Student Name and ID Number

MATH 3012 Final Exam, December 11, 2014, WTT

1. Consider the 11-element set X consisting of the three capital letters $\{A, B, C\}$ and the eight digits $\{0, 1, 2, \ldots, 7\}$.

a. How many strings of length 9 can be formed if repetition of symbols is *not* permitted?

b. How many strings of length 9 can be formed if repetition of symbols is permitted?

c. How many strings of length 9 can be formed using exactly three A's, four B's and two 7's?

d. How many strings of length 9 can be formed if exactly three characters are letters and exactly two of the remaining six characters are 5's? Here, repetition is allowed.

2. How many integer valued solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 58$ when:

a. $x_i > 0$ for $i = 1, 2, 3, 4, 5$.

b. $x_i \geq 0$ for $i = 1, 2, 3, 4, 5$.

c. $x_i > 0$ for $i = 1, 3, 5$, $x_2 > 7$ and $x_4 \geq 8$.

d. $x_i > 0$ for $i = 1, 2, 3, 4, 5$ and $x_2 \leq 7$.

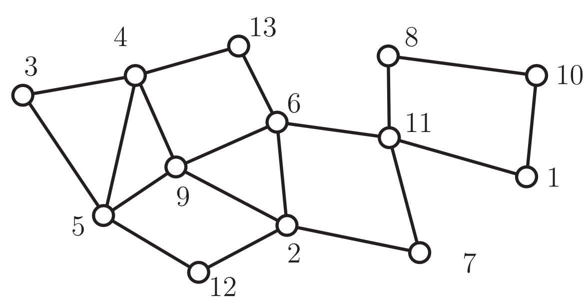
3 a. Use the Euclidean algorithm to find $d = \gcd(3465, 819)$.

b. Use your work in the first part of this problem to find integers a and b so that $d = 3465a + 819b$.

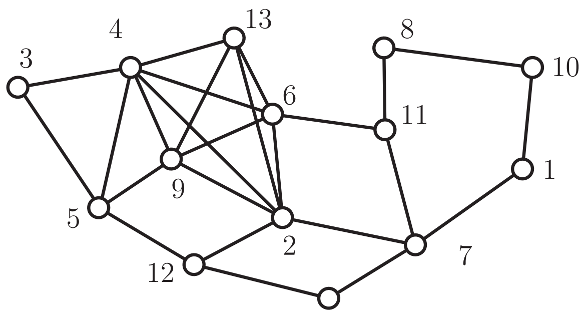
c. Using your previous work, factor 819 completely into a product of primes. You will need this answer later on this test.

4. For a positive integer n, let t_n count the number of ternary strings of length n which do not contain a substring of the form: $(1,2,0)$ (in consecutive positions). Note that $t_1 = 3$, $t_2 = 9$ and $t_3 = 26$. Develop a recurrence equation for t_n and use it to compute t_4, t_5 and t_6.

5. Use the algorithm developed in class (always go to the least vertex adjacent via an unused edge) to find an Euler circuit in the graph G shown below. Use node 1 as root:



6. Consider the following graph.



a. Explain why $\{3, 4, 5\}$ is a maximal clique.

b. Find the maximum clique size $\omega(G)$ for this graph, and find a set of vertices that form a maximum clique.

c. Show that $\chi(G) = \omega(G)$ by providing a proper coloring of G. You may indicate your coloring by writing directly on the figure.

d. Despite the fact that $\chi(G) = \omega(G)$, the graph G is not perfect. Explain why.

7. Show that the graph G from the preceding problem is hamiltonian by providing an appropriate listing of the vertices, starting with $1, 10, 8$ and ending with 1.

