KEY

Student Name and ID Number

MATH 3012 Quiz 1, September 24, 2010, WTT

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4×3

1. Consider the 10-element set consisting of the four digits $\{0, 1, 2, 3\}$ and the six capital letters $\{A, B, C, D, E, F\}$.

a. How many strings of length 7 can be formed if repetition of symbols is *not* permitted?

$P(10,7)$ or $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

b. How many strings of length 7 can be formed if repetition of symbols is permitted?

10^7

c. How many strings of length 7 can be formed using exactly two 0's, three A's and two C's?

$\binom{7}{2,3,2}$ or $\frac{7!}{2!\,3!\,2!}$

d. How many strings of length 7 can be formed if exactly three characters are letters and exactly two of the remaining four characters are 3's?

$\binom{7}{3} 6^3 \binom{4}{2} 3^2$

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2. How many lattice paths from $(3, 4)$ to $(17, 19)$ pass through $(8, 10)$?

$\binom{11}{5}\binom{18}{9}$

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6×2

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 = 27$, all $x_i > 0$.

$\binom{26}{2}$

b. $x_1 + x_2 + x_3 = 27$, all $x_i \geq 0$.

$\binom{29}{2}$

c. $x_1 + x_2 + x_3 < 27$, all $x_i > 0$.

$\binom{26}{3}$

d. $x_1 + x_2 + x_3 \leq 27$, all $x_i \geq 0$.

$\binom{30}{3}$

e. $x_1 + x_2 + x_3 = 27$, all $x_i > 0$, $x_2 \geq 5$.

$\binom{22}{2}$

f. $x_1 + x_2 + x_3 = 27$, all $x_i > 0$, $x_2 \leq 4$.

$\binom{26}{2} - \binom{22}{2}$

4. Use the Euclidean algorithm to find $d = \gcd(630, 495)$.

$630 = 1 \cdot 495 + 135$
$495 = 3 \cdot 135 + 90$
$135 = 1 \cdot 90 + 45$
$90 = 2 \cdot 45 + 0$

$\boxed{45 = \gcd(630, 495)}$

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5. Use your work in the preceding problem to find integers a and b so that $d = 630a + 495b$.

$135 = 1 \cdot 630 - 1 \cdot 495$
$90 = 1 \cdot 495 - 3 \cdot 135$
$45 = 1 \cdot 135 - 1 \cdot 90$

$45 = 1 \cdot 135 - 1(1 \cdot 495 - 3 \cdot 135)$
$= -1 \cdot 495 + 4 \cdot 135$
$= -1 \cdot 495 + 4(1 \cdot 630 - \cancel{3}1 \cdot 495)$
$= 4 \cdot 630 - 5 \cdot 495$

$\boxed{a = 4 \qquad b = -5}$

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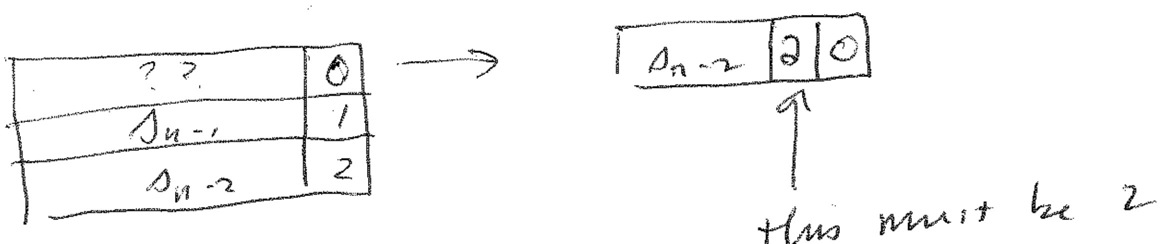
6. For a positive integer n, let s_n count the number of ternary strings of length n that do not contain 00 or 10 as a substring. Note that $s_1 = 3$ and $s_2 = 7$. Develop a recurrence relation for s_n and use it to compute s_3, s_4 and s_5.



