Solutions

Student Name and ID Number

MATH 3012 Final Exam, December 8, 2015, WTT

16

4+4

1. Consider the 36-element set X consisting of the twenty-six capital letters of the English alphabet and the ten digits $\{0, 1, 2, \ldots, 9\}$. In the four parts of this question, you are asked for the number of strings of length 19 subject to various conditions.

a. Repetition of symbols is permitted: 36^{19}

b. Repetition of symbols is *not* permitted: $P(36,19)$ OR $36 \cdot 35 \cdot 34 \cdot \ldots \cdot 18$

c. The string uses exactly five A's, two B's, eight 3's and four 6's:

$\binom{19}{5}\binom{14}{2}\binom{12}{8}$ OR $\frac{19!}{5!\,2!\,8!\,4!}$ OR $\binom{19}{5,2,8,4}$

d. Exactly eight characters are digits, exactly five characters are B's and the remaining six characters are capital letters is (repetition is allowed):

$\binom{19}{8} 10^8 \binom{11}{5} 25^6$

16

4+4

2. How many integer valued solutions are there to the equation $y_1 + y_2 + y_3 < 80$ when:

a. $y_i > 0$ for $i = 1, 2, 3$: $\binom{79}{3}$

b. $y_i \geq 0$ for $i = 1, 2, 3$: $\binom{82}{3}$

c. $y_i > 0$ for $i = 1, 2$ and $y_3 > 9$: ______ $\binom{70}{3}$

d. $y_i > 0$ for $i = 1, 2$ and $0 < y_3 \leq 9$: ______ $\cancel{\binom{79}{3}-\binom{76}{3}}$ $\binom{79}{3}-\binom{70}{3}$

13

5,5,3

3 a. Use the Euclidean algorithm to find $d = \gcd(204, 1190)$.

$$204 \overline{)1190} \quad \text{quotient } 5$$
$$1020$$
$$170$$

$$170 \overline{)204} \quad \text{quotient } 1$$
$$170$$
$$\boxed{34}$$

$$34 \overline{)170} \quad \text{quotient } 5$$
$$170$$
$$0$$

$34 = g.c.d(204, 1190)$

b. Use your work in the first part of this problem to find integers a and b so that $d = 204a + 1190b$.

$170 = 1 \cdot 1190 - 5 \cdot 204$

$34 = 1 \cdot 204 - 1 \cdot 170$

$= 1 \cdot 204 - 1(1 \cdot 1190 - 5 \cdot 204)$

$= 6 \cdot 204 - 1 \cdot 1190$ so $a = 6$ $b = -1$

c. Use your answer to part (a) to factor 204 into primes. You will need this answer later in the test.

204 is divisible by 34 In fact $204 = 34 \cdot 6$

so $204 = 2 \cdot 17 \cdot 2 \cdot 3$

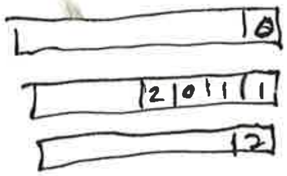
$= 2^2 \cdot 3 \cdot 17$

4. For a positive integer n, let t_n count the number of ternary sequences which do not contain 2011 as four consecutive characters. Develop a recurrence for t_n and use it to find t_6.

18

$t_1 = 3$ $t_2 = 9$ $t_3 = 27$ $t_4 = 80 = 81 - 1$

For $n \geq 5$



t_{n-1}

$t_{n-1} - t_{n-4}$

t_{n-1}

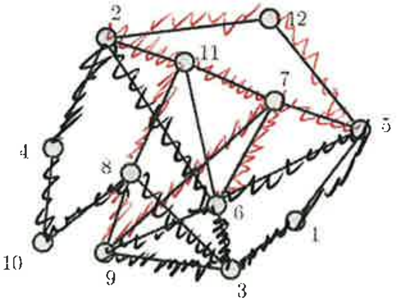
$t_n = 3t_{n-1} - t_{n-4}$

$t_5 = 3t_4 - t_1 = 3 \cdot 80 - 3 = 240 - 3 = 237$

$t_6 = 3t_5 - t_2 = 3 \cdot 237 - 9 = 711 - 9 = 702$

5. Use the algorithm developed in class (from node i, always take the edge ij not traversed previously, where j is minimum) to find an Euler circuit in the graph G shown below. Use node 1 as root.

