Complete Solutions for MATH 3012 Quiz 1, September 27, 2011, WTT

Note. The answers given here are more complete than is expected on an actual exam. It is intended that the more comprehensive solutions presented here will be valuable to students in studying for the final exam. In a few places, the wording of a problem is changed slightly to reflect the modifed layout. A table providing point values for the problems is given at the very end.

1. Consider the 16-element set consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the six capital letters $\{A, B, C, D, E, F\}$.

a. How many strings of length 9 can be formed if repetition of symbols is *not* permitted?

This is just a permutation problem. Answer: $P(16, 9)$.

b. How many strings of length 9 can be formed if repetition of symbols is permitted?

Strings from an alphabet of fixed size. Answer: 16^9.

c. How many strings of length 9 can be formed using exactly two 6's, three B's and four D's?

This is a "Mississippi" problem. Answer: $\binom{9}{2,3,4}$.

d. How many strings of length 9 can be formed using exactly two 6's, three B's and four D's if the four $D's$ are required to occur consecutively in the string?

We consider the block of four $D's$ as a single letter. The resulting string has only $2+3+1=6$ letters. Answer: $\binom{6}{2,3,1}$.

2. How many lattice paths from $(2, 8)$ to $(27, 39)$ do *not* pass through $(18, 23)$?

The basic fact we need to remember here is that the number of lattice paths from $(0, 0)$ to (m, n) is $\binom{m+n}{m}$, assuming of course that $m, n \geq 0$. More generally, when $a \leq c$ and $b \leq d$, the number of lattice paths from (a, b) to (c, d) is $\binom{(c-a)+(d-b)}{c-a}$. In this problem, the number of lattice paths from $(2, 8)$ to $(27, 39)$ is $\binom{25+31}{25} = \binom{56}{25}$. Of these, $\binom{16+15}{16}$ pass through $(18, 23)$. Answer: $\binom{56}{25} - \binom{31}{15}$. Note: Be careful as the answer can be rewritten in four different ways using the identity $\binom{a}{b} = \binom{a}{a-b}$.

3. How many integer valued solutions to the following equations and inequalities:

a. $x_1 + x_2 + x_3 + x_4 = 32$, all $x_i > 0$.

This is the starting point for all problems of this type. We are counting the number of distributions of m non-distinct objects into n distinct cells, with the additional requirement that each cell must receive at least one object (all cells are non-empty). This requires $m \geq n$. We consider the m objects placed on a line—so that there are $m - 1$ "gaps". We then choose $n - 1$ of these gaps, a process which separates the objects into n non-empty groups, which appear in left to right order on the line. So the number of distributions is $\binom{m-1}{n-1}$.

In this problem, $m = 32$ and $n = 4$, so the answer is: $\binom{31}{3}$.

b. $x_1 + x_2 + x_3 + x_4 = 32$, all $x_i \geq 0$.

Suppose we are talking about distributing 32 apples, which we assume are non-distinct, and we are distributing them to four variables, Alice, Bob, Carlos and Dave. Alice receives x_1 apples, Bob gets x_2, etc. We are assuming here that we can tell the difference between the four individuals, which in the problem means that x_1, x_2, x_3 and x_4 are distinct variables.

We add to our collection four "artificial" apples resulting in a total of $36 = 32 + 4$ apples. We then distribute these 36 apples to the four individuals. After, this distribution is made, we "tax" each of them and make them give back one apple. Now it may happen that any one (and perhaps as many as three of the four) individuals are left with none. Regardless, it is clear that we have counted the number of distributions with empty cells allowed. In general the answer would be $\binom{m+n-1}{n-1}$. In this specific problem, the answer is $\binom{35}{3}$.

c. $x_1 + x_2 + x_3 + x_4 < 32$, all $x_i > 0$.

Add a new "slack" variable x_5 which is positive. Then the problem becomes one just like part (a). Answer: $\binom{31}{4}$.

d. $x_1 + x_2 + x_3 + x_4 \leq 32$, all $x_i \geq 0$.

