KEY

Student Name and ID Number

MATH 3012 Quiz 1, February 8, 2007, WTT

15

5x3

1. Let X be the 36 element set consisting of the 10 digits in $\{0, 1, 2, \ldots, 9\}$ and the 26 lower case letters in the English alphabet $\{a, b, c, \ldots, z\}$. Then consider strings of length 20 whose symbols come from the set X.

a. How many strings of length 20 can be formed if repetition of symbols is *not* permitted?

$$P(36, 20) = 36 \cdot 35 \cdot 34 \cdot \ldots \cdot 17$$

b. How many strings of length 20 can be formed if repetition of symbols is permitted?

$$36^{20}$$

c. How many strings of length 20 can be formed using exactly three 0's, six 3's, four x's, two y's and five z's?

$$\binom{20}{3,6,4,2,5} = \frac{20!}{3!\,6!\,4!\,2!\,5!}$$

d. Let $G = \{g, e, n, i, u, s\}$. How many strings of length 20 are possible if repetition is allowed and *exactly* 8 of the 20 symbols in the string come from G.

$$\binom{20}{8} 6^8\, 30^{12}$$

e. How many strings of length 20 are possible if repetition is *not* allowed and *exactly* 3 of the 20 symbols in the string come from the set G defined in part d?

$$\binom{20}{3} P(6,3)\, P(30,17)$$

5

2. How many lattice paths from $(0, 0)$ to $(23, 17)$ pass through $(8, 6)$?

$23-8=15$
$17-6=11$

$$\binom{14}{8}\binom{26}{15}$$

15

5x3

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i > 0$.

$$\binom{62}{4}$$

62 gaps, choose 4

b. $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$.

$$\binom{67}{4}$$

Add 1 artificial apple for each

c. $x_1 + x_2 + x_3 + x_4 + x_5 \leq 63$, all $x_i \geq 0$. First add $x_6 \geq 0$ and make equation. Then its like part b

$$\binom{68}{5}$$

d. $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \geq 10$.

Same as $x_1 + x_2 + x_3 + x_4 + x_5 = 53$
all $x_i \geq 0$

$$\binom{57}{4}$$

e. $x_1 + x_2 + x_3 + x_4 + x_5 = 63$, all $x_i \geq 0$, $x_2 \leq 9$.

(b) - (d)

$$\binom{67}{4} - \binom{57}{4}$$

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4. Define a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ by $f(1) = 7$ and $f(k+1) = f(k) + 2k + 3$. Use math induction to prove that $f(n) = n^2 + 2n + 4$, for all $n \geq 1$.

When $n = 1$ $f(1) = 7$ and $1^2 + 2 \cdot 1 + 4 = 7$ so formula holds.

Now let $k \geq 1$ and assume $f(k) = k^2 + 2k + 4$.

Then
$$\begin{aligned} f(k+1) &= f(k) + 2k + 3 \\ &= (k^2 + 2k + 4) + 2k + 3 \\ &= k^2 + 4k + 7 \\ &= (k^2 + 2k + 1) + 2(k+1) + 4 \end{aligned}$$

$\therefore$ the formula holds for all $n \geq 1$.

5. Use the Euclidean algorithm to find $d = \gcd(693, 45)$.

8

$$45 \overline{)693} = 15, \quad 693 - 45 \cdot 15: \; 693,\ 45,\ 243,\ 225,\ 18$$

$$18 \overline{)45} = 2, \quad 45 - 36 = 9$$

$$9 \overline{)18} = 2, \quad 18 - 18 = 0$$

so $\boxed{g.c.d.(693, 45) = 9}$

$$\left.\begin{aligned} 693 &= 15 \cdot 45 + 18 \\ 45 &= 2 \cdot 18 + 9 \end{aligned}\right\} \quad \begin{aligned} 9 &= 1 \cdot 45 - 2 \cdot 18 \\ &= 1 \cdot 45 - 2(693 - 15 \cdot 45) \\ &= -2 \cdot 693 + 31 \cdot 45 \end{aligned}$$

6. Use your work in the preceding problem to find integers a and b so that $d = 693a + 45b$.

8

$\boxed{a = -2 \qquad b = 31}$

Note: $-2 \cdot 693 + 31 \cdot 45 = -1386 + 1395 = 9$

8

7. For a positive integer n, let t_n count the number of ways to tile a $3 \times n$ checkerboard with rectanges of size 1×3 and 3×1. It is easy to see that $t_1 = t_2 = 1$ while $t_3 = 2$. Find a recurrence equation satisfied by t_n and use it to calculate t_7.

