Translation of *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée.*

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On a Cantor curve which contains a one-to-one and continuous image of any given curve.

Note by **W. Sierpiński**, presented by Émile Picard.

The goal of this note is to construct *a (planar) Cantor curve* C_0 *such that, if* C *is a given (planar) Cantor curve, there always exists a one-to-one and continuous image* C' *of the curve* C *where all of the points are points of the curve* C_0.

The curve C_0 will be defined as follows. Let Q be a given square, for example the square whose vertices are the points $(0,0), (0,1), (1,0)$ and $(1,1)$. Divide the square Q into nine smaller squares and exclude the interior of the one which contains the center of the square Q. For each of the eight remaining squares, do the same process ad infinitum. Together, all of the points of Q which are not excluded obviously constitute a Cantor line; this is the curve C_0.

Let C now be an arbitrarily given Cantor curve: I claim that there exists a curve C', completely contained within C_0, which is the one-to-one and continuous image of the curve C. To demonstrate this, it is obvious that it suffices to show that there exists a curve K which is a one-to-one and continuous image of the curve C_0 and which contains all of the points of the curve C.

To define the curve K, first construct a square U such that the curve C is contained in the interior of U. Like the coordinates axes, take the sides of the square U. Divide the square U into nine new squares: let V be the one of them which contains the center of the square U.

We say, to be brief, that *a rectangle* R *has the property* P, if the sides are parallel to the coordinate axes and it does not contain any point of the curve C in its interior.

*All footnotes have been added by the translator. This is a rough translation of the article by this name published in the 'Comptes Rendus De L'Academie Sciences' in 1916. It can be found at gallica.bnf.fr/ark:/12148/bpt6k3115n/

Let R be a rectangle that has the property P which is in the interior of the square V (such a rectangle obviously exists, since the curve C is nowhere dense in the plane). Denote by x_1, the abscissa[1] of the left side of the rectangle R; by x_2 for the right side; by y_1 the ordinate[2] of the bottom, and by y_2 for the top. Also set $x_0 = 0$, $y_0 = 0$. The four lines

$$x = x_1, \qquad x = x_2, \qquad y = y_1, \qquad y = y_2$$

divide the square U into 9 rectangles whose lower left points are $(x_\alpha, y_\beta)(\alpha = 0,1,2; \beta = 0,1,2)$; we denote by $U_{\alpha,\beta}$ the rectangle which has the point (x_α, y_β) in the lower left (obviously, we have that $U_{1,1} = R$). Divide each of the nine rectangles $U_{\alpha,\beta}$ into nine equal rectangles; let $V_{\alpha,\beta}$ always be the one which contains the center of the rectangle $U_{\alpha,\beta}$.

We are going to now construct nine rectangles $S_{\alpha,\beta}$ that have the property P. In general, if σ denotes a symbol defining a rectangle S_σ having sides that are parallel to the coordinate axes, we will always denote by x'_σ the abscissa of the left side of the rectangle S_σ, by x''_σ the abscissa of the right side, by y'_σ the ordinate of the bottom and by y''_σ for the top.

The nine rectangles $S_{\alpha,\beta}$ will now be defined by recursion as follows. In the interior of $V_{\alpha,\beta}$, we will choose a rectangle $S_{\alpha,\beta}$ having the property P at contained in one part between the parallel lines $y = y'_{\alpha-1,\beta}, y = y''_{\alpha-1,\beta}$ and the other part between the parallel lines $x = x'_{\alpha,\beta-1}, x = x''_{\alpha,\beta-1}$ (if $\alpha = 0$ or $\beta = 0$, one can omit one or the other of these conditions). We have

$$x_{\alpha_1} = x'_{\alpha,2}, \qquad x_{\alpha_2} = x''_{\alpha,2}, \qquad y_{\beta_1} = y'_{2,\beta}, \qquad y_{\beta_2} = y''_{2,\beta}$$

where $\alpha = 0,1,2; \beta = 0,1,2$ and we denote by $R_{\alpha,\beta}$ the rectangle formed by the lines

$$x = x_{\alpha_1}, \qquad x = x_{\alpha_2}, \qquad y = y_{\beta_1}, \qquad y = y_{\beta_2}.$$

One easily sees that the rectangle $R_{\alpha,\beta}$ will be contained in $S_{\alpha,\beta}$; thus it does not contain any point of the curve C in its interior. We also have

$$x_{\alpha_0} = x_\alpha, \qquad y_{\beta_0} = y_\beta$$

where $\alpha, \beta = 0,1,2$. The lines

$$x = x_{\alpha_1\alpha_2}, \qquad y =_{\beta_1\beta_2} \qquad (\alpha_1, \alpha_2, \beta_1, \beta_2 = 0,1,2)$$

divide the square U into 81 rectangles whose lower left points are $(x_{\alpha_1\alpha_2}, y_{\beta_1\beta_2})$; we denote by $U_{\alpha_1\alpha_2,\beta_1\beta_2}$ the rectangle whose lower left is the point $(x_{\alpha_1\alpha_2}, y_{\beta_1\beta_2})$.