18



(1, 3, 6, 2, 4, 10, 8, 3, 9, 6, 5, 1

$\uparrow$

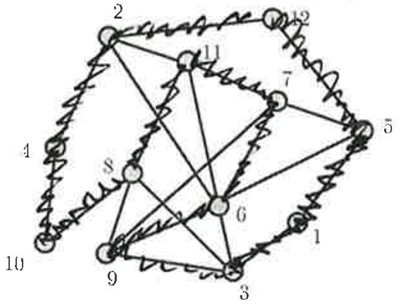
(6, 7, 5, 12, 2, 11, 7, 9, 8, 11, 6)

Insert to get:

(1, 3, 6, 7, 5, 12, 2, 11, 7, 9, 8, 11, 6, 2, 4, 10, 8, 3, 9, 6, 5, 1)

10

6. Show that the graph below is hamiltonian. You may give your answer by darkening appropriate edges on the figure, or by giving an appropriate permutation of the vertex set.



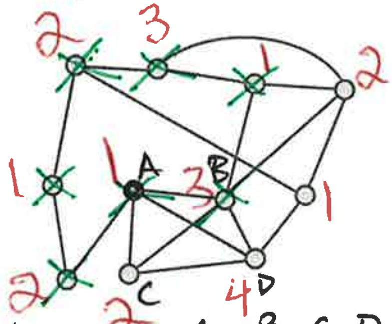
(1, 5, 12, 2, 4, 10, 8, 11, 7, 6, 9, 3)

other correct answers are possible

12

3×4

7. Consider the graph G shown below.



Note: There are many 4-colorings

a. For this graph, $\omega(G)$ is: 4 and A, B, C, D is a maximum clique

b. Show that $\chi(G) = \omega(G)$ by providing a proper coloring of G. You may indicate your coloring by writing directly on the figure.

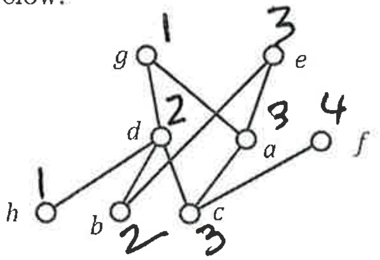
c. Explain why the graph G is *not* perfect.

The vertices marked in green form an induced C_7. Note that $\chi(C_7) = 3$ while $\omega(C_7) = 2$ so G is not perfect.

12

3×4

8. Consider the poset P shown below.



Many correct Dilworth partitions

a. The poset P is not an interval order. By inspection, find four points which determine a subposet isomorphic to $\mathbf{2} + \mathbf{2}$:

{a, d, e, h}

b. List a set of elements which forms a maximum antichain in P:

{a, b, f, h}

24 4

c. By inspection, find a Dilworth partition of the poset P. You may provide your answer by writing directly on the figure.

16

4+4

9. For the subset lattice $\mathbf{2}^9$,

a. The total number of elements is: 2^9

b. The total number of maximal chains is: $9!$

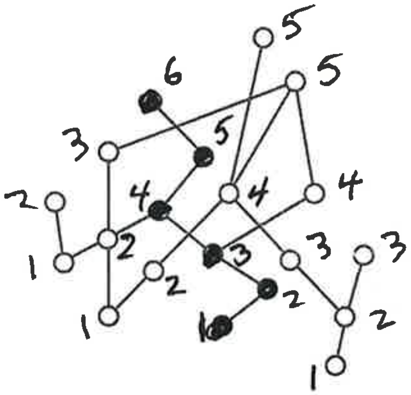
c. The number of maximal chains through $\{3, 5, 8\}$ is: $3!\,(9-3)! = 3!\,6!$

d. The width of $\mathbf{2}^9$ is: $\binom{9}{4}$ or $\binom{9}{5}$ or $C(9,4)$ or $C(9,5)$

