Solutions

Student Name and ID Number

MATH 3012 Final Exam, December 15, 2011, WTT

20

1. Consider the 16-element set X consisting of the six capital letters $\{A, B, C, D, E, F\}$ and the ten digits $\{0, 1, 2, \ldots, 9\}$.

a. How many strings of length 11 can be formed if repetition of symbols is permitted?

$$16^{11}$$

b. How many strings of length 11 can be formed if repetition of symbols is *not* permitted?

$$P(16,11) \quad \text{or} \quad 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

c. How many strings of length 11 can be formed using exactly four 3's, two B's and five D's?

$$\binom{11}{4,2,5} \quad \text{or} \quad \frac{11!}{4!\,2!\,5!}$$

d. How many strings of length 11 can be formed if exactly four characters are letters and exactly three of the remaining seven characters are 8's? Here, repetition is allowed.

$$\binom{11}{4} 6^4 \binom{7}{3} 9^4$$

e. How many symmetric binary relations are there on X?

$$2^{11} \cdot 2^{\binom{11}{2}}$$

f. How many symmetric and reflexive binary relations are there on X?

$$2^{\binom{11}{2}}$$

g. How many equivalence relations are there on X with class sizes 4, 4, 4, 2, 1 and 1?

$$\frac{11!}{4!\,4!\,4!\,2!\,1!\,1!\,3!\,1!\,2!}$$

15

2. How many integer valued solutions are there to the equation $x_1 + x_2 + x_3 + x + 4 = 52$ when:

a. $x_i > 0$ for $i = 1, 2, 3, 4$.

$$\binom{51}{3}$$

b. $x_i \geq 0$ for $i = 1, 2, 3, 4$.

$$\binom{55}{3}$$

c. $x_i > 0$ for $i = 1, 3, 4$ and $x_2 > 7$.

$$\binom{44}{3}$$

d. $x_i > 0$ for $i = 1, 3, 4$ and $x_2 \leq 7$. Assume $0 < x_2 \leq 7$

$$\binom{51}{3} - \binom{44}{3}$$

10

3 a. Use the Euclidean algorithm to find $d = \gcd(420, 792)$.

$$792 = 1 \cdot 420 + 372, \quad 420 = 1 \cdot 372 + 48, \quad 372 = 7 \cdot 48 + 36, \quad 48 = 1 \cdot 36 + \boxed{12}, \quad 36 = 3 \cdot 12 + 0$$

$$\gcd(420, 792) = 12$$

b. Use your work in the first part of this problem to find integers a and b so that $d = 420a + 792b$.

$792 = 1 \cdot 420 + 372$
$420 = 1 \cdot 372 + 48$
$372 = 7 \cdot 48 + 36$
$48 = 1 \cdot 36 + 12$

$12 = 48 - 1 \cdot 36$
$36 = 372 - 7 \cdot 48$
$48 = 420 - 1 \cdot 372$
$372 = 792 - 1 \cdot 420$

$$\begin{aligned} 12 &= 48 - 1 \cdot 36 \\ &= 48 - 1(372 - 7 \cdot 48) \\ &= 8 \cdot 48 - 1 \cdot 372 \\ &= 8(420 - 1 \cdot 372) - 1 \cdot 372 \\ &= 8 \cdot 420 - 9 \cdot 372 \\ &= 8 \cdot 420 - 9(792 - 1 \cdot 420) \\ &= -9 \cdot 792 + 17 \cdot 420 \end{aligned}$$

so $\boxed{a = 17 \quad b = -9}$

c. Using your previous work, factor 792 completely into a product of primes. You will need this answer later on this test.

$12 \overline{)792}$ = 66; 72, 72

$66 = 2 \cdot 3 \cdot 11$

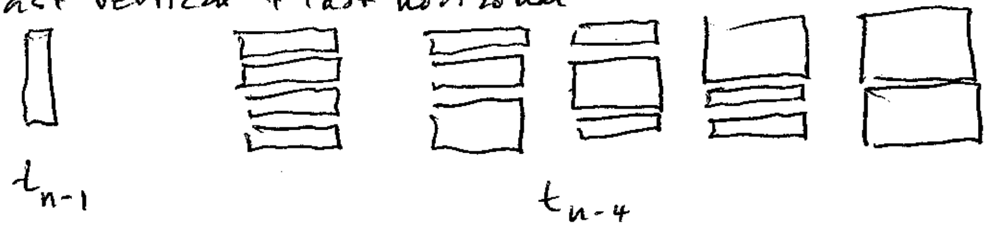
$792 = 12 \cdot 2 \cdot 3 \cdot 11$

$\boxed{792 = 2^3 \cdot 3^2 \cdot 11}$

15

4. For a positive integer n, let t_n count the number of ways to tile a $4 \times n$ array with dominoes of the following three sizes: 4×1, 1×4 and 2×4. Note that dominoes of size 4×2 are not permitted. Then $t_1 = t_2 = t_3 = 1$ and $t_4 = \cancel{5}\ 6$. Develop a recurrence for t_n and use it to find t_6.