We now suppose that we have already defined the rectangles

$$S_{\alpha_1\alpha_2\ldots\alpha_{n-1},\beta_1\beta_2\ldots\beta_{n-1}} \quad \textit{and} \quad U_{\alpha_1\alpha_2\ldots\alpha_n,\beta_1\beta_2\ldots\beta_n}$$

where $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n = 0,1,2$.

[1] the perpendicular distance from a point to the vertical axis, ie, the horizontal coordinate
[2] the perpendicular distance from a point to the horizontal axis, ie, the vertical coordinate

Divide each of these rectangles $U_{\alpha_1\alpha_2...\alpha_n,\beta_1\beta_2...\beta_n}$ into nine equal rectangles and let $V_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ always be the one in the middle.

The indices $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n$ have been given, we denote by $U_{\gamma_1...\gamma_n,\beta_1...\beta_n}$ the rectangles $U_{\xi_1...\xi_n,\eta_1...\eta_n}$ whose right edge coincides with the left edge of rectangle $U_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ and by $U_{\alpha_1...\alpha_n,\delta_1...\delta_n}$ the ones whose top edge coincides with the bottom edge of rectangle $U_{\alpha_1...\alpha_n,\beta_1...\beta_n}$.

In the interior of the rectangle $V_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ we find a rectangle $S_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ which possesses property P and is contained in one part between the parallels $y = y'_{\gamma_1...\gamma_n,\beta_1...\beta_n}$ and $y = y''_{\gamma_1...\gamma_n,\beta_1...\beta_n}$ and in the other part between the parallels $x = x'_{\alpha_1...\alpha_n,\delta_1...\delta_n}$ and $x = x''_{\alpha_1...\alpha_n,\delta_1...\delta_n}$. (In the case $\alpha_1 = ... = \alpha_n = 0$ omit the first condition and in the case $\beta_1 = ... = \beta_n = 0$, the other.) The rectangles $S_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ will thus be determined by recurrence. We have

$$x_{\alpha_1...\alpha_n 1} = x'_{\alpha_1...\alpha_n 2...2}, \qquad x_{\alpha_1...\alpha_n 2} = x''_{\alpha_1...\alpha_n 2...2}$$
$$y_{\beta_1...\beta_n 1} = y'_{2...2\beta_1...\beta_n}, \qquad y_{\beta_1...\beta_n 2} = y''_{2...2\beta_1...\beta_n}$$

and we denote by $R_{\alpha_1...\alpha_n,\beta_1...\beta_n}$ a rectangle formed by the lines

$$x = x_{\alpha_1...\alpha_n 1}, \quad x = x_{\alpha_1...\alpha_n 2}, \quad y = y_{\beta_1...\beta_n 1}, \quad y = y_{\beta_1...\beta_n 2}.$$

The lines

$$x = x_{\alpha_1...\alpha_n\alpha_{n+1}}, \quad y = y_{\beta_1...\beta_n\beta_{n+1}}, \quad (\alpha_1, ..., \alpha_{n+1}, \beta_1, ..., \beta_{n+1} = 0, 1, 2)$$

divide the square U into 3^{2n+2} rectangles whose bottom left sides are respectively the points $(x_{\alpha_1...\alpha_{n+1}}, y_{\beta_1...\beta_{n+1}})$: which we respectively denote by $U_{\alpha_1...\alpha_{n+1},\beta_1...\beta_{n+1}}$.

Thus, having already defined (for a given value n) the rectangles $U_{\alpha_1...\alpha_n,\beta_1...\beta_n}$, we can always define the rectangles $U_{\alpha_1...\alpha_{n+1},\beta_1...\beta_{n+1}}$.

We now exclude from the square U the interior of the rectangle R and of all the rectangles $R_{\alpha_1...\alpha_n,\beta_1...\beta_n}(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n = 0, 1, 2; n = 1, 2, 3, ...)$. We denote by K the set of all the points of square U which remain : this is obviously a Cantor curve and any point of the curve C is a point of the curve K.

Now let t be an element of the interval $(0, 1)$ and

$$t = (0.c_1c_2c_3...)_3$$

be the representation of the fraction in base 3. We have

$$\varphi(t) = \lim_{n=\infty} x_{c_1c_2...c_n}, \quad \psi(t) = \lim_{n=\infty} y_{c_1c_2...c_n}.$$

One easily sees that the functions $\varphi(t)$ and $\psi(t)$ are well-defined in the interval $(0, 1)$ and that in this interval they are continuous and increasing functions of the variable t.

By making the correspondence of any point (x, y) of the curve C_0 to the point $[\varphi(x), \psi(y)]$ of the curve K, we have, as one easily sees, a one-to-one and continuous map from the curve C_0 to the curve K. The property of the curve C_0 is thus demonstrated.

We remark that we have demonstrated, as observed by M. E. Mazurkiewicz, that any point of C_0 is a ramification point of infinite order.