12

10+2

10. Consider the poset P shown below.



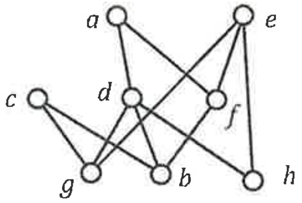
a. Determine a minimum partition of P into antichains by recursively stripping off the set of minimal elements. You may display your answer by writing directly on the diagram. Then darken a set of points that form a maximum chain.

b. The height h of the poset P is: 6

18

8+8+2

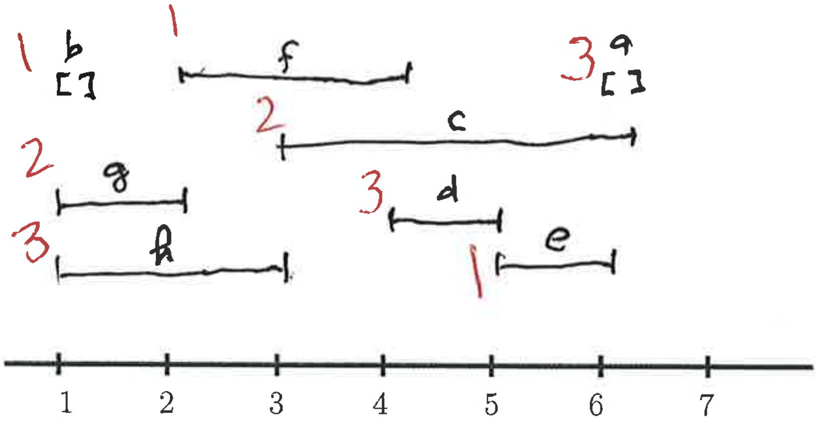
11. The poset P shown below is an interval order:



a. On the next page, find the down sets and the up sets. Then use these answers to find an interval representation of P that uses the least number of end points.

$D(a) =$	bdfgh	6	$U(a) =$	∅	6	$I(a) =$	[6, 6]
$D(b) =$	∅	1	$U(b) =$	acdef	1	$I(b) =$	[1, 1]
$D(c) =$	bg	3	$U(c) =$	∅	6	$I(c) =$	[3, 6]
$D(d) =$	bgh	4	$U(d) =$	a	5	$I(d) =$	[4, 5]
$D(e) =$	bfgh	5	$U(e) =$	∅	6	$I(e) =$	[5, 6]
$D(f) =$	b	2	$U(f) =$	ae	4	$I(f) =$	[2, 4]
$D(g) =$	∅	1	$U(g) =$	acde	2	$I(g) =$	[1, 2]
$D(h) =$	∅	1	$U(h) =$	ade	3	$I(h) =$	[1, 3]

b. In the space below, draw the representation you have found. Then use the First Fit Coloring Algorithm for interval graphs to solve the Dilworth Problem for this poset, i.e., find the width w and a partition of P into w chains. You may display your answers by writing the colors directly on the intervals in the diagram.



c. The following points form a maximum antichain in P:

{b,g,h} {f,a,h}, {f,c,h} etc. many correct answer

10

5+5

12 a. Write all the partitions of the integer 5.

5 = 5 O D

= 4+1 D

= 3+2 D

= 3+1+1 O

= 2+2+1

= 2+1+1+1

= 1+1+1+1+1 O

Note: 3 into odd parts
3 into distinct parts

b. Returning to your answer to part a, mark all the partitions of 5 into odd parts with a capital O. Also mark all partitions of 5 into distinct parts with a capital D.

