Continuum Theory, Cantor Sets, and the Topology of Dimension

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1 Abstract

Continuum Theory is an area of Topology concerning the study of compact and connected metric spaces. While this might seem to simplify things, there are still some interesting examples and open problems. We will cover some examples of continua and the Hahn-Mazurkiewicz Theorem, which states that there exist continuous maps from the closed interval onto Peano continua.

2 Cantor Sets and Dimension

2.1 Cantor Sets

First, we will define the middle-thirds Cantor set. Let C_0 be the interval $[0, 1]$. Then remove the open middle third of the interval to get

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Continue this process again by removing the open middle thirds out of both portions of C_1 to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

This can then be continued iteratively to define all the sets C_n. Then the classical middle-thirds Cantor set is

$$K = \bigcap_{n=0}^{\infty} C_n$$

The first five steps of the construction are given geometrically in Figure 1.

The measure of this set, in terms of Lebesgue measure, is zero. This can be seen by summing the measures of the removed parts:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots + \frac{2^n}{3^{n+1}} + \ldots = \frac{1}{3} \cdot \frac{1}{1 - 2/3} = 1$$

To get a rough idea of what remains in the Cantor set, it is clear that for any C_n, the endpoints of each interval in C_n will be in K.

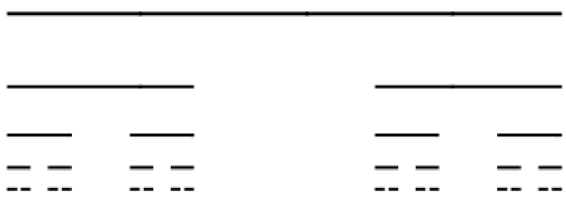

Figure 1

There are several topological properties that the Cantor set has. The first is that it is compact. This is seen easily since only open sets are being removed, so the complement of the open sets will remain closed. It is perfect since it has no isolated points. No matter how small of a ball that is restricted around a point, at some step of the process, some interval piece of a C_n was contained inside that ball, and so there are other points of the final Cantor set which will also be inside any ball. Also, the Cantor Set is totally disconnected since any set containing more than one point will be disconnected. Since the Cantor set is a subset of $\mathbb{R}$, this is equivalent to the Cantor set not containing any intervals. The lengths of the intervals in C_n are $1/3^n$ and as n goes to infinity, the lengths of the intervals will approach zero. So the end result will have no intervals contained in it, so the Cantor set is totally disconnected.

What is key about these properties is that they are also sufficient to describe the Cantor set. Any set that is compact, perfect and totally disconnected is homeomorphic to the middle-thirds Cantor set.[1] Since we are mostly concerned in this paper about the Cantor set from a topological perspective, we will refer to any set having these three properties as a Cantor set.

So if instead of taking out the middle third of each interval, we can take out the middle-fifth of each interval and end with a set that is homeomorphic to the middle-thirds Cantor set. The measure of this set is still zero since the removed intervals sum to

$$\frac{1}{5} + \frac{4}{25} + \ldots + \frac{4^n}{5^{n+1}} + \ldots = \frac{1}{5} \cdot \frac{1}{1 - 4/5} = 1$$

One particular example of a Cantor set that does not have measure zero is the Smith-Volterra-Cantor set. This particular set is created by removing the middle $1/4$ in the first step and by removing an interval of length $1/4^n$ from each of the intervals during the following steps. The first few steps are shown in Figure 2.

Figure 2

The total measure of the intervals removed from interval $[0, 1]$ is

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^{n+1}} + \ldots = \frac{1}{4} \cdot \frac{1}{1 - 1/2} = \frac{1}{2}$$

Thus the measure of this Cantor set is $1/2$. So these two sets have completely different measures, but are homeomorphic. In fact, by adjusting the width of the intervals, the Cantor set can have any measure in $[0, 1)$.

[1] [6] theorem 2-97, p.99

In fact, by removing intervals of length $1/2^n k^n$ out of each interval at the n-th step, for natural numbers $k \geq 2$, the Lebesgue measure of the removed parts will approach

$$\frac{1}{2k} \cdot \frac{1}{1 - 1/2k} = \frac{1}{2k - 1} \to 0 \text{ as } k \to \infty$$

So the Lebesgue measure of these Cantor sets approaches one, though it can never equal one since an interval of some measure is always removed during the first step.

Cantor sets are not restricted to subsets of $\mathbb{R}$. The first few steps of one example, often referred to as a Cantor dust, is given in Figure 3. This extension of the middle-thirds Cantor set to the plane can also easily be extended to any number of dimensions.

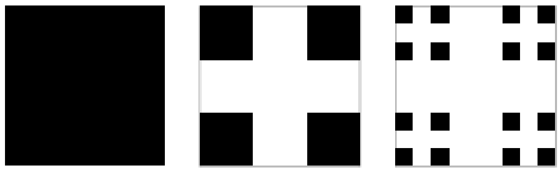

Figure 3

One particularly interesting example of a Cantor set in $\mathbb{R}^3$ is Antoine's necklace. It is constructing by first taking a torus, and then within that torus, taking a chain of tori. These smaller tori do not intersect. For the next step, within each of those smaller tori, take another chain of even smaller tori. Continue this process indefinitely. The intersection of all of these tori is Antoine's necklace. See Figure 4.[2] Since each of the links of this necklace do not intersect one another, then the necklace is totally disconnected. But even as a totally disconnected set in $\mathbb{R}^3$, its complement is not simply connected.[3] The chains are interlinked so much that there is no way for a line to pass through them.

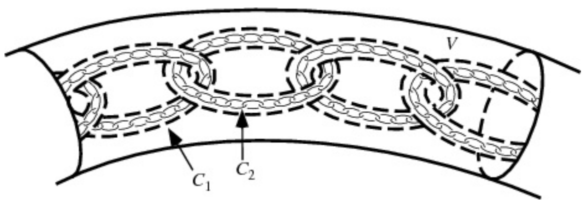

Figure 4

Another fascinating property of the middle-third Cantor set is that there exists a one-to-one map of the Cantor set whose image is the interval [0,1]. It is easiest to show this fact by thinking of the Cantor set in a different way.

It is possible to write an element n of the interval $[0, 1]$ in the form

$$n = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots, \quad \text{where } a_i = 0, 1, \text{or } 2, \text{ for all } i$$

This is often called *ternary* or *base 3* and can also be expressed as

$$n =_3 0.a_1 a_2 a_3 ...$$

[2]Weisstein, Eric W. "Antoine's Necklace." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/AntoinesNecklace.html

[3]*Simply connected*: Every image of S^1 is homotopic to a point

Conveniently, the middle third Cantor set is constructed at each step by splitting the intervals into thirds and choosing only the first and last. This is the same as restricting the a_i's to only 0 or 2, for all i. So another way to represent the Cantor set is the set of numbers n such that $n =_3 0.a_1a_2...$, where $a_i = 0$ or 2, for all i.