Last vertical + last horizontal



$t_n = t_{n-1} + 5t_{n-4}$

$t_5 = t_4 + 5t_1 = 6 + 5 = 11$

$\boxed{t_6 = t_5 + 5t_2 = 11 + 5 \cdot 1 = 16}$

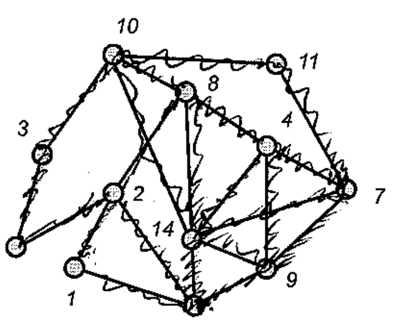
10

5. Use the algorithm developed in class to find an Euler circuit in the graph G shown below (use node 1 as root):

(1, 2, 5, 1)

(2, 6, 3, 10, ~~14, 4, 7, 5~~, 8, 2)

(1, 2, 6, 3, 10, 8, 2, 5, 1)

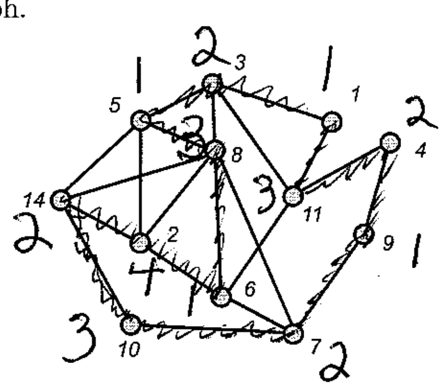


10, 11, 7, 4, 8, 14, 4, 9, 5, 14, 7, 9, 14, 10

FINAL (1, 2, 6, 3, 10, 11, 7, 4, 8, 14, 4, 9, 5, 14, 7, 9, 14, 10, 8, 2, 5, 1)

10

6. Consider the following graph.



a. Explain why $\{3, 5, 8\}$ is a maximal clique. It is a clique because $\{3,5\}$, $\{5,8\}$ and $\{3,8\}$ are edges. It is maximal because no other vertex is adjacent to 3, 5 and 8.

b. Find the maximum clique size $\omega(G)$ for this graph, and find a set of vertices that form a maximum clique. $\omega(G) = 4$ $\{2, 5, 8, 14\}$ is a maximum clique

c. Show that $\chi(G) = \omega(G)$ by providing a proper coloring of G. You may indicate your coloring by writing directly on the figure.

d. Despite the fact that $\chi(G) = \omega(G)$, the graph G is not perfect. Explain why.

G is perfect $\iff \chi(H) = \omega(H)$ for every induced subgraph H. But $\{4, 6, 7, 9, 11\}$ induce C_5
$\omega(C_5) = 2$ and $\chi(C_5) = 3$

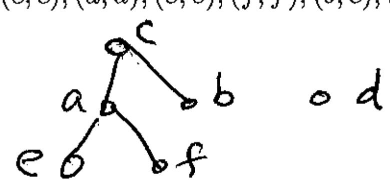
10

7. Show that the graph G from the preceding problem is hamiltonian by providing an appropriate listing of the vertices, starting with $1, 11, 4$ and ending with 1.

$(1, 11, 4, 9, 7, 10, 14, 2, 6, 8, 5, 3, 1)$ is a hamiltonian cycle.