Now the slack variable x_5 is non-negative. So this problem becomes just like part (b). Answer: $\binom{36}{4}$.

e. $x_1 + x_2 + x_3 + x_4 = 32$, all $x_i > 0$, $x_2 \geq 8$.

Here the principle is to "set aside" some apples before the distribution is made. In this case, we give seven apples to Bob (corresponding to the variable x_2) in advance. There are then $25 = 32 - 7$ apples remaining. We distribute these 25 apples with all four of Alice, Bob, Carlos and Dave getting at least one. Now $x_i > 0$ for all i and $x_2 \geq 8$. So the answer is $\binom{24}{3}$.

f. $x_1 + x_2 + x_3 + x_4 = 32$, all $x_i > 0$, $x_2 \leq 13$.

We work two related problems and determine the answer for our problem as the difference. The first problem is the number of distributions with all $x_i > 0$. This is part (a) and the answer if $\binom{31}{3}$. The second problem is the number of distributions with all $x_i > 0$ and $x_2 \geq 14$. This is part (e), with different parameters, so the answer is $\binom{18}{3}$. Clearly, the answer to our problem is then $\binom{31}{3} - \binom{18}{3}$.

4. Use the Euclidean algorithm to find $d = \gcd(630, 495)$.

We carry out a series of long divisions, stopping when the remainder is zero:

$$\begin{aligned}
630 &= 1 \cdot 495 + 135 \\
495 &= 3 \cdot 135 + 90 \\
135 &= 1 \cdot 90 + 45 \\
90 &= 2 \cdot 45 + 0
\end{aligned}$$

Accordingly, $\gcd(630, 495) = 45$.

5. Use your work in the preceding problem to find integers a and b so that $d = 630a + 495b$.

Start by rewriting the results of the long division done previously (all but the last one):

$$\begin{aligned}
135 &= 1 \cdot 630 - 1 \cdot 495 \\
90 &= 1 \cdot 495 - 3 \cdot 135 \\
45 &= 1 \cdot 135 - 1 \cdot 90
\end{aligned}$$

Now substitute, starting at the bottom:

$$\begin{aligned} 45 &= 1 \cdot 135 - 1 \cdot 90 \\ &= 1 \cdot 135 - 1 \cdot (1 \cdot 495 - 3 \cdot 135) \\ &= 4 \cdot 135 - 1 \cdot 495 \\ &= 4(1 \cdot 630 - 1 \cdot 495) - 1 \cdot 495 \\ &= 4 \cdot 630 - 5 \cdot 495 \end{aligned}$$

So a correct answer is $a = 4$ and $b = -5$. Note: It is easy to see that there are infinitely many correct answers. For every integer n, you may verify that $a = 4 - 495n$ and $b = -5 + 630n$ works.

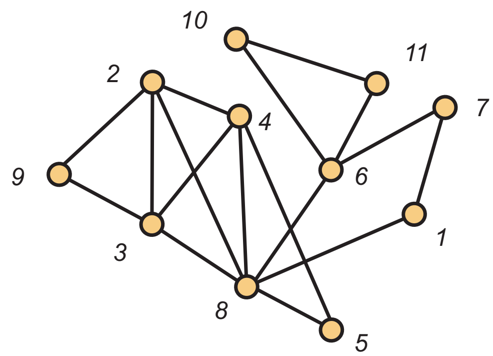
6. For a positive integer n, let s_n count the number of ternary strings of length n that do not contain 00 or 01 as a substring. Note that $s_1 = 3$ and $s_2 = 7$. Develop a recurrence relation for s_n and use it to compute s_3, s_4 and s_5.