13

6+7

22

13a. Find the general solution to the advancement operator equation:

$A^4(A-5+4i)^3(A-1)^2(A+8)(A-9)f = 0$

$f(n) = c_1(5-4i)^n + c_2 n(5-4i)^n + c_3 n^2(5-4i)^n$

$+ c_4 + c_5 n$

$+ c_6(-8)^n$

$+ c_7 9^n$

Note: The solution space is a vector space of dimension 7

b. Write the form of a particular solution of the non-homogeneous advancement operator equation (do not carry out the work necessary to evaluate any constants in your answer):
$A^4(A-5+4i)^3(A-1)^2(A+8)(A-9)f = 17 \cdot 3^n - 28 \cdot 2^n$.

$$f(n) = c\,3^n + d\,2^n$$

c. Find the solution to the advancement operator equation:
$(A^2 - 11A + 28)f(n) = 0, \quad f(0) = -2$ and $f(1) = 1$.

$$(A^2 - 11A + 28) = (A-4)(A-7)$$

general solution is $f(n) = c_1 4^n + c_2 7^n$

$n = 0 \qquad c_1 4^0 + c_2 7^0 = -2 = c_1 + c_2$

$n = 1 \qquad c_1 4^1 + c_2 7^1 = 1 = 4c_1 + 7c_2$

$-8 = 4c_1 + 4c_2$

$9 = 3c_2 \Rightarrow c_2 = 3$

$-2 = c_1 + 3 \Rightarrow c_1 = -5$

$$f(n) = -5 \cdot 4^n + 3 \cdot 7^n$$

20

5x4

14a. Write the inclusion/exclusion formula for the number $S(n,m)$ of onto functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$.

$$S(n,m) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (m-i)^n$$

b. Evaluate your answer to part a when $n = 5$ and $m = 3$ (reduce your answer so that only arithmetic is required to obtain the final answer).

$$S(5,3) = 3^5 - \binom{3}{1} 2^5 + \binom{3}{2} 1^5 - \binom{3}{3} 0^5$$

$$= 243 - 3 \cdot 32 + 3 \cdot 1$$

c. Write the inclusion/exclusion formula for the number d_n of derangements on $\{1, 2, \ldots, n\}$.

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

d ~~c~~. Evaluate your formula for d_n when $n = 7$ (reduce your answer so that only arithmetic is required to get the final answer).

$$d_7 = 7! - \binom{7}{1} 6! + \binom{7}{2} 5! - \binom{7}{3} 4! + \binom{7}{4} 3! - \binom{7}{5} 2! + \binom{7}{6} 1! - \binom{7}{7} 0!$$

$$= 5040 - 5040 + 21 \cdot 120 - 35 \cdot 24 + 35 \cdot 6 - 21 \cdot 2 + 7 \cdot 1 - 1 \cdot 1$$

e ~~d~~. Use your answer to Problem 3(c) to find the value of the Euler ϕ-function $\phi(n)$ when $n = 204$ (reduce your answer so that only arithmetic is required to get the final answer.

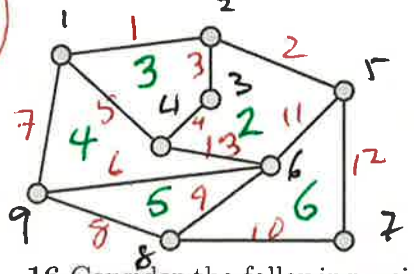
$$204 = 2^2 \cdot 3 \cdot 17 \Rightarrow$$

$$\phi(204) = 204\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\left(1 - \tfrac{1}{17}\right)$$

$$= 2^2 \cdot \cancel{3} \cdot \cancel{17} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{16}{\cancel{17}} = 4 \cdot 16 = 64$$

8

15 In the space to the right, verify Euler's formula for the following planar graph:



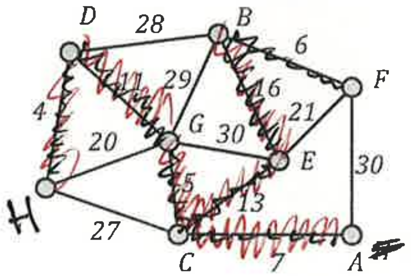
n - e + f = 2

9 - 13 + 6 = 15 - 13 = 2 ✓

16

8+8

16 Consider the following weighted graph:



In the space below, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex A as the root.