10

8. Draw an order diagram for the poset whose ground set is $\{a, b, c, d, e, f\}$ and whose order relation is: $\{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (b, c), (f, c), (e, c), (e, a), (a, c), (f, a)\}$



10

9. For the subset lattice $\mathbf{2}^{15}$,

a. The total number of elements is: 2^{15}

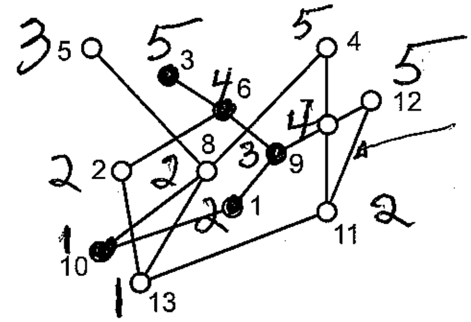
b. The total number of maximal chains is: $15!$

c. The number of maximal chains through $\{2, 6, 7, 9, 11\}$ is: $5!\ 10!$

d. The width of $\mathbf{2}^{15}$ is: $\binom{15}{7}$ OR $\binom{15}{8}$

10

10. For the poset P shown below,



erase this edge

a. List all elements comparable with 7.

$\{4, 12, 9, 1, 10, 11, 13\}$

(Some would include 7) This is a matter of convention.

b. List all elements covered by 7.

$\{9, 11\}$

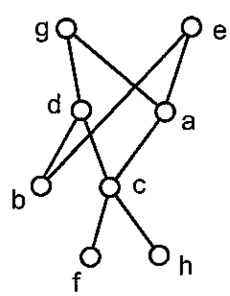
c. By inspection (not by algorithm), explain why this poset is not an interval order.

$\{5, 8\} \cup \{3, 6\}$ form $\underline{2} + \underline{2}$. This is forbidden in an interval order

d. Find the height h and a partition into h minimal elements by recursively stripping off the set of minimal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.

25

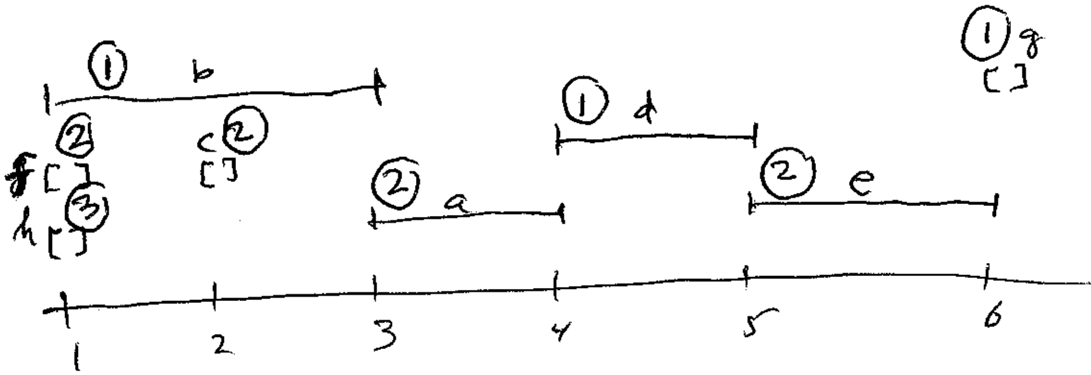
11. The poset P shown below is an interval order:



a. Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

3	$D(a) =$ c f h	4	$U(a) =$ g e	$I(a) =$	[3, 4]
1	$D(b) =$ ∅	3	$U(b) =$ d g e	$I(b) =$	[1, 3]
2	$D(c) =$ f h	2	$U(c) =$ a d g e	$I(c) =$	[2, 2]
4	$D(d) =$ b c f h	5	$U(d) =$ g	$I(d) =$	[4, 5]
5	$D(e) =$ a b c f h	6	$U(e) =$ ∅	$I(e) =$	[5, 6]
1	$D(f) =$ ∅	1	$U(f) =$ a c d g e	$I(f) =$	[1, 1]
6	$D(g) =$ a b c d f h	6	$U(g) =$ ∅	$I(g) =$	[6, 6]
1	$D(h) =$ ∅	1	$U(h) =$ a c d g e	$I(h) =$	[1, 1]

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.



width = 3

c. Find a maximum antichain in P: $\{b, f, h\}$

10

12 a. Write all the partitions of the integer 8 into odd parts:

$8 = 7+1$
$= 5+3$
$= 5+1+1+1$
$= 3+3+1+1$
$= 3+1+1+1+1+1$
$= 1+1+1+1+1+1+1+1$

b. Write all the partitions of the integer 8 into distinct parts:

$8 = 8$
$= 7+1$
$= 6+2$
$= 5+3$
$= 5+2+1$
$= 4+3+1$

c. Write all partitions of the integer 14 associated with the coefficient of x^{14} in the generating series expansion of $f(x) = (1 + x^3 + x^6)(1 + x^5)/(1 - x^2)$.