This non-geometric representation allows an easy one-to-one map to be described between the Cantor set and a closed interval. The map is given by

$$f(\frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots) = \frac{a_1/2}{2} + \frac{a_2/2}{2^2} + \ldots + \frac{a_n/2}{2^n} + \ldots$$

This is the binary representation of the numbers in the interval $[0, 1]$. Since the a_i for the Cantor sets are either 0 or 2, then $a_i/2$ will be 0 or 1.

This map also shows very concretely that two sets with different measures can still have a one-to-one map between them. This brings up another concern when trying to define a topologically invariant concept of dimension.

2.2 Fractal and Topological Dimensions

The dimension of the Cantor set does not seem to fall easily into any naturally numbered dimension. It has too many points to be zero dimensional, but has a zero measure, which makes it too small to be one-dimensional. The study of fractals necessitated a dimension to be defined that gives an idea on how close to a naturally numbered dimension a particular fractal is.

This particular definition[4] is called the box dimension. The reason for this name is that one way to think of this dimension is by counting the number of small boxes that it takes to cover an object.

Definition 1. Let A be a subset of $\mathbb{R}^n$ and $\epsilon > 0$. Denote by $N(\epsilon)$ the minimal number of n-cubes of side length ϵ that are required in order to cover A. Then the *box dimension of A* is given by

$$d = -\lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon}$$

If the fractal in question is self-similar, then by fixing $\epsilon = r$, where r is the ratio of each smaller pieces to the original. Then there will be a fixed number of balls of radius r to cover it, and this r will be fixed as well. So the box dimension can be simplified to

$$d = \frac{\log N}{\log 1/r}$$

and no limit is necessary.

To try out these dimensions, lets look at the first two Cantor sets that were discussed. The fractal dimension of the middle thirds Cantor set is

$$d = \frac{\log 2}{\log 3} \approx .6309$$

But the dimension of the middle-fifths Cantor set is

$$d = \frac{\log 2}{\log 5/2} \approx .7565$$

[4] [5]

Since you are removing less from the middle-fifths Cantor set then its dimension is closer to the dimension of the line than the middle-thirds Cantor set.

In order to calculate the dimension of the Smith-Volterra-Cantor set, we must use the full definition of the box dimension as the intervals vary. The full calculation of this dimension is in the appendix.

$$\begin{aligned} -\frac{\log 2}{\log 3/8} &\approx .7067 \\ -\frac{\log 4}{\log 5/32} &\approx .7468 \\ &\vdots \\ \lim_{n\to\infty} -\frac{\log 2^n}{\log\left(\frac{2^n+1}{2^{2n+1}}\right)} &= 1 \end{aligned}$$

So the Smith-Volterra-Cantor set, which is still homeomorphic to the middle-thirds Cantor set, has a dimension exactly equal to one. If we consider the dimension Cantor dust in Figure 3, it is

$$d = \frac{\log 4}{\log 3} \approx 1.2619$$

So the dimension of Cantor sets can even be greater than one. If we look at the dimension of the n-dimensional extension of the Cantor dust, we find that it is

$$d = \frac{\log 2^n}{\log 3} \to \infty \text{ as } n \to \infty$$

So the Cantor sets, which are all homeomorphically equivalent, can have dimension from less than one all the way to infinity. Each of these variations of Cantor set has a different fractal dimension. One thing that this makes very clear is that this definition of dimension is not a topological property.

This conclusion might make one concerned on whether the dimensions of $\mathbb{R}^n$ and $\mathbb{R}^m$ have any meaning topologically. Fortunately, Brouwer[5] showed that if such a homeomorphism exists between $\mathbb{R}^n$ and $\mathbb{R}^m$, then n equals m. This hints at the fact that there is a way to define a dimension inductively which is preserved under homeomorphisms.

It is necessary to start the definition of an inductive dimension[6] by asking what dimension is the dimension of the empty set. It is clear that a single point should be zero dimensional. Setting the dimension of the empty set to -1 works well with the inductive definition for dimension.

Definition 2. A space X has *dimension 0 at a point* p if there are arbitrarily open neighborhoods of p with empty boundaries.

If every point of a nonempty set X has dimension 0, then X has *dimension 0.*

Theorem 2.1. *If a space X is compact, then X has dimension 0 if and only if it is totally disconnected.*

Definition 3. A space X has *dimension* $\leq n$ *at a point* p if p has arbitrarily small open neighborhoods whose boundaries have dimension $\leq n-1$.

[5] [1]
[6] [7]

X has *dimension* $\leq n$, written dim $X \leq n$, if every point of X has dimension $\leq n$.
X has *dimension* n if dim $X \leq n$ is true and dim $X \leq n-1$ is false.
X has *dimension* ∞ if dim $X \leq n$ is false for all n.

This definition of dimension will be constant under homeomorphisms since empty boundaries of open sets will still be empty boundaries of open sets after going through a homeomorphism. The definition builds inductively up from there.

3 Continuum Theory

The term continuum was first formaly defined by Georg Cantor in order to try and capture the essence of the real line $\mathbb{R}$. Cantor said that a subset of a Euclidean space is a continuum provided that for any two points a and b of the subset and a given $\epsilon > 0$, there exist a finite number of points $p_0 = a, p_1, ..., p_n = b$ such that the distance between consecutive p_i's is less than ϵ.[7]. Prior to this, it was thought that a perfect set was enough structure to describe a continuum. His Cantor set is a counterexample to that reasoning. Since Cantor, the definition has evolved to the one presented below.

3.1 Some basics

Definition 4. A topological space is a *continuum* if it is a nonempty, compact, connected metric space.

If a continuum contains only one point, then it is called *degenerate*. As degenerate continuum are not that interesting, we will concern ourselves with nondegenerate continuum for the rest of the paper.

Now, we will look at some examples of continuum to get a scope of what this definition includes.

The first and simplest example is an arc. It is clear that a closed interval will satisfy the definition of a continuum, so an arc will also be a continuum.

The next example of a continuum is the topologist's sine curve, shown in Figure 5. It is the graph of $y = \sin 1/x$ for x-values in $(0, 1]$ combined with the closure arc from $(0, -1)$ to $(0, 1)$. This set is connected, but not locally connected nor arc-connected.

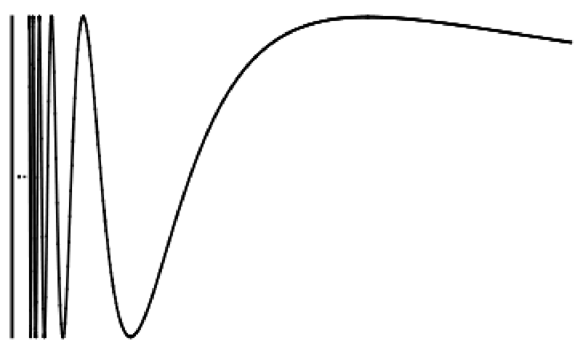

Figure 5

[7][8]

Figure 6 shows the Warsaw circle, which is a way to adjust the topologist's sine curve to be arc-connected. But note that the Warsaw circle is still not locally connected.