Let $n \geq 3$. We develop a recurrence relation satisfied by the sequence s_n. Look at the good strings of length n and divide them into G_0, G_1 and G_2, where G_i is the set of all good strings that start with character i in the first position. We claim that $|G_1| = |G_2| = s_{n-1}$. This follows from the fact that if the first character is a 1 or a 2, then the last $n-1$ characters form a good string of length $n-1$. Furthermore, we can take any good string of length $n-1$ and prepend a 1 or a 2 at the start and obtain a good string of length n.
On the other hand, if the first character is a 0, then the second character *must* be a 2. Afterwards, in the last $n-2$ positions, we have a good string of length $n-2$ and all of these may occur. It follows that $|G_0| = s_{n-2}$. So the recurrence equation becomes $s_n = 2s_{n-1} + s_{n-2}$.
Using this recurrence, we make the following computations:

$$\begin{aligned} s_3 &= 2 \cdot 7 + 3 = 17 \\ s_4 &= 2 \cdot 17 + 7 = 41 \\ s_5 &= 2 \cdot 41 + 17 = 99 \end{aligned}$$

There is a fascinating subtlety to this problem. If you work it “left to right”, as we have just described, it is quite easy. But you could have grouped the good strings according to their last character, i.e., taking a “right to left” approach. If you try this, you will find that it more challenging to find a simple recurrence. It can be done and I showed you how in class, but regardless, it is more complicated than the solution presented above.
To understand this comment fully, just work the problem of counting all ternary strings that don’t have a 00 or a 10, and try to take the same “left to right” approach illustrated above.

7. Use the algorithm developed in class, with vertex 1 as root, to find an Euler circuit in the following graph:



The rule is to always take the first available neighbor joined by an edge on which we have not yet walked. So the initial string becomes:

$$(1, 7, 6, 8, 1)$$

Now when we reach the last 1 in this sequence, there are no available edges. So starting at the first 1, we scan left to right looking for a vertex which has edges incident to it that we have not yet visited. The first such vertex is 6, so we start with this vertex and begin again. Now the sequence becomes:

$$6, 10, 11, 6)$$

This string is inserted into the first, yielding

$$(1, 7, 6, 10, 11, 6, 8, 1)$$

We repeat the process, scanning left to right for the first vertex incident with an edge we have not yet visited. Now it is vertex 8. Starting from 8, we get:

$$8, 2, 3, 4, 2, 9, 3, 8, 4, 5, 8)$$

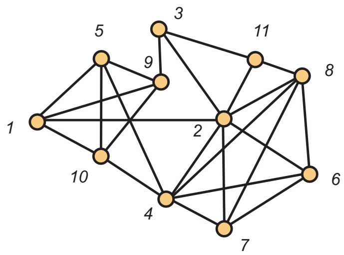
Inserting, we get:

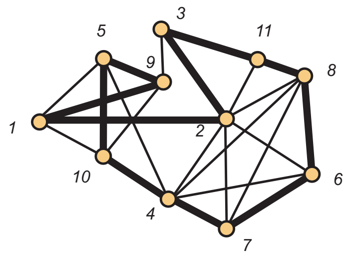
$$(1, 7, 6, 10, 11, 6, 8, 2, 3, 4, 2, 9, 3, 8, 4, 5, 8, 1)$$

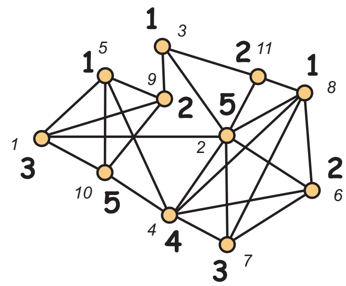
When we scan left to right, we see that no vertex is incident with any edge we haven't already visited. Furthermore, when we check against the list of all edges, we see that the current circuit includes all edges of the graph. So we have an Euler circuit.

Note that this last check, insuring that we have visited all edges, is necessary since the graph might be disconnected. Visually, it is clear that the graph in this problem is connected and this is not an issue. But if we are thinking about how this might be programmed and used on a graph presented as a large data file, the connectivity check is essential.