zero, one or two 3's
zero or one 5
Arbitrary # of 2's

$14 = 2+2+2+2+2+2+2$
$= 3+3+2+2+2+2$
$= 5+3+2+2+2+2$

10

13. Find the general solution to the advancement operator equation:
$A^2(A-5)^3(A+2)^2(A-7)f = 0$

$f(n) = c_1 5^n + c_2 n 5^n + c_3 n^2 5^n + c_4 (-2)^n + c_5 n(-2)^n + c_6 7^n$

10

14. Find the solution to the advancement operator equation:
$(A^2 + 8A + 12)f(n) = 0, \quad f(0) = 16$ and $f(2) = -68$.

$(A+2)(A+6)f(n)=0$

general solution $f(n) = c_1(-2)^n + c_2(-6)^n$

$f(0) = 16 = c_1 + c_2$

$f(1) = -68 = -2c_1 - 6c_2$

$32 = 2c_1 + 2c_2$

$34 = -4c_2$

$\frac{34}{-4} = c_2 = -\frac{17}{2}$

10

15 a. Write the inclusion/exclusion formula for the number of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

$$f(n,m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^n$$

b. Evaluate your formula when $n = 6$ and $m = 3$.

$$f(6,3) = \binom{3}{0}3^6 - \binom{3}{1}2^6 + \binom{3}{2}1^6 - \binom{3}{3}0^6$$

$$= 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6$$

$$= 729 - 192 + 3 = 540$$

10

16.

a. Write the inclusion/exclusion formula for the number of derangements on $\{1, 2, \ldots, n\}$.

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

b. Evaluate your formula when $n = 7$.

$$\binom{7}{0}7! - \binom{7}{1}6! + \binom{7}{2}5! - \binom{7}{3}4! + \binom{7}{4}3! - \binom{7}{5}2! + \binom{7}{6}1! - \binom{7}{7}0!$$

$$7! - 7! + 21 \cdot 120 - 35 \cdot 24 + 35 \cdot 6 - 21 \cdot 2 + 7 \cdot 1 - 1 \cdot 1$$

$$1854$$

10

17. Previously, you factored 792 into a product of primes. Using this factorization, evaluate the euler ϕ-function $\phi(792)$.

$792 = 2^3 \cdot 3^2 \cdot 11$

$\phi(792) = 2^3 \cdot 3^2 \cdot 11 \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{11}\right)$

$= 2^3 \cdot 3^2 \cdot \cancel{11} \cdot \frac{1}{\cancel{2}} \cdot \frac{2}{\cancel{3}} \cdot \frac{10}{\cancel{11}} = \cancel{120}\ 240$

10

18 a. Let G be a graph on 11 vertices in which every vertex has 6 neighbors. Explain why G is hamiltonian but not planar.

Dirac's theorem asserts that a graph on n vertices is hamiltonian when $\deg(x) \geq \lceil \frac{n}{2} \rceil$ for every $x \in G$. In this case $6 = \lceil 11/2 \rceil$ so G is hamiltonian. Also G has $11 \cdot 6 / 2 = 33$ edges. But the maximum number of edges in a planar graph on n vertices is $3n-6$ but $33 > 3 \cdot 11 - 6 = 27$.

a. Show that there is a non-hamiltonian graph on 11 vertices in which every vertex has degree at least 5.

$K_{5,6}$ is non-hamiltonian

deg 6

deg 5

10

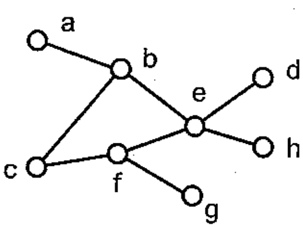
19. Verify Euler's formula for the planar graph shown below.

$V = 8$

$E = 8$

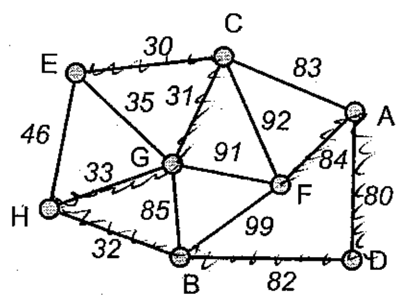
$F = 2$

$V - E + F = 2$ ✓



15

20. Consider the following weighted graph:



In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.