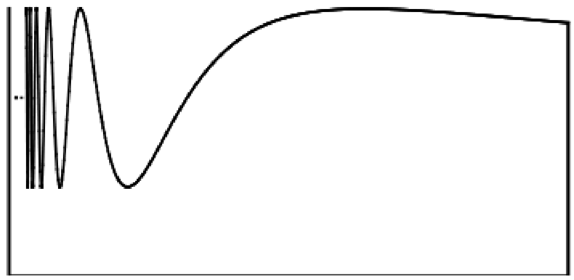

Figure 6

Nothing in the definition of a continuum limits the number of dimensions. Spaces which are homeomorphic to the closed balls $B^n = \{(x_i)_{i=1}^n \in \mathbb{R}^n : \|x\| \leq 1\}$ are continua. Even spaces homeomorphic to the Hilbert cube, defined as

$$\prod_{i=1}^{\infty} I_i, \text{ where } I_i = [0,1]$$

with the product topology[8], are continua.

Continua can have many different dimensions, from only one or two dimensions, all the way to n or even infinite dimensions.

3.2 Interesting constructions

Besides these examples, there are two main constructions to make more interesting and complex continua. The first is to take infinite intersections of nested continua and the other route is a structure called an inverse limit.

Theorem 3.1. *Let $X_1 \supset X_2 \supset \ldots$ where each of the X_i are continua. Then $X = \cap_{i=1}^{\infty} X_i$ is a continuum.*

Proof. To show that $X \neq \varnothing$, we will actually show that if $U \subset X_1$ is an open subset and $X \subset U$, then there exists some N such that $X_i \subset U$, for all $i \geq N$.
Let $U \subset X_1$ open subset such that $X \subset U$ Suppose, for a contradiction that, that for each $i = 1, 2, 3, ...$ that there exists an $x_i \in X_i - U$. Then $\{x_i\}$ is a sequence in X_1, which is a compact metric space. So we can choose a sequence which converges to some point $p \in X_1$. Also, since U is open in X_1, then $p \notin U$. Since for each m, $x_i \in X_m$, for all $i \geq m$, then $p \in X_m$. Thus $p \in X \subset U$. This is a contradiction.

X is compact since for each X_i, $X_1 - X_i$ is an open set in X_1. So

$$X = X_1 - \bigcup_{i=2}^{\infty} (X_1 - X_i)$$

[8]Basis elements are of the form $\prod_{i=1}^{\infty} U_i$, where U_i is open in $[0,1]$, and all but finitely many of the U_i are equal to $[0,1]$. [12]

is a closed subset of X_1. Since X_1 is compact, then X is compact.

X is clearly metric since every point of X is also a point of X_1. So the metric on X_1 can be simply restricted to the points of X.

To show that X is connected, suppose U and V are disjoint open subsets of X such that $U \cup V = X$. Since X is compact, then U and V are closed and compact. Since X_1 is normal, then there exist open disjoint sets A and B of X_1 such that $U \subset A$ and $V \subset B$. From the first part of the proof, $A \cup B$ satisfies the conditions of U. Thus there exists an N such that $X_i \subset A \cup B$, for all $i \geq N$. For such an X_i, $U \subset X_i \cap A$ and $V \subset X_i \cap B$ are nonempty disjoint open sets. Thus X_i is not connected. But X_i is a continuum, so this is a contradiction. Thus X is connected. $\square$

The first example of a continua that can be created using the intersections of nested continua is the Sierpinski Universal Curve. Let X_1 be the unit square, $[0,1] \times [0,1]$. We can find X_2 by dividing X_1 into nine equal squares and removing the middle one. Similarly, divide each of the smaller squares in X_2 into nine more squares and remove the middles to get X_3. Continue this process for the rest of the X_i's. Figure 7 shows this process a few steps down the line.

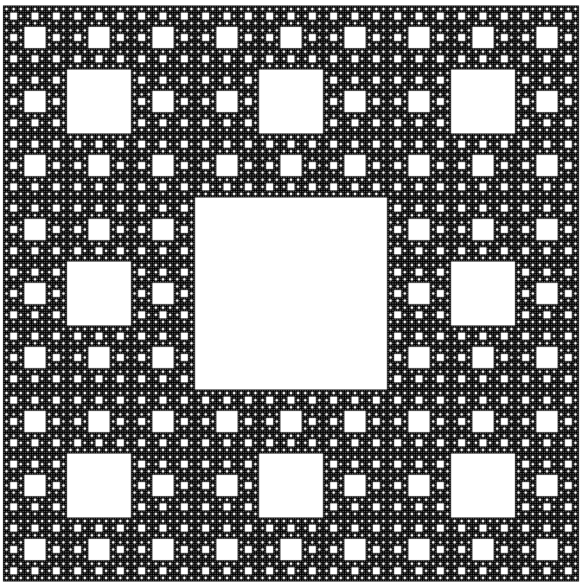

Figure 7

Definition 5. A *curve* is a one-dimensional continuum.

The reason that the Sierpinski curve is called universal is that it contains a homeomorphic image of every one-dimensional continua in the plane.[9]

Another type of continua that can come from this construction are indecomposable continua.

Definition 6. A continuum is said to be *decomposable* if it can be written as the union of two subcontinuum.

A continuum is *indecomposable* if it is not decomposable.

[9]See the appendix for a translation of Sierpinski's 1916 proof of this fact.

Definition 7. A *simple chain from x to y* is a collection of sets $\{U_1, U_2, ..., U_n\}$, called *links*, such that
(i) $a \in U_i$ if and only if $i = 1$,
(ii) $b \in U_i$ if and only if $i = n$, and
(iii) $U_i \cap U_j \neq \varnothing$ if and only if $|i - j| \leq 1$.

For the example of an indecomposable continuum, let a, b, and c be distinct points in $\mathbb{R}^2$. Let $\mathcal{A}_n$ be a simple chains whose links are closed balls with radii less than $1/2^n$ such that
(i) if $n = 3k+1$, then A_n is a chain from a to c through b,
(ii) if $n = 3k+2$, then A_n is a chain from b to a through c,
(iii) if $n = 3k$, then A_n is a chain from c to b through a, and
(iv) $\cup\mathcal{A}_n \subset \cup\mathcal{A}_{n-1}$.
Then let X_n be the union of the links in $\mathcal{A}_n$. Thus $X = \cap_{i=1}^{\infty} X_n$ will be a continuum, since the X_n are nested continuum. Figure 8 has the first three simple chains of this construction.

