequalities for conv$(\mathscr{P}^2_k)$, with explicit forms and facet conditions.
- Full linear characterization of conv$(\mathscr{P}^2_2)$.
- A computational study using proposed inequalities in a branch-and-cut framework.
- Extensions to (i) the case of $m \geq 3$; (ii) mixed-binary conic optimization.

Notation

- $[j] = \{1,2,\ldots,j\}$ and $[i,j] = \{i, i+1, \ldots, j\}$.
- In $\mathscr{P}^2_k$, - weights in a: a_L, a_H, with $0 \leq a_L \leq a_H$;
  - $\mathscr{I}_c := \{i \in N : a_i = a_c\}, c \in \{L,H\}$;
  - c^t: first t lower-/higher-weighted items, $c \in \{L,H\}$;
  - $d_L = |\mathscr{I}_L \cap [k-1]|$, $d_H = |\mathscr{I}_H \backslash [k-1]|$;
  - $\mathscr{I}_L \cap [k-1] = \mathscr{L} = (\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{d_L})$;
  - $\mathscr{I}_H \backslash [k-1] = \mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_{d_H})$.
- Marginal return: $\rho_i(X) = g(X \cup \{i\}) - g(X), X \subseteq N, i \in N \backslash X$.
- For any $B \subseteq N$,
$$\mathscr{P}^m_k(B) = \left\{ (w,x) \in \mathbb{R} \times \{0,1\}^{|B|} : w \geq f\left(\sum_{i\in B} a_i x_i\right), \sum_{i\in B} x_i \leq k \right\}$$

Lifting Problems

- $B = [b] \subseteq N$, lift $w \geq \pi^\top x$ valid for conv$(\mathscr{P}^2_k(B))$ to obtain
$$w \geq \sum_{i=1}^b \pi_i x_i + \sum_{j=b+1}^n \phi_j x_j$$
  valid for conv$(\mathscr{P}^2_k)$.
- Lifting problem for $j \in [b+1, n]$:
$$\begin{aligned} \phi_j := \min \quad & w - \sum_{i=1}^b \pi_i x_i - \sum_{i=b+1}^{j-1} \phi_i x_i \\ \text{s.t.} \quad & w \geq f\left(a_j + \sum_{i=1}^{j-1} a_i x_i\right), \\ & \sum_{i=1}^{j-1} x_i \leq k-1, \\ & x \in \{0,1\}^{j-1}. \end{aligned}$$
- Objective value with x, or its support X:
$$\phi^X_j = f\left(a_j + \sum_{i\in X} a_i\right) - \sum_{i\in X\cap B} \pi_i - \sum_{i\in X\backslash B} \phi_i.$$

Extended Polymatroid Inequalities (EPIs)

- Given submodular $G : 2^N \to \mathbb{R}$, a permutation δ of N,
$$\text{(EPI)} \quad w \geq \sum_{i=1}^n \rho_{\delta_i} x_{\delta_i},$$
  $\rho_{\delta_1} = G(\{\delta_1\})$ and $\rho_{\delta_i} = \rho_{\delta_i}(\{\delta_1, \ldots, \delta_{i-1}\})$ for $i \in [2,n]$.
- facet-defining for conv$(\mathscr{P}^2_k(S))$ with $S \subseteq N$, $|S| \leq k$

Separation Inequalities (SIs)

- Given permutation $\delta = \{\delta_1, \ldots, \delta_n\}$, $i_0 \in \{0,1,\ldots,k-1\}$,
$$\text{(SI)} \quad w \geq \sum_{i=1}^{i_0} \rho_{\delta_i} x_{\delta_i} + \sum_{i=i_0+1}^n \psi x_{\delta_i}$$
$$\psi = [f(k\alpha) - f(i_0\alpha)]/(k - i_0), \rho_{\delta_i} = f(i\alpha) - f((i-1)\alpha).$$
- define conv$(\mathscr{P}^1_k)$ [1]
- valid for conv$(\mathscr{P}^1_k(\mathscr{I}_c))$ with $c \in \{L,H\}$