Kruskal's Algorithm

DH	4
CG	5
BF	6
AC	7
DG	11
CE	13
BE	16

Prim's Algorithm

AC	7
CG	5
DG	11
DH	4
CE	13
BE	16
BF	6

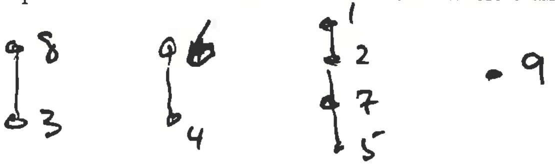
10

7+3

17. Consider a poset P whose ground set is $X = \{1, 2, 3, \ldots, 9\}$. Network flows (and the special case of bipartite matchings) are used to find the width w of P and a minimum chain partition. When the labelling algorithm halts, the following edges are matched:

$$3'8'' \quad 4'6'' \quad 5'7'' \quad 2'1'' \quad 7'2''$$

a. Find the chain partition of P that is associated with this matching.



b. Find the width w of the poset P.

w = 4

20

18. A data file `digraph_data.txt` has been read for a digraph whose vertex set is [7]. The weights on the directed edges are shown in the matrix below. The entry $w(i,j)$ denotes the length of the edge from i to j. If there is no entry, then the edge is not present in the graph. Apply Dijkstra's algorithm to find the distance $d(x)$ from vertex 1 to vertex x for all vertices x in G. Also, for each x, find a shortest path $P(x)$ from 1 to x.

W	1	2	3	4	5	6	7
1	0	20	17	28	23	44	48
2		0	4	6	3	8	16
3		9	0		41	12	20
4	27	20	28	0		2	10
5	9	1	2	2	0	4	9
6	82	5	3	2	2	0	7
7		8	22	4	18	12	0