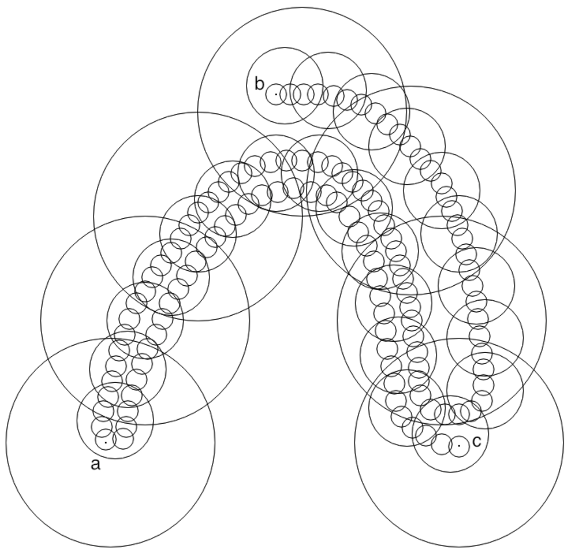

Figure 8

This continuum is indecomposable since any subcontinuum Y which contains the points a and b will also be contained within all of the X_n, where $n = 3k$. So Y will also contain all of X since it is connected and removing any of the links of X_n would disconnect it.. So no proper subcontinuum of X will contain two of a, b, or c.

Another construction to create more continua is inverse limits.

Definition 8. An *inverse sequence* is the sequence $\{X_i, f_i\}_{i=1}^{\infty}$, where the X_i are *coordinate spaces*, $f_i : X_{i+1} \to X_i$ are *bonding maps*. The sequence is often written

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \cdots .$$

The *inverse limit*, written $\varprojlim\{X_i, f_i\}_{i=1}^{\infty}$, is defined as

$$\varprojlim\{X_i, f_i\}_{i=1}^{\infty} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i, \text{ for all } i\}$$

Often, the inverse limit is often abbreviated to $X_\infty = \varprojlim\{X_i, f_i\}_{i=1}^{\infty}$. Similarly to nested intersections, if all of the X_i are continua, then the inverse limit is a continuum as well. If the X_i are compact metric spaces, then the inverse limit is also a compact metric space.[10]

One example of a continuum that can be described by an inverse limit is the p-solenoid. The coordinate spaces are each simple closed curves S^2 and the bonding maps are each $f : x \to x^p$. This is homeormorphic to the geometric representation of a p-solenoid which consists of nested torii which wrap around p times at each step.

3.3 Extra connectivity

Now, we will look at continua which have some 'extra strong' connectedness. This will allow us to find some elegant and strong results on these types of continua.

Definition 9. Let S be a topological space and $p \in S$. Then S is *connected im kleinen (cik) at p* if every neighborhood of p contains a connected neighborhood of p.

Note that the neighborhoods in this definition are not necessarily open. The restriction to open neighborhood leads to the next definition.

Definition 10. Let S be a topological space and $p \in S$. Then S is *locally connected at p* if every open neighborhood of p contains a connected open neighborhood of p.

The difference between these two definitions can be seen at point P of the example in Figure 9.[11] Within each layer, there are countably many trapezoids are less than $1/2^n$ away from the largest trapezoid in the layer. Then this set is cik at P, since each closed layer is connected. But it is not locally connected at P, since an open set containing a layer would also contain the disconnected tops of the trapezoids of the next layer.

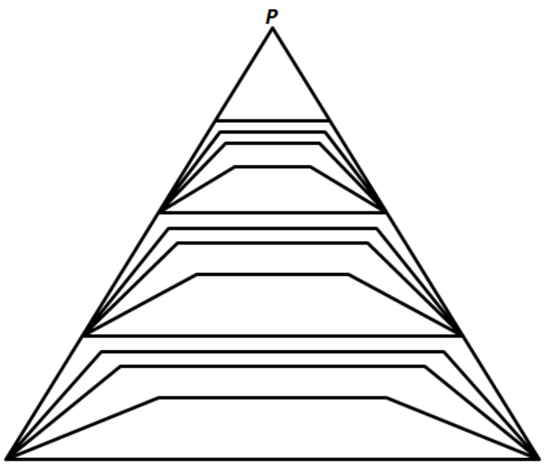

Figure 9

So when looking at individual points, these two definitions are different. But the next theorem considers what happens globally for these two definitions.

[10] [13]
[11] [11]

Theorem 3.2. *Let S be a topological space.The following are equivalent:*
(i) S is locally connected at every point.
(ii) the components of each open subset of S are open.
(iii) S is cik at every point.

3.4 Peano Continua

The class of continua that are locally connected at every point have a particularly strong theorem, called the Hahn-Mazurkiewicz Theorem. First though, we will give a name to this type of continua.

Definition 11. A metric space X is a *Peano space* if and only if
(i) X is locally connected at every point,
(ii) the components of each open subset of X are open, or
(iii) X is cik at every point.
A *Peano continuum* is a Peano space which is a continuum.

Theorem (Hahn-Mazurkiewicz). *Every Peano continuum is a continuous image of the closed interval* $[0, 1]$.

The proof of the Hahn-Mazurkiewicz Theorem will require a few lemmas first.

Lemma 3.3. *For any compact metric space X, there exists a continuous map f from the Cantor set onto X.*

Proof. Choose finite covers $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, ...$ such that
(1) each $A \in \mathcal{A}_n$ is closed
(2) for all $A \in \mathcal{A}_n$, $\operatorname{diam} A < 1/2^n$
(3) $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$.

Then make $\mathcal{A}_0 = \{A^0_1, A^0_2, ..., A^0_{k_0}\}$ disjoint by defining $B^0_i = A^0_i \times \{i\}$. Let $B_0 = \bigcup_{i=1}^{k_0} B^0_i$

For $\mathcal{A}_1 = \{A^1_1, A^1_2, ..., A^1_{k_1}\}$, if $A^1_j \subset A^0_i$, then define $B^1_{ij} = A^1_j \times \{i\} \times \{j\}$. Again, define B_1 to be the union of these disjoint sets.Then, define the bonding map$f_1 : B_1 \to B_0$ by $f_1(a, i, j) = f(a, i)$.

Continue to disjoint the $\mathcal{A}_n$ in the same manner. We have created an inverse sequence. Create a second inverse sequence by setting $X_n = X$ and identity bonding maps g_n. Then define maps $h_n : B_n \to X_n$ by $h_n(a, i, j, ..., p) = a$. With all of this we have the following diagram:

$$\begin{tikzcd}
B_0 \arrow[d, "h_0"] & B_1 \arrow[l, "f_1"'] \arrow[d, "h_1"] & B_2 \arrow[l, "f_2"'] \arrow[d, "h_2"] & \cdots \arrow[l, "f_3"'] \\
X_0 & X_1 \arrow[l, "g_1"] & X_2 \arrow[l, "g_2"] & \cdots \arrow[l, "g_3"]
\end{tikzcd}$$

Then $X_\infty = X$ and since each of the h_n are continuous and onto, then h_∞ is also continuous and onto. B_∞ is totally disconnected.

