On the polyhedrality of Chvátal-Gomory closure

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Overview

- *Chvátal-Gomory procedure* was the first cutting-plane procedure has ever been introduced: Given closed convex set K and a valid inequality $cx \le \delta$ with $c \in \mathbb{Z}^n$, $cx \le \lfloor \delta \rfloor$ is called the *CG cut* of K.
- *CG closure* of a closed convex set K is defined as

$$K' := \bigcap_{\substack{(c,\delta)\in\mathbb{Z}^n\times\mathbb{R}, \\ K\subseteq\{cx\le\delta\}}} \{cx \le \lfloor \delta \rfloor\}.$$

- Seminal result by Schrijver [1980]: If K is a rational polyhedron, then K' is a rational polyhedron. He further asked: **whether K' is a (rational) polytope if K is an irrational polytope**?
- This long-standing open problem was answered in the affirmative independently by Dunkel and Schulz [2010] and Dadush et al. [2011]. In particular, Dadush et al. [2011] proved the stronger result: **K' is a rational polytope if K is a compact convex set**.
- **Question:** *Can the boundedness assumption be further relaxed? What is the necessary condition for K' to be a rational polyhedron?*

Main result

Theorem 1

Given a closed convex set K, TFAE:

- K' is a rational polyhedron;
- K' is finitely-generated;
- there exists a rational polyhedron P such that $K' \subseteq P \subseteq K$.

Theorem 2

If K is the sum of a compact convex set and a rational polyhedral cone, then K' is a rational polyhedron.

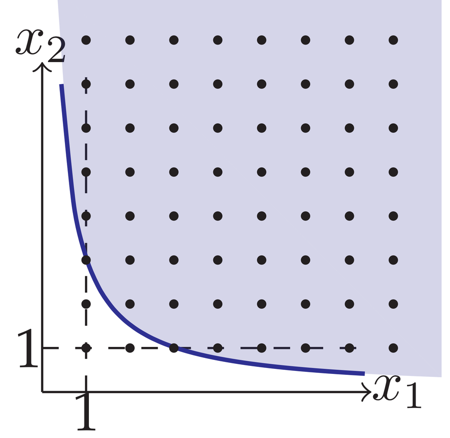
Corollary 1

If K is a polyhedron with $\mathrm{int}(K) \cap \mathbb{Z}^n \neq \emptyset$, then K' is a rational polyhedron if and only if $\mathrm{conv}(K \cap \mathbb{Z}^n)$ is a rational polyhedron.

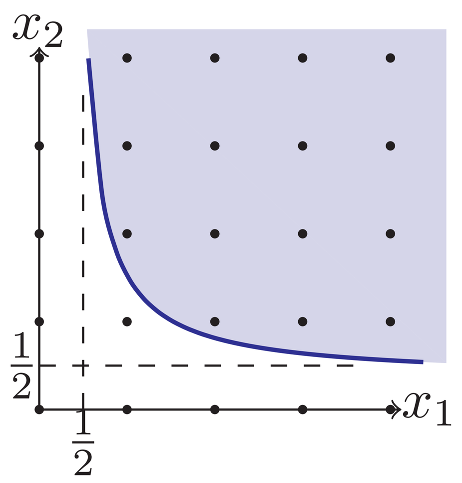
- Theorem 2 generalizes the same result for compact convex set and rational polyhedron.
- Fundamentally different proof technique than that of Dunkel and Schulz [2010], Dadush et al. [2011] and Braun and Pokutta [2014], which lends itself to potential applications to more classes of cutting planes.

Counterexamples

- K has rational polyhedral recession cone, but K cannot be written as the sum of a compact set and its recession cone:
  - $K = \{x \in \mathbb{R}^2_+ \mid x_1 \cdot x_2 \ge 1\}$.



  - $\mathrm{conv}(K \cap \mathbb{Z}^2)$ is a rational polyhedron, but K' is not a polyhedron: $x_1 \ge 1, x_2 \ge 1$ are not CG cuts.
  - $K = \{x \in \mathbb{R}^2_+ \mid (x_1 - 0.5)(x_2 - 0.5) \ge 1\}$.



  - Both $\mathrm{conv}(K \cap \mathbb{Z}^2)$ and K' are rational polyhedra.
- K can only be written as the sum of a compact convex set and an irrational polyhedral cone:
  - $K = \{x \in \mathbb{R}^2 \mid \sqrt{2}x_1 - x_2 = 0\}$.
  - $K' = K$ which is an irrational polyhedron, even though $\mathrm{conv}(K \cap \mathbb{Z}^2) = \{(0,0)\}$ is a rational polyhedron.

Key lemma for Theorem 1

Lemma 1: Gordan's Lemma (Hilbert basis)

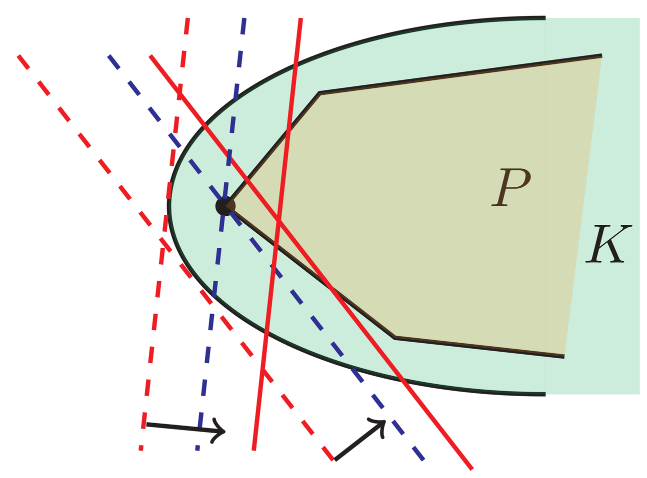
$C \subseteq \mathbb{R}^n$ is a rational polyhedral cone, then $C \cap \mathbb{Z}^n$ is finitely generated: there exists $\{g^1, \dots, g^m\} \subseteq C \cap \mathbb{Z}^n$ such that every $x \in C \cap \mathbb{Z}^n$ is an **integral conical combination** of these points.