Kruskal's Algorithm

EC 30
CG 31
BH 32
GH 33
AD 80
BD 82
AF 84

Prim's Algorithm

AD 80
BD 82
BH 32
GH 33
CG 31
EC 30
AF 84

20

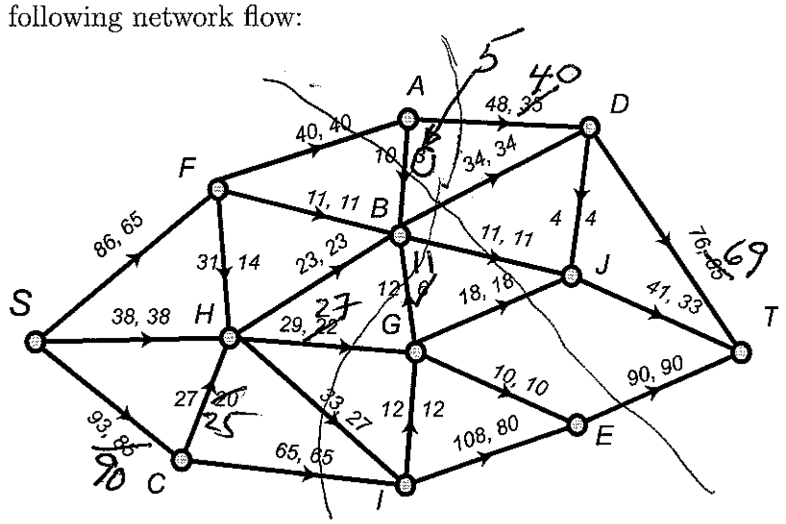
21. A data file `digraph_data.txt` has been read for a digraph whose vertex set is [7]. The weights on the directed edges are shown in the matrix below. The entry $w(i,j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x.

W	1	2	3	4	5	6	7
1	0	38	42	17	38	64	29
2		0	4		30	23	10
3			0		41	18	
4	27	20	28	0		45	
5			2		0	21	9
6	82	5	3	2	2	0	
7		8	22	4	18	12	0

2	3	4	5	6	7
(1,2) 38	(1,3) 42	1 $\boxed{(1,4)\ 17}$	4 $\boxed{(1,5)\ 38}$	(1,6) 64	2 $\boxed{(1,7)\ 29}$
~~(1,4,2) 17+20=37~~	~~(1,4,2,3) 37+4=41~~			~~(1,4,6) 17+45=62~~	
3 $\boxed{(1,4,2)\ 17+20=37}$	(1,4,2,3) 37+4=41			~~1,4~~	
~~(1,7,2)~~	5 $\boxed{1,5,3\ 38+2=40}$			6 $\boxed{(1,7,6)\ 29+12=41}$	

30

22. Consider the following network flow:



a. What is the current value of the flow?

$65+38+85 = 188 = 65+33+90$

b. What is the capacity of the cut $V = \{S, A, B, C, F, H\} \cup \{D, E, G, \cancel{D}, \cancel{G}, I, J, T\}$.

$48+34+11+29+33+65$ (BG is backwards)

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

S (*, +, ∞)
C (S, +, 8)
F (S, +, 21)
H (C, +, 7)
G (H, +, 7)
I (H, +, 6)
B (G, +, 6)
E (I, +, 6)
A (B, −, 5)
D (A, +, 5)
T (D, +, 5)

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

(S, C, H G, B, A, D, T)

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled.

S (*, +, ∞)
C (S, +, 3)
F (S, +, 21)
H (C, +, 2)
G (H, +, 2)
I (H, +, 2)
B (G, +, 1)
E (I, +, 2)

f. Find a cut whose capacity is equal to the value of the updated flow.

{S, B, C, E, F, G, H, I} ∪ {A, D, J, T}

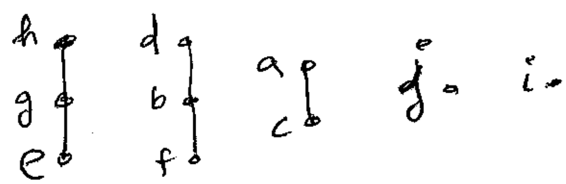
capacity = 40 + 34 + 11 + 18 + 90 = 193 ✓

10

23. Consider a poset P whose ground set is $X = \{a, b, c, d, e, f, g, h, i, j\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition. When the labelling algorithm halts, the following edges are matched:

$$e'g'' \quad b'd'' \quad c'a'' \quad f'b'' \quad g'h''$$

a. Find the chain partition of P that is associated with this matching. Also find the value of w.



$W = 5$

b. Explain why element i belongs to every maximum antichain in P.

Every maximum antichain contains an element from each of these 5 chains. The only element in C_5 is i.

SCORING : Total is 300 points