B_∞ is not perfect, but $B_\infty \times K$ is perfect (K is the Cantor set). Thus $B_\infty \times K$ is homeomorphic to K (since perfect and totally disconnected). Consider the following maps:

$$\begin{array}{rll} H: & K \to B_\infty \times K & \text{, a homeomorphism} \\ \pi: & B_\infty \times K \to B_\infty & \text{, a projection map (continuous and onto)} \\ h_\infty: & B_\infty \to X_\infty & \text{, continuous and onto} \\ \Phi: & X_\infty \to X & \text{, a homeomorphism} \end{array}$$

Then $f = \Phi h_\infty \pi H$ is a continuous and onto map from K to X. □

We will need a few definitions for the next few lemmas.

Definition 12. Let X be a connected space and $x \in X$. If $X \setminus \{x\}$ is not connected, then x is a *cut point.* If $X \setminus \{x\}$ is connected, then x is a *non-cut point.*

Definition 13. Let a and b be points in X. The image of a one-to-one continuous mapping $f : [0, 1] \to X$, where $f(0) = a$ and $f(1) = b$, is an *arc.*
If, for any $x, y \in X$, there exists an arc in X from x to y, then X is *arc-connected.*

Lemma 3.4. *If M is a continuum with exactly two non-cut points, then M is an arc.*

The proof of this Lemma involves defining a partial order on the cut points between particular points p and q. Then it can be found that restricting to only the set of these cut points and p and q results in a linear order. From there it is straightforward to define a homeomorphism on a dense subset of M to the interval $I = [0, 1]$.[12]

Lemma 3.5. *Peano continua are arc-connected.*

Proof. Let P be a Peano continuum and a, b distinct points in P. Since P is both connected and locally connected, then there exists a simple chain $\mathcal{C}_1$ from a to b such that $\mathcal{C}_1 = \{U_{11}, U_{12}, ..., U_{1k_1}\}$ where each link U_{1i} is connected, open, and has a diameter less than 1.[13]

Define $\mathcal{C}_1^* = \bigcup_{i=1}^{k_1} U_{1i}$. Since each of the U_{1i} is connected and $U_{1i} \cap U_{1,i+1} \neq \varnothing$, then $\mathcal{C}_1^*$ is connected. For any $x \in \mathcal{C}_1^*$, x is in at most two links of the chain $\mathcal{C}_\infty$. So there exists an open connected neighborhood V_x of x such that $\operatorname{diam} V_x < 1/2$ and $\overline{V_x}$ is completely contained within the links that contain x. Since $\mathcal{C}_1^*$ is connected, it is possible to extract from the family of V_x's a simple chain from a to b. Name this chain $\mathcal{C}_2 = \{U_{21}, ..., U_{2k_2}\}$, and repeat this product inductively such that if U_{ni} is a link of $\mathcal{C}_n$, then

(i) $\operatorname{diam} U_{ni} < \dfrac{1}{2^n}$

(ii) $\overline{U_{ni}} \subset U_{n-1,j}$, for some link $U_{n-1,j} \in \mathcal{C}_{n-1}$.

Let each

$$\mathcal{C}_n^* = \bigcup_{i=1}^{k_n} \overline{U_{ni}}$$

. Then each $\mathcal{C}_n^*$ is a continuum and $\mathcal{C}_n^* \subset \mathcal{C}_{n-1}^*$, for all n. Thus

$$A = \bigcap_{n=1}^{\infty} \mathcal{C}_n^*$$

is a continuum.

[12] [3] , [13]

[13] [3]

The next thing to show is that every point besides a and b are cut points.[14] Let $x \in A \setminus \{a, b\}$. Then x is contained in at most two links of $\mathcal{C}_n$. Let D_n denote the union of the links occurring before these two links and by E_n the union of the links occurring after. Then $(\bigcup_{n=1}^{\infty} D_n) \cap A$ and $(\bigcup_{n=1}^{\infty} E_n) \cap A$ are a separation of $A \setminus \{x\}$. Thus x is a cut point.

So A is an arc between a and b. Thus P is arc-connected. □

Lemma 3.6. *Let P is a Peano continuum and $\epsilon > 0$. There exists a $\delta > 0$ such that if $x, y \in P$ with $d(x, y) < \delta$, then there exists an arc from x to y with diameter less than ϵ.*

Proof. For each $x \in P$, let V_x be a connected open neighborhood of x lying in the ball $B(x, \epsilon/5)$. Then the family of V_x's is an open cover of P. Since P is compact, we can refine this to a finite cover $\{V_1, ..., V_n\}$.

Define A_i as the union of all the V_j such that the distance between the sets V_i and V_j is greater than 0. Let $\delta_i = d(V_i, A_i)$ and $\delta = \inf_{A_i \neq \varnothing} \delta_i$. Since there are only finitely many V_i, then δ_i is only zero if $A_i = \varnothing$. Thus $\delta > 0$.

Choose an $x, y \in P$ such that $d(x, y) < \delta$. Without loss of generality, suppose that $x \in V_i$. Since $d(x, y) < \delta$, then $y \notin A_i$. Either $y \in V_i$ or $y \in V_j$ where $d(V_i, V_j) = 0$.

If $y \in V_i$, then $\overline{V_i}$ is a Peano continuum containing x and y. Thus there is an arc between x and y contained in the ball $\underline{B(x, \epsilon/5)}$, which has a diameter less than ϵ.

If $y \in V_j$ where $d(V_i, V_j) = 0$, then $\overline{V_i} \cup \overline{V_j}$ is a Peano continuum with a diameter less than ϵ and which contains x and y. So there is an arc between x and y with a diameter less than ϵ. □

Finally, we arrive at the proof of our big theorem.[15]

Proof of Hahn-Mazurkiewicz Theorem. Suppose P is a Peano continuum. Then by Lemma 3.3, there exists a map $f : K \to P$ (K is the Cantor set). $I \setminus K$ consists of a countable collection of disjoint open intervals, $\mathcal{I} = \{I_1, I_2, ...\}$. Specify the intervals I_n by (a_n, b_n). Also, we can assume that $\mathcal{I}$ is linearly ordered by decreasing size.

If, for I_n, $f(a_n) = f(b_n)$, then extend f by defining $f(x) = f(a_n)$, for all $x \in I_n$. If $f(a_n) \neq f(b_n)$, then by Lemma 3.6, there exist δ_n corresponding to $\epsilon = \frac{1}{2^n}$ such that if $d(x, y) < \delta_n$, then there exists an arc between x and y with diameter less than $\frac{1}{2^n}$. Since f is uniformly continous on K, [16] then for δ_n, one can choose a η_n such that if $d(x, y) < \eta_n$ for $x, y \in K$, then $d(f(x), f(y)) < \delta_n$ in P. We will choose the η_n such that $\eta_1 > \eta_2 > \dots$.

For all I_j that have a diameter greater than or equal to η_1, denote by α_j the arc in P with a domain of $[a_j, b_j]$ going from $f(a_j)$ to $f(b_j)$. Extend the map of f by setting $f(x) = \alpha_j(x)$ for all $x \in [a_j, b_j]$.