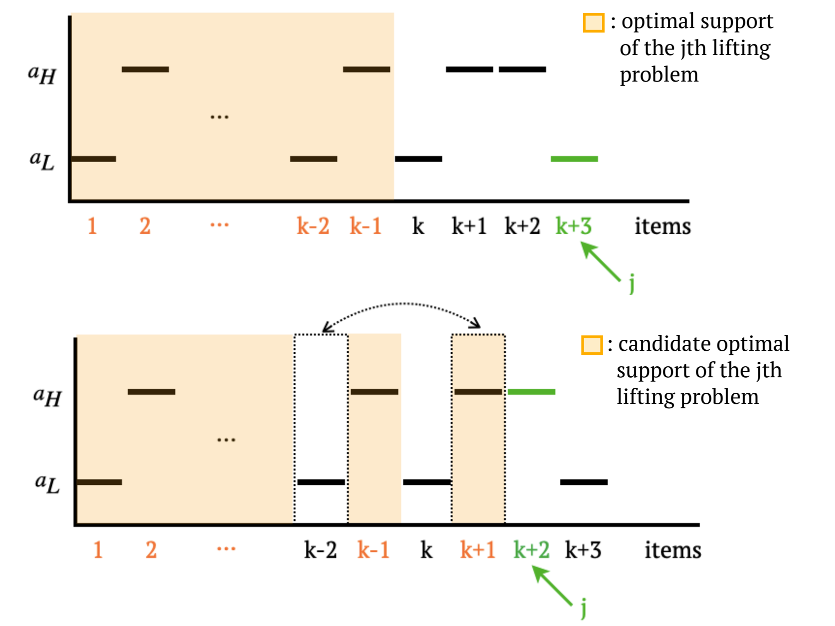
Exact Lifting of EPIs

- **Lifted-EPIs**: $w \geq \sum_{i=1}^k \rho_i x_i + \sum_{i=k+1}^n \zeta_i x_i$
- $w \geq \sum_{i=1}^k \rho_i x_i$ an EPI for $\mathscr{P}^2_k([k])$.

Proposition. The lifting coefficients of a lifted-EPI are
$$\zeta_j = \begin{cases} \zeta_j^{[k-1]}, & \text{if } j \in \mathscr{I}_L \backslash [k], \\ \min\left\{ \zeta_{\mathscr{H}_{i-1}}, \zeta_j^{\mathscr{H}(\min\{i-1,d_L\}) \cup \mathscr{L}(d_L - i + 1) \cup (\mathscr{I}_H \cap [k-1])} \right\}, & \text{if } j = \mathscr{H}_i, i \in [d_H], \end{cases}$$
where $\zeta_{\mathscr{H}_0} = \zeta_j^{\mathscr{H}(\min\{0,d_L\}) \cup \mathscr{L}(d_L - 0) \cup (\mathscr{I}_H \cap [k-1])} = \zeta_j^{[k-1]}$.

- facet-defining for conv$(\mathscr{P}^2_k)$
- stronger than approximate lifted inequalities (ALIs) proposed in [4]

Example.



Full Linear Characterization of conv$(\mathscr{P}^2_2)$

Super-average inequality:
$$w \geq \sum_{i\in\mathscr{I}_L} f(2a_L) x_i / 2 + \sum_{i\in\mathscr{I}_H} f(2a_H) x_i / 2$$

Assumption. $f(a_L + a_H) - f(a_L) \leq f(2a_H)/2$.

Theorem. Under the given assumption, the set of $(w,x) \in \mathbb{R}^{n+1}$ constructed by the *lifted-EPIs*, *lower-SIs*, *higher-SIs*, the *super-average inequality*, together with the *trivial bounds* and the *cardinality constraint*, is conv$(\mathscr{P}^2_2)$.

