Student Name and ID Number

MATH 3012 Final Exam, December 8, 2003, WTT

1. Consider the following seven letter alphabet: $\{A, B, C, D, E, F, G\}$.

a. How many strings of length 19 have exactly 6 B's?

b. How many strings of length 19 contain exactly 4 A's, 2 B's, 7 D's and 6 G's (and no C's, E's or F's)?

c. Of the strings described in part b, how many have all 4 A's before the 7 D's?

d. Of the strings described in part c, how many have the 4 A's and the 7 D's occuring together as a block of 11 consecutive characters?

2. How many integer value solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 = 59$, all $x_i > 0$.

b. $x_1 + x_2 + x_3 < 59$, all $x_i > 0$.

c. $x_1 + x_2 + x_3 = 59$, all $x_i \geq 0$.

d. $x_1 + x_2 + x_3 \leq 59$, all $x_i \geq 0$.

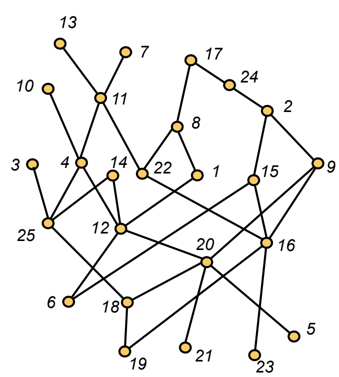
e. $x_1 + x_2 + x_3 = 59$, all $x_i > 0$, $x_3 > 12$.

3. In three space, consider moves from one point with integer coordinates to another formed by adding one of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. How many paths from $(0, 0, 0)$ to $(8, 5, 9)$ can formed with such moves?

4. Use the Euclidean algorithm to find $d = \gcd(780, 2772)$.

Then find integers x and y so that $d = 780x + 2772y$.

5. Consider the partially ordered set (poset) shown below:



a. Find a minimum partition of this poset into antichains.

b. Find the height h of this poset (maximum number of points in a chain).

c. Find a chain of h points in this poset.

6. Note that $2772 = 2^2 \times 3^2 \times 7 \times 11$. Compute $\phi(2772)$, where ϕ is the Euler ϕ function.

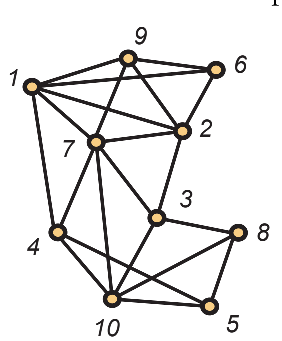
7. Let A denote the advancement operator, i.e., $Af(n) = f(n+1)$. Find the general solution of the following recurrence equation:

$$(A-1)^4(A-3)^2(A+7)^3 f(n) = 0$$

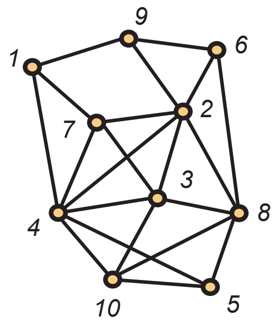
8. Given $f(0) = 15$ and $f(1) = -23$, find the particular solution of the following recurrence equation:

$$(2A^2 + 7A - 15)f(n) = 0$$

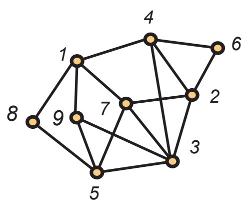
9. Show that G is planar.



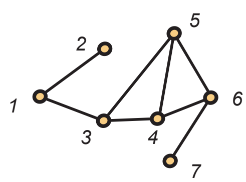
10. Show that G is hamiltonian by listing the vertices in an appropriate order, starting with vertex 1.



11. Explain why the following graph does not have an Euler circuit. Then show that all the edges can be traced exactly once starting at one vertex and ending at another by listing the vertices (with repetition allowed) in an appropriate order.

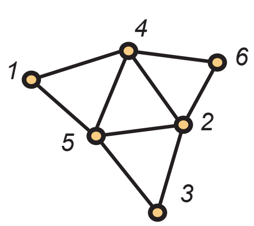


12. Show that G is an interval graph and has a representation in which the order of the left endpoints matches the labels on the vertices. Then color the graph using First Fit and find the chromatic number $\chi(G)$ and the maximum clique size $\omega(G)$.



13. Draw a set of 8 intervals and label them with the numbers from $\{1, 2, \ldots, 8\}$ so that (a) the associated interval graph has maximum clique size 2; and (b) if the vertices of the graph are colored using First Fit in the order they have been labelled, then 4 colors will be used.