If I_k is an interval such that $\eta_2 \leq \operatorname{diam} I_k < \eta_1$, then by choice of η_1, there exists an arc α_k from $f(a_k)$ to $f(b_k)$ with a diameter less than $\frac{1}{2}$. Extend the map f onto α_k just as before. This process can be continued inductively. For each I_m, where $\eta_{n+1} \leq \operatorname{diam} I_m < \eta_n$, extend the map f from $[a_m, b_m]$ onto an arc with diameter less than $\frac{1}{2^n}$. Denote the finale extension as the map F.

Let $\epsilon > 0$. Since $f : K \to P$ is uniformly continuous, there exists a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < \delta$. Also, there exists some N such that $\frac{1}{2^N} < \epsilon$. Then there are only finitely many I_m such that $\operatorname{diam} I_m > \eta_N$. The arc α_m in P is

[14] Every continuum has at least two non-cut points.[13]

[15] [3]

[16] Heine-Cantor Theorem: If $f : M \to N$ is continuous, M and N are metric spaces and M is compact, then f is uniformly continuous. [12] Thm 27.6

the image of I_m and by Lemma 3.6, there exists a δ_m such that the diameter of the arc between two points $F(x_m), F(y_m) \in \alpha_m$ is less than ϵ, whenever $d(x_m, y_m) < \delta_m$. Let δ^* denote the minimum between δ, $\frac{1}{2^N}$, and all the δ_m for which $\operatorname{diam} I_m > \eta_N$. By using δ^*, F is shown to be uniformly continuous, and thus F is continuous.

□

3.5 Space-filling curves

In particular, we can utilize the Hahn-Mazurkiewicz Theorem to know that the product space $[0,1] \times [0,1]$ is a continuous image of the interval $[0,1]$. Unfortunately, the Hahn-Mazurkiewicz Theorem does not give any information as to how this map is created. The first map that accomplished this was described by Giuseppe Peano in 1890.[17]

These types of maps are classified as space-filling curves. Note that the interval $[0,1]$ and the product space $[0,1] \times [0,1]$ have different topological dimensions. This does not contradict the topological invariance of dimension, since this map does not have to be one-to-one. In fact, it is necessarily not one-to-one, since the map will be continuous and onto. Space-filling curves are not even limited to finite dimensions; the infinite dimensional Hilbert cube is a Peano continuum.

The first three steps of the map that Peano described from $[0,1]$ onto the unit square $[0,1] \times [0,1]$ are shown in Figure 10. To get from the first step to the second step of this curve, cut the unit square into nine equal squares. By dividing the interval $[0,1]$ also into nine parts, we can place a copy of the curve in the first step into each of these squares. In the first interval $[0,1/9]$, map the first curve to the lower left square. In the second interval, map a rotated version of the first curve into center left curve.

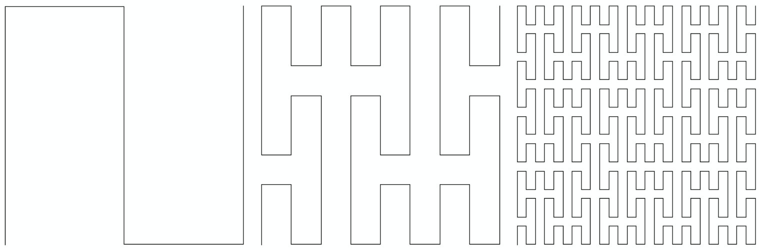

Figure 10

Continue this mapping to fill the other seven subsquares, rotating as needed to keep the curve connected. Then this process can be continued further by subdividing the square infinitely and the resulting map will be a continuous map from $[0,1]$ onto the unit square $[0,1] \times [0,1]$.

[17] [2]

4 Open Problems

Within the area of Continuum Theory, there are several open problems. These have been collected for many years in a published collection, called the Houston Problem Book.[18] More problems, including the one that we will be looking at are on the website "Open Problems in Continuum Theory."[19]

4.1 Is every aposyndetic homogeneous curve mutually aposyndetic?

Remember that a curve is a one-dimensional continuum.

Definition 14. A space X is said to be *aposyndetic* if for any two points x and y of X there is a subcontinuum A such that x is in the interior of A and y is in the complement of A.
A space X is *mutually aposyndetic* if for any two points x and y of X there are disjoint subcontinuum A and B such that x is in the interior of A and y is in the interior of B.

Definition 15. A space X is *homogeneous* if for any two points p and q of X there exists a homeomorphism h from X onto itself such that $h(p) = q$.

From an article by F. Burton Jones[20], if a continuum in the plane is both homogeneous and aposyndetic, then it is the simple closed curve, i.e., homeomorphic to a circle. Since a simple closed curve is mutually aposyndetic, then any counterexample to this open problem must be a curve that is not embeddable in the plane. Now we will define a term which relates to aposyndetic.

Definition 16. A space X is said to be *semi-locally-connected*, written *SLC*, if for any point x of X and neighborhood N of x, there exists an open subset V of N containing x such that the complement of V in X has finitely many components.

When looking at a space locally, SLC is very different from aposyndetic. But the next theorem shows that these are equivalent globally.

Theorem 4.1. *A continuum is aposyndetic iff it is SLC.*[21]

Proof. Suppose that X is aposyndetic and $x \in X$. Let N be a neighborhood of x. Then $x \notin \overline{X - N}$.

If $y \in \overline{X - N}$, define U_y to be a subcontinuum of X such that $y \in \operatorname{Int} U_y$ and $x \notin U_y$. Then the collection of sets $\operatorname{Int} U_y$ is an open cover of $\overline{X - N}$. Since $\overline{X - N}$ is compact, then the U_y's can be reduced to a finite cover $\{U_1, U_2, ...U_n\}$.

So $V = \cap_{i=1}^{n}(X - U_i)$ is an open subset of N containing x such that $X - V = \cup_{i=1}^{n} U_i$ is a connected set. Thus X is SLC.

Suppose that X is SLC. Let $x, y \in X$ and $\epsilon > 0$ such that $y \notin B = B(x, \epsilon)$.

So $X - B$ is a neighborhood of y. Thus there exists an open subset V of $X - B$ containing y such that $X - V$ has finitely many components. Since theses components

[18] [4]
[19] [14]
[20] [10]
[21] [9]

are also closed and disjoint, then each of these components is open in $X - V$. Let A be the component which contains x. Since $x \in B$ and $B \cap V = \varnothing$, then x is not in the boundary of $X - V$.

So $x \in \operatorname{Int} A$ in X, $y \notin A$, and A is connected and closed. Thus X is aposyndetic.

□

Next, we will introduce weak triods and a result about them.[22]

Definition 17. A continuum X is be the *essential sum* of the subcontinua X_i $(1 \leq i \leq n)$, written

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n,$$

provided that

$$X = \bigcup_{i=1}^{n} X_i \quad \text{and} \quad X_k \not\subset \bigcup_{i \neq k} X_i \text{ for each } k = 1, ..., n.$$

Definition 18. A continuum X is called a *weak triod* provided that $X = X_1 \oplus X_2 \oplus X_3$ where $\cap_{i=1}^{3} X_i \neq \varnothing$.