- **Dickson's Lemma:** Poset $(\mathbb{N}^n, \le)$ has no infinite antichain. Here *Antichain (chain)* is a subset of a poset that no (every) two elements are comparable with each other.

Lemma 2

Given a rational polyhedral cone $C \subseteq \mathbb{R}^n$, a sequence $\{v^i\}_{i\in\mathbb{N}} \subseteq C \cap \mathbb{Z}^n$ and a vector $q \in \mathbb{Q}^n$. Then there exist $a, b \in \mathbb{N}$ such that $v^a - v^b \in C$ and $v^a q - \lfloor v^a q \rfloor = v^b q - \lfloor v^b q \rfloor$.

- For each $v^i \in C \cap \mathbb{Z}^n$, by Gordan's Lemma it can be written as: $v^i = G \cdot \lambda^i$, for $\lambda^i \in \mathbb{N}^m$.
- ***Folklore***: An infinite poset contains either an infinite chain or an infinite antichain.
- Within this infinite poset $\{\lambda^i\}_{i\in\mathbb{N}}$, by Gordan's Lemma, there exists an infinite chain $\{\lambda^i\}_{i\in I'}$.
- For rational vector q, write it as $q = \frac{1}{D}z$ for $D \in \mathbb{N}, z \in \mathbb{Z}^n$. Hence $\{v^i q - \lfloor v^i q \rfloor \mid i \in I'\} \subseteq \{0, \frac{1}{D}, \dots, \frac{D-1}{D}\}$, which is a finite set. So there exists $a, b \in I'$ such that $v^a q - \lfloor v^a q \rfloor = v^b q - \lfloor v^b q \rfloor$.
- Moreover, λ^a and λ^b are comparable, say $\lambda_a \ge \lambda_b$, then $v^a - v^b = G \cdot (\lambda^a - \lambda^b) \in C$.



- Assume $\sigma_P(c^i) > \lfloor \sigma_K(c^i) \rfloor$, then: $\sigma_K(c^i) \ge c^i q^i := \sigma_P(c^i) > \lfloor \sigma_K(c^i) \rfloor$, for some extreme point $q^i \in P$.
- We obtain $\lfloor \sigma_K(c^i) \rfloor = \lfloor \sigma_P(c^i) \rfloor$: CG cuts $c^i x \le \lfloor \sigma_K(c^i) \rfloor$ for K are the same as the CG cuts $c^i x \le \lfloor \sigma_P(c^i) \rfloor$ for P.
- Since extreme point $q^i \in P$ is finitely many, there exists $q \in P$ and infinite index set $I \subseteq \mathbb{N}$, such that $c^i q = \sigma_P(c^i)$ for $i \in I$.
- $cq = \sigma_P(c)$ if and only if $c \in C$ for some rational polyhedral cone C.
- By Lemma 3, there exist two CG cuts $c^a x \le \lfloor \sigma_K(c^a) \rfloor$ and $c^b x \le \lfloor \sigma_K(c^b) \rfloor$, such that $\sigma_P(c^a - c^b) = (c^a - c^b) q \in \mathbb{Z}$.

- So $c^a x \le \lfloor \sigma_K(c^a) \rfloor = \lfloor c^a q \rfloor$ is dominated by $c^b x \le \lfloor \sigma_K(c^b) \rfloor = \lfloor c^b q \rfloor$ and $(c^a - c^b)x \le (c^a - c^b)q$, the second inequality is valid to $P \supseteq K'$, hence $c^a x \le \lfloor \sigma_K(c^a) \rfloor$ is dominated by finitely many CG cuts of K.

Proof sketch of Theorem 2 for polyhedral K

Theorem 3: extension of Kronecker [1884]

Let $n, N_0 \in \mathbb{N}$ and $\pi \neq 0 \in \mathbb{R}^n$. Then $\mathbb{Z}^n - \pi\mathbb{Z}_{\ge N_0}$ contains a dense subset of $V_\pi := \{\alpha^T x = 0 \text{ for any } \alpha \in \mathbb{Q}^n \text{ s.t. } \alpha^T \pi \in \mathbb{Q}\}$.

- Any valid inequality $\alpha x \le \beta, \beta \in \mathbb{Z}$, for large N_0 and small ϵ, find $c^i - n_i \cdot \alpha \in V_\pi \cap (\mathbb{Z}^n - \alpha\mathbb{Z}_{\ge N_0})$: $0 = \sum \lambda_i (c^i - n_i \alpha), \|c^i - n_i \alpha\| < \epsilon$.
- $\forall v \in \mathrm{rec}(\{x \in K \mid \alpha x = \beta\}) : \alpha^T v = 0, v \in \mathbb{Q}^n$. Hence Theorem 4 implies: $(c^i - n_i \alpha) \cdot v = 0$.
- Additional argument implies $\lfloor \sigma_K(c^i) \rfloor \le n_i \beta$:

$$\{c^i x \le \lfloor \sigma_K(c^i) \rfloor, \forall i\} \subseteq \{c^i x \le n_i \beta, \forall i\} \subseteq \{\alpha x \le \beta\}.$$

- Argue for all finitely many facet-defining inequality $\alpha x \le \beta$ of K, concludes the proof.