$\boxed{P(1) = (1) \quad d_1 = 0}$

$P(2) = (1,2) \quad d_2 = 20$

$P(3) = (1,3) \quad d_3 = 17$

$P(4) = (1,4) \quad d_4 = 28$

$P(5) = (1,5) \quad d_5 = 23$

$P(6) = (1,6) \quad d_6 = 44$

$P(7) = (1,7) \quad d_7 = 48$

shortest temp path is P(3). Mark permanent and scan

$\boxed{P(3) = (1,3) \quad d_3 = 17}$

updates

$P(6) = (1,3,6) \quad d_6 = 17+12 = 29$

$P(7) = (1,3,7) \quad d_7 = 17+20 = 37$

shortest temp path is P(2). Mark Permanent and scan

$\boxed{P(2) = (1,2) \quad d_2 = 20}$

updates

$P(4) = (1,2,4) \quad d_4 = 20+6 = 26$

$P(6) = (1,2,6) \quad d_6 = 20+8 = 28$

$P(7) = (1,2,7) \quad d_7 = 20+16 = 36$

$\boxed{P(5) = (1,5) \quad d_5 = 23}$

updates

$P(4) = (1,5,4) \quad d_4 = 23+2 = 25$

$P(6) = (1,5,6) \quad d_6 = 23+4 = 27$

$P(7) = (1,5,7) \quad d_7 = 23+9 = 32$

$\boxed{P(4) = (1,5,4) \quad d_4 = 25}$

No updates

$\boxed{P(6) = (1,5,6) \quad d_6 = 27}$

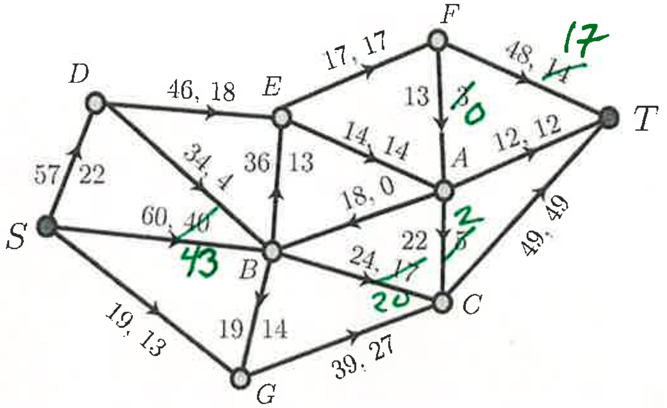
No update

$\boxed{P(7) = (1,5,7) \quad d_7 = 32}$

22

3+3+5+3+5+3

19. Consider the following network flow:



a. The value of the current flow is: 22+40 +13 = 75

b. The capacity of the cut $\{S, B, D, G\} \cup \{A, C, E, F, T\}$ is: 46+36 + 24 + 39

Note: Do NOT INCLUDE EDGE AB

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

S (*, +, ∞)
B (S, +, 20)
D (S, +, 35)
G (S, +, 6)
C (B, +, 7)
E (B, +, 20)
A (C, −, 5)
F (A, −, 3)
T (F, +, 3)

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram. BACKTRACKING: S, B, C, A, F, T

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled.

S (*, +, ∞)
B (S, +, 17)
D (S, +, 35)
G (S, +, 6)
C (B, +, 4)
E (B, +, 17)
A (C, −, 2)

f. Find a cut whose capacity is equal to the value of the updated flow.

$\mathcal{L} = \{S, A, B, C, D, E, G\}$ $\mathcal{U} = \{F, T\}$

capacity = 17+12+49 = 78 = 75+3 ✓

20

20+1

20. **True–False.** Mark in the left margin.

$2^{40} = (2^{10})^4 > (1024)^4 > (10^3)^4 = 10^{12}$

T 1. $2^{40} > 100,000,000$.

T 2. The coefficient of $x^6y^8z^2$ in $(2x - 4y^3 + 8z)^{29}$ is 0.

F 3. There is a planar graph G on 328 vertices with $\chi(G) = 39$.

F 4. If G is a planar graph, then $\chi(G) = \omega(G)$.

F 5. All graphs with 1024 vertices and 2973 edges are planar.

T 6. Every graph on 684 vertices in which every vertex has degree 402 is hamiltonian.

T 7. Every connected graph on 986 vertices in which every vertex has degree 42 has an Euler circuit.

T 8. A cycle on 739 vertices is a homeomorph of the complete bipartite graph $K_{2,2}$.

T 9. When $n \geq 3$, the shift graph S_n is a triangle-free graph with $\binom{n}{2}$ vertices and $\binom{n}{3}$ edges. Furthermore, $\chi(S_n) = \lceil \lg n \rceil$.

T 10. The number of lattice paths from $(0, 0)$ to (n, n) which do not pass through a point above the diagonal is the Catalan number $\binom{2n}{n}/(n + 1)$.

T 11. Any modern computer can accept a file of 1,000 positive integers, each at most 5,000, and quickly determine whether 2,742 is the sum of three integers in the file.

T 12. Any modern computer can accept a file of 1,000 positive integers, each at most 5,000, and quickly determine whether 385,742 is the product of two integers in the file.

T 13. Any modern computer can accept a file of 1,000 positive integers, each at most 5,000, and quickly factor each of the numbers into primes.

F 14. There is a graph on 837 vertices in which no two vertices have the same degree.

F 15. There is a poset with 623 points having width 29 and height 19.

F 16. There is a sequence of 523 distinct positive integers which does not have an increasing subsequence of size 38 nor a decreasing subsequence of size 27.

F 17. The permutation $(7, 1, 3, 8, 5, 2, 4, 6)$ is a derangement.

F 18. Linear programming problems with integer coefficient constraints always have integer valued solutions.

F 19. Every linear programming problem is also a network flow problem.

F 20. The Ramsey number $R(3, 3)$ is 18.