8. For the subset lattice $\mathbf{2}^{12}$,

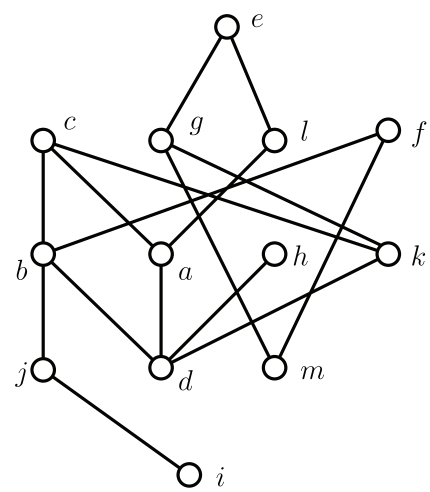
a. The total number of elements is: ____________________

b. The total number of maximal chains is: ____________________

c. The number of maximal chains through $\{2, 4, 6, 9\}$ is: ____________________

d. The width of $\mathbf{2}^{12}$ is: ____________________

9. For the poset P shown below,



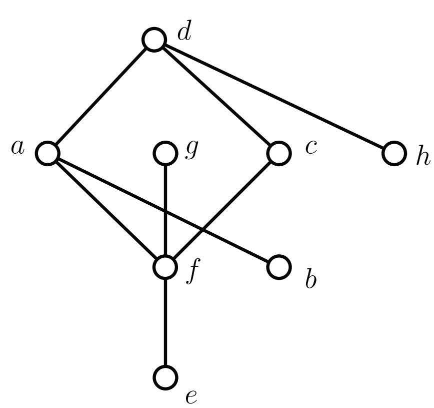
a. List all elements comparable with a. ____________________

b. List all elements covered by a. ____________________

c. By inspection (not by algorithm), explain why this poset is not an interval order.

d. Find the height h and a partition into h minimal elements by recursively stripping off the set of minimal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.

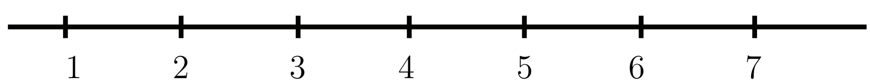
10. The poset P shown below is an interval order:



a. Find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

$D(a) =$	$U(a) =$	$I(a) =$
$D(b) =$	$U(b) =$	$I(b) =$
$D(c) =$	$U(c) =$	$I(c) =$
$D(d) =$	$U(d) =$	$I(d) =$
$D(e) =$	$U(e) =$	$I(e) =$
$D(f) =$	$U(f) =$	$I(f) =$
$D(g) =$	$U(g) =$	$I(g) =$
$D(h) =$	$U(h) =$	$I(h) =$

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.



c. Find a maximum antichain in P:

11 a. Write all the partitions of the integer 7 into odd parts:

b. Write all the partitions of the integer 7 into distinct parts:

12 a. Interpret the coefficient a_n of x^n in the following generating function in terms of partitions of an integer n:
$f(x) = (1+x)(1+x^3)(1+x^5)(1+x^7)(1+x^9)\ldots$

b. Then write all partitions of the integer 16 associated with the coefficient of x^{16} for the function $f(x)$ in part c.

13. Find the general solution to the advancement operator equation:
$A^3(A+5)^2(A-3)^2(A-7)(A-3)f = 0$

14. Find the solution to the advancement operator equation:
$(A^2 - 6A + 9)f(n) = 0, \quad f(0) = 8$ and $f(1) = 9$.

15 a. Write the inclusion/exclusion formula for the number of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

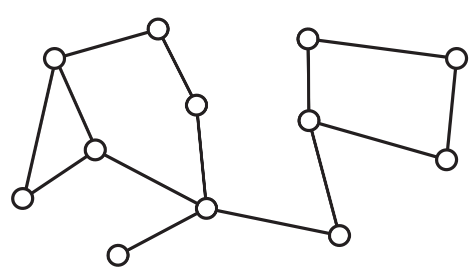
b. Evaluate your formula when $n = 6$ and $m = 4$.

16. a. Write the inclusion/exclusion formula for the number of derangements on $\{1, 2, \ldots, n\}$.

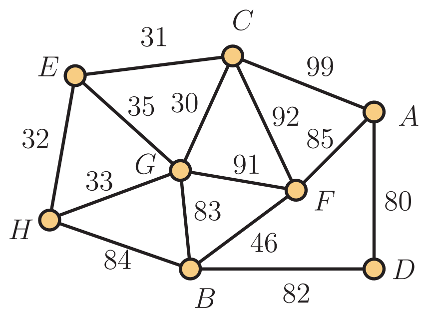
b. Evaluate your formula when $n = 6$.