8. Consider the graph on the left:







a. Explain why the graph shown does not have an Euler circuit.

A graph has an Euler circuit if and only if it is connected and every vertex has even degree. This graph is connected, but there are four vertices that have odd degree: vertices 2, 3, 8 and 11. Note that in any graph, the number of vertices having odd degree is always even.

b. Provide a listing of the vertices that constitutes a Hamiltonian cycle *starting* with vertices 1, 2 and 3 in that order.

Here is a listing of the vertices that forms a hamiltonian cycle: $(1, 2, 3, 11, 8, 6, 7, 4, 10, 5, 9)$. All vertices appear exactly once in this list. Each consecutive pair forms an edge in the graph and the last vertex is adjacent to the first. As specified, the list begings with $(1, 2, 3)$. For visual clarity, we illustrate this cycle with the middle figure above. Note that there are several correct answers to this problem. All of them start out as $(1, 2, 3, 11, 8)$. The next two vertices are 6 and 7 but they can be visited in either order. Then you must visit vertex 4. After that you can do $(10, 5, 9)$, $10, 9, 5)$, $(5, 9, 10)$ or $(5, 10, 9)$.

It is important to remember that we do not have an effective algorithm for finding hamiltonian cycles. Here, we just proceeded by inspection.

c. Find a set of vertices that forms a maximal clique but not a maximum clique.

Recall that a set S of vertices forms a *clique* when any distinct pair of vertices forms an edge, and a clique is *maximal* when there is no vertex x which is not in S for which $S \cup \{x\}$ is also a clique. A clique is *maximum* when there is no clique which has more vertices.

There are many, many correct answers here. One is $\{2, 3, 11\}$. Note that these three vertices form a triangle (a clique of size 3 is typically called a triangle), but there is no other vertex adjacent to all three of them. This clique is not maximum, since $\{1, 5, 9, 10\}$ is also a clique and it has size 4.

d. What is $\omega(G)$ for this graph?

The notation $\omega(G)$ denotes the maximum clique size of the graph G. Here, we note that $\omega(G) = 5$ (see answer to next question).

e. Find a set of vertices which forms a maximum clique in this graph.

The set $\{2, 4, 6, 7, 8\}$ forms a clique of size 5. By inspection, there are no cliques of size 6. Note that we have no effective algorithm for finding the maximum clique size $\omega(G)$ for a graph G. If G has n vertices, we can dertermine whether there is a clique of size 5 by examining each of the $\binom{n}{5}$ subsets to see if they form a clique. If $n = 1,000$, this means examing approximately 10^{15} sets, which is barely doable. But this technique is not likely to be of much use when $n = 1,000,000$ and certainly won't work when $n = 1,000,000,000$. On the other hand, determing whether G has a clique of size $\lfloor n/2 \rfloor$ is already beyond reach when $n = 1,000$.

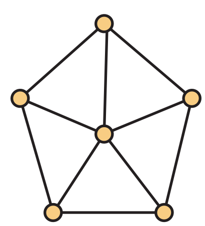
f. Show that $\chi(G) = \omega(G)$ for this graph by providing an optimum coloring. You may write directly on the figure.

The notation $\chi(G)$ is used to denote the *chromatic number* of G, i.e., the fewest number of colors required so that we can assign a color to each vertex so that adjacent vertices never receive the same color. For any graph G, we always have the trivial inequality $\chi(G) = \omega(G)$. In class, we showed that for every $t \geq 3$, there is a graph G_t for which $\omega(G) = 2$ and $\chi(G) = t$. In other words, there are triangle-free graphs which have arbitrarily large chromatic number.

On the other hand, there are important classes of graphs for which $\chi(G) = \omega(G)$. The graph shown here satisfies $\chi(G) = \omega(G) = 5$, and we illustrate in the figure on the right a 5-coloring. The colors are the bold-face numbers. Again, we comment that we have an effective algorithm for coloring a graph only when it is 2-colorable, i.e., it has no odd cycles. But for any $t \geq 3$, it is apparently very difficult to answer whether a graph has chromatic number at most t.