Theorem 4.2. *If $X = Y \oplus Z$ and X is not a weak triod, then $X - Y$ and $X - Z$ are connected.*

Note that $X = Y \oplus Z$ is the same as saying X is a decomposable continuum. The next theorem will show that aposyndetic continua are also decomposable.

Definition 19. A continuum X is said to be *freely decomposable* if for any two distinct points $x, y \in X$, then X is the sum of two continua, neither of which contains both x and y.

Theorem 4.3. *Every aposyndetic continuum is freely decomposable.*[23]

Clearly, a freely decomposable continuum is also decomposable. With the previous work, we now arrive at our final theorem related to this open problem.

Theorem 4.4. *If X is an aposyndetic continuum and not a weak triod, then X is mutually aposyndetic.*

Proof. Let X be an aposyndetic continuum and not a weak triod. Let $a, b \in X$. By Theorem 4.3, there exist subcontinua Y and Z such that $X = Y \oplus Z$ with $a \in Y$ and $b \in Z$. Also, $a \notin Z$ and $b \notin Y$.

By Theorem 4.2, $X - Y$ and $X - Z$ are connected. Also, note that $a \in X - Z$ and $X - Z$ is open. Since X is SLC by Theorem 4.1, then there exists an open subset V of $X - Z$ whose complement has finitely many components. Also, choose V small enough such that $\overline{V} \subsetneq X - Z$. Denote the components of $X - V$ by C_i, $i = 1, ..., n$. Without loss of generality, let C_1 be the component which contains b.

Define $K = (\overline{X - Z}) \cup (\cup_{i=2}^{n} C_i)$ and $M = C_1$. Since $X - Z$ is connected and the C_i are closed and intersect the boundary of V[24], both K and M are subcontinua of X. So $X = K \oplus M$ and thus $X - K$ and $X - M$ are connected.

Also, $b \in \operatorname{Int}(Z) \cap C_1 \subset X - K$ and $a \in V \subset X - M$.

[22][13] p 208
[23][9]
[24][13] p. 73

Note that $\overline{X-K} = C_1 - \text{Int}(K)$ and $\overline{X-M} = (\cup_{i=2}^n C_i) \cup \overline{V}$.

Since $\overline{V} \subset \text{Int}(K)$, and components are always disjoint, then $\overline{X-K}$ and $\overline{X-M}$ are disjoint subcontinua of X. Also, $a \in X - M$ and $b \in X - K$ are open, so a and b are in the respective interiors of $\overline{X-M}$ and $\overline{X-K}$. Since a and b were chosen arbitrarily, X is mutually aposyndetic. $\square$

So any counterexample to the open problem must be a weak triod. This means that it must also contain a triod of type 3 and type 4.[25]

[25] [16]

5 Appendix

5.1 Calculation

$$\begin{aligned}
\lim_{n\to\infty} -\frac{\log 2^n}{\log\left(\dfrac{2^n+1}{2^{2n+1}}\right)} &= \lim_{n\to\infty} \frac{\log 2^n}{\log\left(\dfrac{2^{2n+1}}{2^n+1}\right)} \\
&= \lim_{n\to\infty} \frac{\dfrac{1}{2^n}\cdot 2^n \cdot \log 2}{\dfrac{2^n+1}{2^{2n+1}}\cdot\left(\dfrac{((2^n+1)\cdot 2\cdot 2^{2n+1}\cdot\log 2)-(2^{2n+1}\cdot 2^n\cdot\log 2)}{(2^n+1)^2}\right)} \\
&= \lim_{n\to\infty} \frac{\log 2}{\dfrac{(2^n+1)\cdot 2\cdot\log 2 - 2^n\cdot\log 2}{2^n+1}} \\
&= \lim_{n\to\infty} \frac{2^n+1}{2^{n+1}-2^n+2} \\
&= \lim_{n\to\infty} \frac{2^n\cdot\log 2}{2^{n+1}\cdot\log 2 - 2^n\cdot\log 2} \\
&= \lim_{n\to\infty} \frac{1}{2-1} \\
&= 1
\end{aligned}$$

5.2 Translation of *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée.*

On a Cantor curve which contains a one-to-one and continuous image of any given curve. Note by **W. Sierpiński**, presented by Émile Picard.[26]

The goal of this note is to construct *a (planar) Cantor curve C_0 such that, if C is a given (planar) Cantor curve, there always exists a one-to-one and continuous image C' of the curve C where all of the points are points of the curve C_0.*

The curve C_0 will be defined as follows. Let Q be a given square, for example the square whose vertices are the points $(0,0),(0,1),(1,0)$ and $(1,1)$. Divide the square Q into nine smaller squares and exclude the interior of the one which contains the center of the square Q. For each of the eight remaining squares, do the same process ad infinitum. Together, all of the points of Q which are not excluded obviously constitute a Cantor line; this is the curve C_0.

Let C now be an arbitrarily given Cantor curve: I claim that there exists a curve C', completely contained within C_0, which is the one-to-one and continuous image of the curve C. To demonstrate this, it is obvious that it suffices to show that there exists a curve K which is a one-to-one and continuous image of the curve C_0 and which contains all of the points of the curve C.

To define the curve K, first construct a square U such that the curve C is contained in the interior of U. Like the coordinates axes, take the sides of the square U. Divide the square U into nine new squares: let V be the one of them which contains the center of the square U.

We say, to be brief, that *a rectangle R has the property P*, if the sides are parallel to the coordinate axes and it does not contain any point of the curve C in its interior.

Let R be a rectangle that has the property P which is in the interior of the square V (such a rectangle obviously exists, since the curve C is nowhere dense in the plane). Denote by x_1, the abscissa[27] of the left side of the rectangle R; by x_2 for the right side; by y_1 the ordinate[28] of the bottom, and by y_2 for the top. Also set $x_0 = 0$, $y_0 = 0$. The four lines

$$x = x_1, \qquad x = x_2, \qquad y = y_1, \qquad y = y_2$$

divide the square U into 9 rectangles whose lower left points are $(x_\alpha, y_\beta)(\alpha = 0,1,2; \beta = 0,1,2)$; we denote by $U_{\alpha,\beta}$ the rectangle which has the point (x_α, y_β) in the lower left (obviously, we have that $U_{1,1} = R$). Divide each of the nine rectangles $U_{\alpha,\beta}$ into nine equal rectangles; let $V_{\alpha,\beta}$ always be the one which contains the center of the rectangle $U_{\alpha,\beta}$.

We are going to now construct nine rectangles $S_{\alpha,\beta}$ that have the property P. In general, if σ denotes a symbol defining a rectangle S_σ having sides that are parallel to the coordinate axes, we will always denote by x'_σ the abscissa of the left side of the rectangle S_σ, by x''_σ the abscissa of the right side, by y'_σ the ordinate of the bottom and by y''_σ for the top.