Exact Lifting of SIs

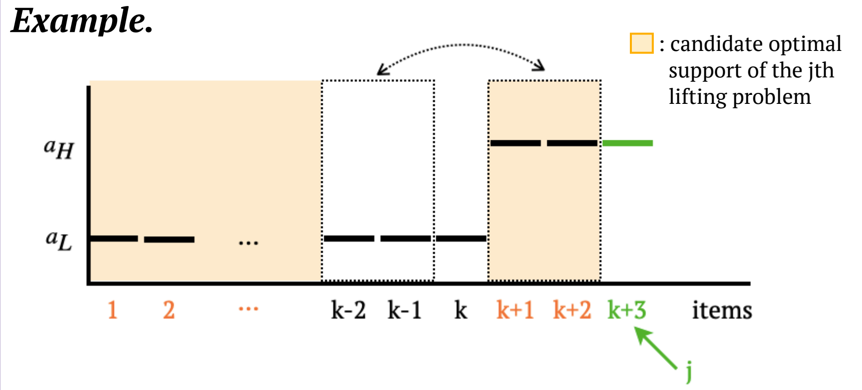
- **Lower-SIs:** lifted SIs of conv$(\mathscr{P}^1_k(\mathscr{I}_L))$
- **Higher-SIs:** lifted SIs of conv$(\mathscr{P}^1_k(\mathscr{I}_H))$

Proposition. The lower-SI lifting coefficients are
$$\eta_j = \begin{cases} \eta^{[k-1]}, & j = |\mathscr{I}_L| + 1, \\ \min\{\eta_{j-1}, \eta^{[k-1-s] \cup H^s}\}, & j = |\mathscr{I}_L| + 1 + s, s \in [n - 1 - |\mathscr{I}_L|]. \end{cases}$$

Proposition. Suppose for a given $i_0 \in \{0,1,\ldots,k-2\}$,
$$f(a_L + (i_0+1)a_H) - f(a_L + i_0 a_H) \leq \frac{f(ka_H) - f(i_0 a_H)}{k - i_0}.$$
The lifting coefficients in higher-SIs are
$$\gamma_j = \begin{cases} \gamma^{[k-1]}, & j = |\mathscr{I}_H| + 1, \\ \min\{\gamma_{j-1}, \gamma^{L^t \cup [k-1-t]}\}, & j = |\mathscr{I}_H| + 1 + t, t \in [n - 1 - |\mathscr{I}_H|]. \end{cases}$$

- facet-defining for conv$(\mathscr{P}^2_k)$ if the base SIs are facet-defining for conv$(\mathscr{P}^1_k(\mathscr{I}_L))$ or conv$(\mathscr{P}^1_k(\mathscr{I}_H))$

Example.



Extensions

1. Multi-weighted Case ($m \geq 3$)

Method 1. If f is monotone, convert $\mathscr{P}^m_k$ to $\mathscr{P}^2_k$.

Example. Given $a = [\ldots]$ such that $\leq \leq \leq$.

The proposed inequalities for conv$(\mathscr{P}^2_k)$ w.r.t. to any of

$\hat{a} = [\ldots]$

$\hat{a} = [\ldots]$

$\hat{a} = [\ldots]$

are valid for conv$(\mathscr{P}^4_k)$ w.r.t. a if f non-decreasing.

Method 2. Approximately lift the proposed inequalities.

Proposition. Given any $w \geq \pi^\top x$ valid for conv$(\mathscr{P}^2_k(S))$,
$$w \geq \sum_{i=1}^s \pi_i x_i + \sum_{i=s+1}^n [f(\mathscr{T}_i + a_i) - f(\mathscr{T}_i)] x_i$$
is valid for conv$(\mathscr{P}^m_k)$,
$$\mathscr{T}_i := \max\left\{ \sum_{i\in T} a_j : T \subseteq [i-1], |T| = k-1 \right\}.$$
$S = [s] \subseteq N$: all items with a pair of distinct weights.