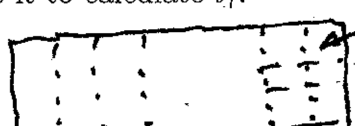
5+3

1

$t_1 = 1$

1 2

$t_2 = 1$

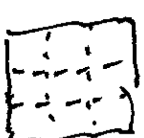


1 2 3 . . . n

either horizontal or vertical

$\boxed{t_n = t_{n-1} + t_{n-3}}$

(If horizontal, so are two others underneath)



1 2 3 $t_3 = 2$

$n \geq 4$

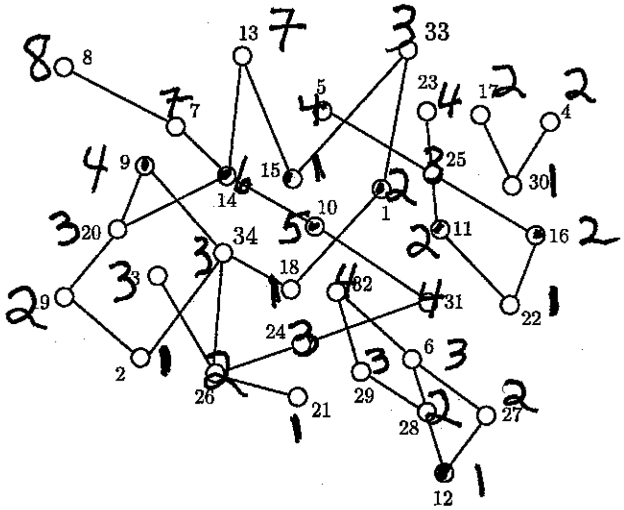
$$\begin{aligned} t_4 &= t_3 + t_1 = 2 + 1 = 3 \\ t_5 &= t_4 + t_2 = 3 + 1 = 4 \\ t_6 &= t_5 + t_3 = 4 + 2 = 6 \end{aligned}$$

$$\boxed{\begin{aligned} t_7 &= t_6 + t_4 \\ &= 6 + 3 = 9 \end{aligned}}$$

8. Consider the partially ordered set (poset) P shown below:

18

9 × 2



a. List the set of minimal elements of P.

MIN(P) = {15, 30, 18, 22, 2, 21, 12}

b. How many elements from $\{1, 2, \ldots, 33\} \cancel{P}$ are comparable with the point labeled 34?

c. How many elements of $\{2, 3, \ldots, 34\}$ are incomparable with the point labeled 1?

5 namely {9, 2, 18, 26, 21}

d. How many components does P have?

4

e. List a set of 5 elements that forms a maximal chain.

{13, 14, 20, 19, 2}

f. Explain why $\{1, 9, 10, 11, 12, 14, 15, 16, 30\}$ is not a maximal antichain.

Because you can add elements to it and still have an antichain.
For example, you can add 3 and 30 or 3, 4, 17

g. Writing directly on the figure, carry out the algorithm we have learned in class for partitioning P into the least possible number of antichains. Start by labeling with integer 1 the minimal elements of P. Thereafter, label with integer $k + 1$ the minimal elements of the subposet remaining when the points labeled 1, 2, ..., k have ben removed.

h. Find the height h of P.

$h = 8$

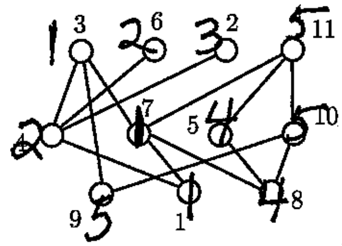
i. List a set of h elements that forms a chain.

{8, 7, 14, 10, 31, 24, 26, 21}

9

3×3

9. Consider the partially ordered set (poset) Q shown below:



Lots of ways to do this.

a. Find the width w of this poset.

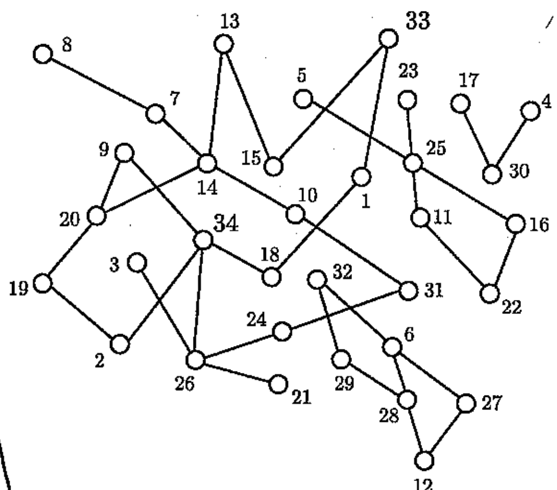
$W = 5$

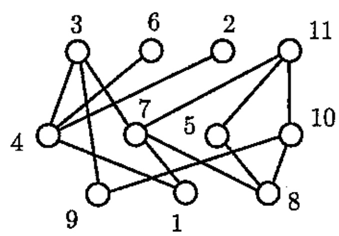
b. List a set of points that form an antichain of size w in Q.

$\{3, 6, 2, 5, 10\}$

c. Determine a partition of P into w chains by labeling directly on the figure each point with an integer from $\{1, 2, \ldots, w\}$ so that for each i, all points labeled i form a chain.

Duplicate Figures. Please note if these are being used.





SCORING

1. $5 \times 3 = 15$
2. $5 = 5$
3. $5 \times 3 = 15$
4. $10 = 10$
5. $8 = 8$
6. $8 = 8$
7. $5 + 3 = 8$
8. $9 \times 2 = 18$
9. $3 \times 3 = 9$

Total 96

Blank test counts as +4