split according to last character

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$s_n = 2s_{n-1} + s_{n-2}$
$s_3 = 2 \cdot 7 + 3 = 17$
$s_4 = 2 \cdot 17 + 7 = 41$
$s_5 = 2 \cdot 41 + 17 = 99$

7. Use the algorithm developed in class to find an Euler circuit in the following graph:

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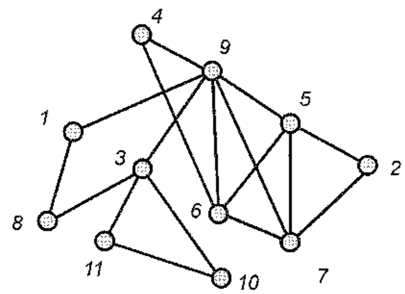

(1)
(1,8,3,9,1)
(3,10,11,3)
1, 8, 3, 10, 11, 3, 9, 1)
9, 4, 6, 5, 2, 7, 5, 9, 6, 7, 9
(1, 8, 3, 10, 11, 3, 9, 4, 6, 5, 2, 7, 5, 9, 6, 7, 9, 1)

8. Consider the following graph:

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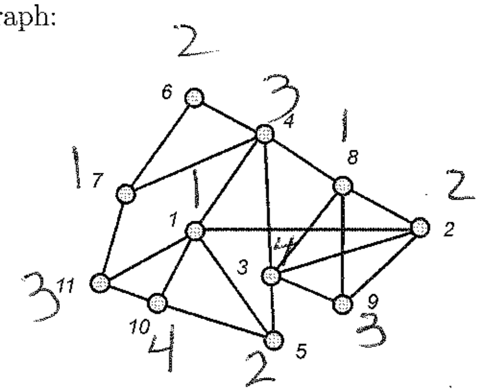
6×2



a. Explain why this graph does not have an Euler circuit. it has vertices of odd degree (such as 7, 1, 3, 9,)

b. Provide a listing of the vertices that constitutes a Hamiltonian cycle *starting* with vertex 1 followed by vertex 2. (1, 2, 9, 8, 3, 4, 6, 7, 11, 10, 5, 1)

c. Find a set of vertices that forms a maximal clique but not a maximum clique.
{6, 4, 7}, {7, 11}, {1, 10, 11}, etc

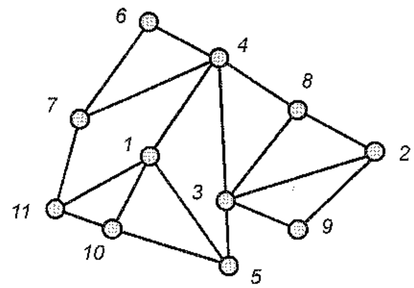
d. What is $\omega(G)$ for this graph?
$\omega(G) = 4$

e. Find a set of vertices which forms a maximum clique in this graph.
{2, 3, 8, 9}

f. Show that $\chi(G) = \omega(G)$ for this graph by providing an optimum coloring. You may write directly on the figure.

9. Verify Euler's formula for the following graph:

(10)



$V - E + F = 2$

$V = 11$

$E = 18$

$F = 9$

$11 - 18 + 9 = 2$ ✓

10. Prove the following identity by Mathematical Induction:

$$4 + 11 + 18 + \ldots 7n - 3 = \frac{n(7n+1)}{2}$$

(10)

When $n = 1$ LHS $= 4$ RHS $= \frac{1 \cdot 8}{2} = 4$ ✓

Assume

$4 + 11 + 18 + \ldots + 7k - 3 = \frac{k(7k+1)}{2}$ for some $k \geq 1$

Then

$4 + 11 + 18 + \ldots + 7k - 3 + 7(k+1) - 3 =$

$$\frac{k(7k+1)}{2} + 7(k+1) - 3$$

$$= \frac{7k^2 + k + 14k + 8}{2}$$

$$= \frac{7k^2 + 15k + 8}{2}$$

$$= \frac{(k+1)(7k+8)}{2}$$

$$= \frac{(k+1)[7(k+1) + 1]}{2}$$ ✓