The nine rectangles $S_{\alpha,\beta}$ will now be defined by recursion as follows. In the interior of $V_{\alpha,\beta}$, we will choose a rectangle $S_{\alpha,\beta}$ having the property P at contained in one part between the parallel lines $y = y'_{\alpha-1,\beta}, y = y''_{\alpha-1,\beta}$ and the other part between the parallel

[26] [15]

[27] the perpendicular distance from a point to the vertical axis, ie, the horizontal coordinate

[28] the perpendicular distance from a point to the horizontal axis, ie, the vertical coordinate

lines $x = x'_{\alpha,\beta-1}, x = x''_{\alpha,\beta-1}$ (if $\alpha = 0$ or $\beta = 0$, one can omit one or the other of these conditions). We have

$$x_{\alpha_1} = x'_{\alpha,2}, \qquad x_{\alpha_2} = x''_{\alpha,2}, \qquad y_{\beta_1} = y'_{2,\beta}, \qquad y_{\beta_2} = y''_{2,\beta}, \qquad (\alpha = 0,1,2; \beta = 0,1,2)$$

and we denote by $R_{\alpha,\beta}$ the rectangle formed by the lines

$$x = x_{\alpha_1}, \qquad x = x_{\alpha_2}, \qquad y = y_{\beta_1}, \qquad y = y_{\beta_2}.$$

One easily sees that the rectangle $R_{\alpha,\beta}$ will be contained in $S_{\alpha,\beta}$; thus it does not contain any point of the curve C in its interior. We also have

$$x_{\alpha_0} = x_\alpha, \qquad y_{\beta_0} = y_\beta \qquad (\alpha, \beta = 0, 1, 2).$$

The lines

$$x = x_{\alpha_1\alpha_2}, \qquad y =_{\beta_1\beta_2} \qquad (\alpha_1, \alpha_2, \beta_1, \beta_2 = 0, 1, 2)$$

divide the square U into 81 rectangles whose lower left points are $(x_{\alpha_1\alpha_2}, y_{\beta_1\beta_2})$; we denote by $U_{\alpha_1\alpha_2,\beta_1\beta_2}$ the rectangle whose lower left is the point $(x_{\alpha_1\alpha_2}, y_{\beta_1\beta_2})$.

We now suppose that we have already defined the rectangles

$$S_{\alpha_1\alpha_2\ldots\alpha_{n-1},\beta_1\beta_2\ldots\beta_{n-1}} \quad \textit{and} \quad U_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n} \quad (\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n = 0, 1, 2).$$

Divide each of these rectangles $U_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n}$ into nine equal rectangles and let $V_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ always be the one in the middle.

The indices $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n$ have been given, we denote by $U_{\gamma_1\ldots\gamma_n,\beta_1\ldots\beta_n}$ the rectangles $U_{\xi_1\ldots\xi_n,\eta_1\ldots\eta_n}$ whose right edge coincides with the left edge of rectangle $U_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ and by $U_{\alpha_1\ldots\alpha_n,\delta_1\ldots\delta_n}$ the ones whose top edge coincides with the bottom edge of rectangle $U_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$.

In the interior of the rectangle $V_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ we find a rectangle $S_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ which possesses property P and is contained in one part between the parallels $y = y'_{\gamma_1\ldots\gamma_n,\beta_1\ldots\beta_n}$ and $y = y''_{\gamma_1\ldots\gamma_n,\beta_1\ldots\beta_n}$ and in the other part between the parallels $x = x'_{\alpha_1\ldots\alpha_n,\delta_1\ldots\delta_n}$ and $x = x''_{\alpha_1\ldots\alpha_n,\delta_1\ldots\delta_n}$. (In the case $\alpha_1 = ... = \alpha_n = 0$ omit the first condition and in the case $\beta_1 = ... = \beta_n = 0$, the other.) The rectangles $S_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ will thus be determined by recurrence. We have

$$\begin{aligned} x_{\alpha_1\ldots\alpha_n 1} &= x'_{\alpha_1\ldots\alpha_n 2\ldots 2}, \qquad & x_{\alpha_1\ldots\alpha_n 2} &= x''_{\alpha_1\ldots\alpha_n 2\ldots 2} \\ y_{\beta_1\ldots\beta_n 1} &= y'_{2\ldots 2\beta_1\ldots\beta_n}, \qquad & y_{\beta_1\ldots\beta_n 2} &= y''_{2\ldots 2\beta_1\ldots\beta_n} \end{aligned}$$

and we denote by $R_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$ a rectangle formed by the lines

$$x = x_{\alpha_1\ldots\alpha_n 1}, \quad x = x_{\alpha_1\ldots\alpha_n 2}, \quad y = y_{\beta_1\ldots\beta_n 1}, \quad y = y_{\beta_1\ldots\beta_n 2}.$$

The lines

$$x = x_{\alpha_1\ldots\alpha_n\alpha_{n+1}}, \quad y = y_{\beta_1\ldots\beta_n\beta_{n+1}}, \quad (\alpha_1, ..., \alpha_{n+1}, \beta_1, ..., \beta_{n+1} = 0, 1, 2)$$

divide the square U into 3^{2n+2} rectangles whose bottom left sides are respectively the points $(x_{\alpha_1\ldots\alpha_{n+1}}, y_{\beta_1\ldots\beta_{n+1}})$: which we respectively denote by $U_{\alpha_1\ldots\alpha_{n+1},\beta_1\ldots\beta_{n+1}}$.

Thus, having already defined (for a given value n) the rectangles $U_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}$, we can always define the rectangles $U_{\alpha_1\ldots\alpha_{n+1},\beta_1\ldots\beta_{n+1}}$.

We now exclude from the square U the interior of the rectangle R and of all the rectangles $R_{\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n}(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n = 0, 1, 2; n = 1, 2, 3, ...)$. We denote by K the set of all the points of square U which remain : this is obviously a Cantor curve and any point of the curve C is a point of the curve K.

Now let t be an element of the interval $(0, 1)$ and

$$t = (0.c_1c_2c_3...)_3$$

be the representation of the fraction in base 3. We have

$$\varphi(t) = \lim_{n=\infty} x_{c_1c_2...c_n}, \quad \psi(t) = \lim_{n=\infty} y_{c_1c_2...c_n}.$$

One easily sees that the functions $\varphi(t)$ and $\psi(t)$ are well-defined in the interval $(0, 1)$ and that in this interval they are continuous and increasing functions of the variable t.

By making the correspondence of any point (x, y) of the curve C_0 to the point $[\varphi(x), \psi(y)]$ of the curve K, we have, as one easily sees, a one-to-one and continuous map from the curve C_0 to the curve K. The property of the curve C_0 is thus demonstrated.

We remark that we have demonstrated, as observed by M. E. Mazurkiewicz, that any point of C_0 is a ramification point of infinite order.

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