2. Mixed-binary conic optimization

$\mathscr{S} := \{(x,y) \in \mathbb{B}^n \times \mathbb{R}^m : \exists w \in \mathbb{R}_+ \text{ s.t. } w \geq F(x), \mathbf{1}^\top x \leq k, Ay + Bw \in \mathscr{K}\}$

where $\mathscr{K}$ is a cone. Inspired by [3], conv$(\mathscr{S})$ is

$\{(x,y) \in [0,1]^n \times \mathbb{R}^m : \exists w \in \mathbb{R}_+ \text{ s.t. } (w,x) \in \text{conv}(\mathscr{P}^2_k), Ay + Bw \in \mathscr{K}\}$

if $F(x) = f(a^\top x)$, f concave, $a \in \mathbb{R}^n_+$ has two weights.

Computational Study

- **Test problem**
  Cardinality-constrained mean-risk minimization.
- **Methods**
  - SOCP;
  - ALI: branch-and-cut with ALIs [4];
  - LEPI-LSI: branch-and-cut with lifted-EPIs and lower-SIs.
- **Implementation**
  Python 3.6
  Gurobi 7.5.1

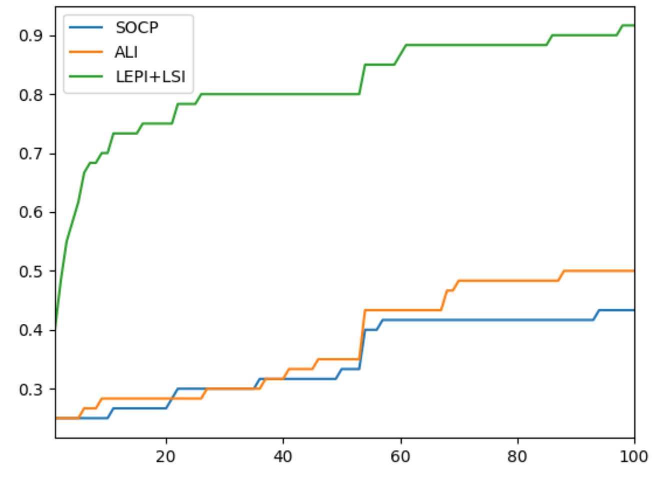


Figure 1. Performance profile of the three solution methods

Table 1: The statistics are averaged across five trials. ^{i_1,i_2}: i_1 instances are solved within 30 min, and i_2 instances exceed the time limit. The average number of total LEPI-LSI cuts is $m = \#$ LEPIs + # LSIs.

n	k	method	time (s)	end gap	# nodes	# total cuts
100	5	LEPI-LSI	$1.45^{5,0}$	–	1468.8	145.8 = 11.8 + 134.0
		ALI	$49.38^{5,0}$	–	22164.8	2214.0
		SOCP	$256.80^{5,0}$	–	113052.6	N/A
	10	LEPI-LSI	$690.43^{5,0}$	–	154966.8	15496.0 = 1768.0 + 13728.0
		ALI	$797.79^{3,2}$	6.40%	176897.4	17689.2
		SOCP	$1093.47^{2,3}$	11.03%	225915.2	N/A
150	5	LEPI-LSI	$3.09^{5,0}$	–	2929.4	291.6=22.4+269.2
		ALI	$128.45^{5,0}$	–	91859.0	9183.6
		SOCP	$1010.90^{3,2}$	1.86%	687479.0	N/A
	10	LEPI-LSI	$1187.80^{3,2}$	6.18%	327231.4	32721.4=7767.0+24954.4
		ALI	$1441.73^{2,3}$	17.50%	332222.0	33220.8
		SOCP	$1620.44^{1,4}$	11.07%	543848.8	N/A
200	5	LEPI-LSI	$92.35^{5,0}$	–	59716.2	5971.2=871.6+5099.6
		ALI	$1287.11^{3,2}$	10.06%	724583.2	72456.2
		SOCP	$1616.85^{1,4}$	19.18%	1842034.0	N/A
	10	LEPI-LSI	$–^{0,5}$	3.69%	476101.4	47608.6=12034.2+35574.4
		ALI	$–^{0,5}$	12.93%	518805.0	51879.4
		SOCP	$–^{0,5}$	15.06%	727939.4	N/A

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