17. Previously, you factored 819 into a product of primes. Using this factorization, evaluate the euler ϕ-function $\phi(819)$.

18. Verify Euler's formula for the planar graph shown below.



19. Consider the following weighted graph:



In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.

Kruskal's Algorithm **Prim's Algorithm**

20. Consider a poset P whose ground set is $X = \{a, b, c, d, e, f, g, h, i, j\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition. When the labelling algorithm halts, the following edges are matched:

$$c'g'' \quad e'a'' \quad d'c'' \quad j'b'' \quad g'h''$$

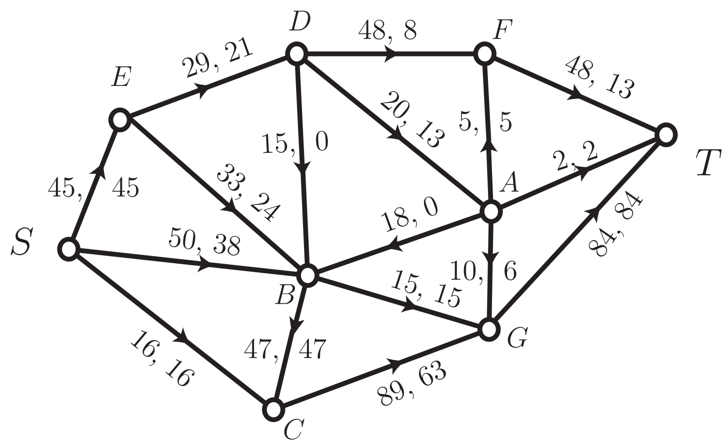
a. Find the chain partition of P that is associated with this matching. Also find the value of w.

b. Explain why elements i and f belong to every maximum antichain in P.

21. A data file `digraph_data.txt` has been read for a digraph whose vertex set is [7]. The weights on the directed edges are shown in the matrix below. The entry $w(i, j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x.

W	1	2	3	4	5	6	7
1	0	19	17	28	22	44	48
2		0	4	6	30	8	10
3		9	0		41	11	20
4	27	20	28	0		45	
5			2		0	21	9
6	82	5	3	2	2	0	10
7		8	22	4	18	12	0

22. Consider the following network flow:



a. What is the current value of the flow?

b. What is the capacity of the cut $V = \{S, B, C, D, E\} \cup \{T, A, F, G\}$.

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled.

f. Find a cut whose capacity is equal to the value of the updated flow.

23. True–False. Mark in the left margin.

1. $2^{100} > 100,000,000,000,000$.
2. There is a planar graph G on 538 vertices with $\chi(G) = 7$.
3. All graphs with 958 vertices and 4273 edges are non-planar.
4. There is a non-hamiltonian graph on 842 vertices in which every vertex has degree 462.
5. Every connected graph on 836 vertices in which every vertex has degree 98 has an Euler circuit.
6. The number of lattice paths from $(7, 9)$ to $(18, 14)$ is $\binom{25}{23}$.
7. The number of spanning trees on $\{1, 2, \ldots, n\}$ is 2^{n-2}.
8. A cycle on 458 vertices is a homeomorph of the complete bipartite graph $K_{2,2}$.
9. The number of lattice paths from $(0, 0)$ to (n, n) which do not pass through a point above the diagonal is the Catalan number $\binom{2n}{n}/(n + 1)$.
10. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly determine whether $2,647$ is the sum of two integers in the file.
11. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly determine whether $475,746$ is the product of two integers in the file.
12. Any modern computer can accept a file of $3,000$ positive integers, each at most $5,000$, and quickly factor each of the numbers into primes.
13. There is a graph on 872 vertices in which no two vertices have the same degree.
14. There is a poset with 623 points having width 58 and height 9.
15. The permutation $(9, 4, 1, 7, 5, 8, 2, 6, 3)$ is a derangement.
16. Linear programming problems with integer coefficient constraints always have integer valued solutions.
17. Every network flow problem is also a linear programming problem.
18. Every linear programming problem is also a network flow problem.