14. Verify Euler's formula for the following planar graph.



15. Suppose G is a graph with 80 vertices and 1837 edges. Explain why G contains a triangle.

16. Suppose G is a graph with 100 vertices and 386 edges. Explain why G is non-planar.

17. Consider the graph G with non-negative edge weights given in the following text file shown below on the left (note that the file has been sorted by edge weights). In the space to the right List the edges *in order* which result from applying Kruskal's algorithm (avoid cycles) to find a minimum weight spanning tree of G. Then set vertex 1 to be the root and list *in order* the edges which result from applying Prim's algorithm (build tree).

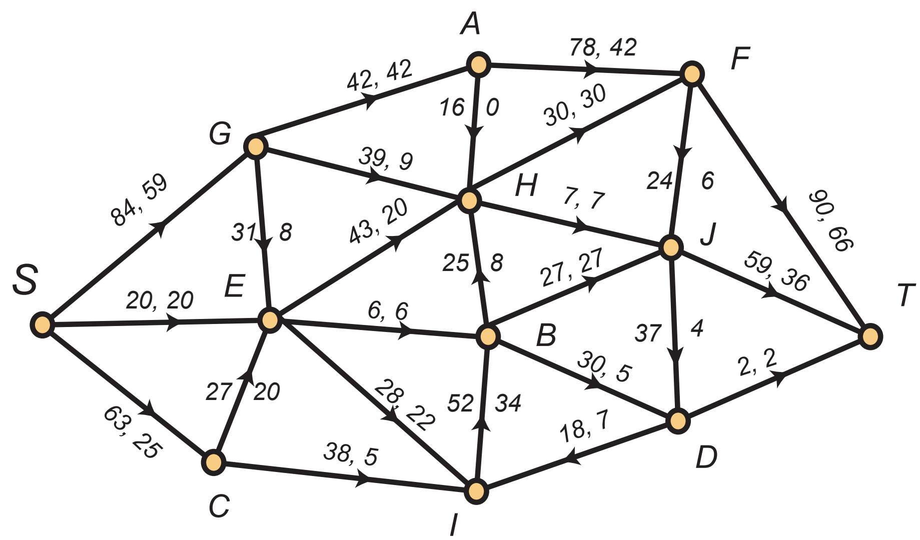
`graph.txt` **Kruskal** **Prim**

3 5 14
4 7 18
5 6 19
3 6 22
1 2 29
2 4 32
1 5 38
2 6 39
3 7 40

18. The matrix given (below left) is the distance matrix for a digraph D whose vertex set is $\{1, 2, 3, 4, 5, 6, 7\}$. The entry $d_{i,j}$ denotes the length of the directed edge from vertex i to vertex j. Edges not present in the graph have infinite length. In the space to the right, apply Dijkstra's algorithm to find all the shortest paths from vertex 1 to all other vertices in the digraph D.

D	1	2	3	4	5	6	7
1	0	25	7	20	8	28	16
2	9	0	12	15	19	∞	8
3	6	40	0	∞	28	16	∞
4	∞	16	∞	0	19	12	∞
5	8	13	24	9	0	12	5
6	4	14	23	∞	18	0	8
7	∞	6	14	∞	28	5	0

19. Consider the following network flow problem.



a. What is the current value of the flow?

b. What is the capacity of the cut: $L = \{S, A, B, C, E, G, H, I\}$; $U = \{T, D, F, J\}$?

c. List *in order* the labels given to vertices labelling algorithm using alphabetic ordering on the nodes and find an augmenting path. Then increase the flow by the binding value of this path, recording the changes directly on the diagram.

d. Repeat the labelling algorithm on the new network flow. It will halt with the sink unlabelled. Find a cut whose capacity is the value of the current flow.

20. Extra Credit Consider the positive integers p_1 and p_2 given below (they each have more than 200 digits so it takes several lines to display them):

p_1 = 643808006803554439230129854961492699151386107534013432918073439524138264842370630061369715394739134090922937332590384720397133335969549256322620979036686633213903952966175107096769180017646161851573147596390153

p_2 = 56125680981752282333498088313568935051383833838594899821664631784577337171193624243181360054669678410455329112434552942717084003541384594864129940145043086760031292483340068923506115878221189886491132772739661669044958531131327771

Both p_1 and p_2 are primes. Now consider the number $n = p_1 p_2$ formed by multiplying them together. Two students, Alice and Bob, are asked to find $\phi(n)$, where ϕ is the Euler ϕ function. But Alice is given p_1 and p_2 while Bob is just given the value of n. Why does Alice have an unfair advantage over Bob? Does it help Bob to be told that n is in fact the product of two primes—without being told the values of these two primes?

21. Extra Credit Show that a tournament always has a directed hamiltonian path.