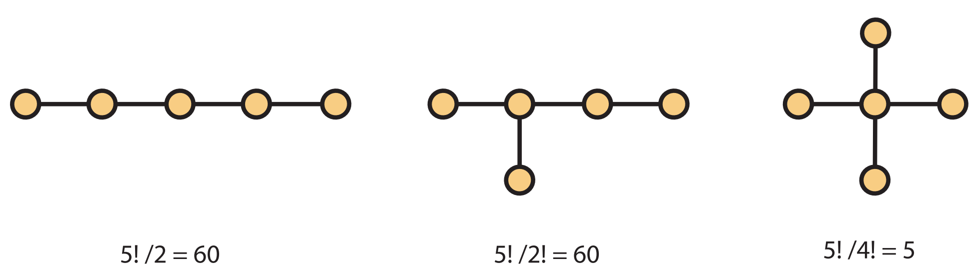
9. Draw a graph G on six vertices with $\omega(G) = 3$ and $\chi(G) = 4$.

We learned in class that the 5-cycle C_5 has $\omega(C_5) = 2$ and $\chi(C_5) = 3$. So if we add a new vertex x adjacent to all five vertices on the 5-cycle, we get a graph G on 6 vertices with $\omega(G) = 3$ and $\chi(G) = 4$. Here is a drawing of that graph.



10. Draw all unlabelled trees on five vertices. Then for each of them, count the number of ways the labels from $\{1, 2, 3, 4, 5\}$ can be applied. Hint: The total number of labeled trees on 5 vertices is $125 = 5^3$.

In the following figure, we draw the three unlabelled trees on 5 vertices. Below each of the trees, we show the number of ways the labels from $\{1, 2, 3, 4, 5\}$ can be applied. The first count is $5!/2 = 60$ since the entire path can be reversed. The second is $5!/2! = 60$ since the two leaves adjacent to the vertex of degree 3 can be permuted. The third count is $5!/4! = 5$ since the four leaves can be permuted. Note that the total number of labeled trees on 5 vertices is $5^3 = 125 = 60 + 60 + 5$ which illustrates the general formula: the number of labeled trees on n vertices is n^{n-2}.



11. Prove the following identity by Mathematical Induction:

$$2 + 8 + 14 + \ldots 6n - 4 = 3n^2 - n \quad \text{when } n \geq 1.$$

Proof. For the *base* step, we note that when $n = 1$, the left hand side consists only of a single term, and that term is 2. On the other hand, when $n = 1$, the right hand side is $3 \cdot 1^2 - 1 = 3 - 1 = 2$. So the formula holds when $n = 1$.
Now for the *inductive* step. We assume that the formula holds for some integer $k \geq 1$. When $n = k + 1$, the left hand side becomes

$$2 + 8 + 14 + \cdots + 6k - 4 + 6(k + 1) - 4.$$

We can then conclude that

$$\begin{aligned} 2 + 8 + 14 + \cdots + 6k - 2 + 6(k + 1) - 4 &= 3k^2 - k + 6(k + 1) - 4 \\ &= 3k^2 + 5k + 3 \\ &= 3k^2 + 6k + 3 - (k + 1) \\ &= 3(k + 1)^2 - (k + 1). \end{aligned}$$

This last computation shows that the formula is also valid when $n = k + 1$. By the Principle of Induction, we have now shown that the formula is valid for all $n \geq 1$.

Point Totals

1. 12 points. 3 parts each worth 4 points.
2. 7 points.
3. 18 points. 6 parts each worth 3 points.
4. 7 points.
5. 7 points.
6. 8 points.
7. 8 points.
8. 12 points. 6 parts each worth 2 points.
9. 5 points.
10. 8 points.
11. 8 points.

Total of 100 points for all 11 questions.