**ISYE/CS 8803: Foundations of Data Privacy** **Fall 2017**

# Lecture 12: Introduction to Game Theory and Mediated Games

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## 1 Introduction

A *game* consists of $n$ players. Each player $i$ has an associated type $t_i \in T$ (payoff relevant information in the game). A player $i$ can choose an action $a_i \in A_i$. There is a *utility function* for each player $u_i : A_1 \times \ldots \times A_n \to \mathbb{R}$,

$$u_i(a_i, a_{-i}) = \text{pay off to player } i \text{ when player } i \text{ plays } a_i \text{ and the rest of players play } a_{-i}$$

Notation: for a vector $v = (v_1, \ldots, v_n)$, we write $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

**Example 1** (single item auction)**.** *Parameters are:* $t_i =$ *value for the good*
$a_i =$ *bid for the good*
$u_i(a_i, a_{-i}) = t_i \cdot \mathbb{1}_{i \text{ wins the auction}} -$ *payment*

**Example 2** (Routing game)**.** *Here we use* $T_i$ *as type, instead of* $t_i$ *to avoid a notation clash.*
$T_i = (s_i, t_i)$ *is the source-sink pair (player* $i$ *wants to go from where to where?)*
$a_i = s_i - t_i$ *path on the graph*
$u_i(a_i, a_{-i} = -$*delay from* $s_i$ *to* $t_i$ *on path* $a_i$ *given traffic by* $a_{-i}$.

Now that games are defined, we need to look at strategy concept before getting to equilibrium.

## 2 Strategy

A *strategy* is $\sigma_i : T \to A_i$ (I know my type, and assume I don't really know anything else, then I have to act something).
If $\sigma_i$ is deterministic, the strategy is called *pure strategy.*
If $\sigma_i$ is randomized, the strategy is called *mixed strategy.*
$\sigma = (\sigma_1, \ldots, \sigma_n)$ is called *strategy profile.*

Strategy $\sigma_i$ is a *best response to* $\sigma_{-i}$ if playing $\sigma_i$ maximizes player $i$'s utility when everyone else plays $\sigma_{-i}$. (In a Nash Equilibrium, to be defined later, everyone is best responding.)
Note: There is a "cool field of study" called *Price of Anarchy*, which studies the efficiency loss due to selfish behaviors (compared to people submitting to a central authority).

# 3 Equilibrium

**Definition 3** (Nash Equilibrium)**.** *A strategy profile $\sigma$ forms an $\eta$-Nash equilibrium for the game defined by a type vector $t$ if for all $i \in [n], a_i \in A_i$,*

$$\mathop{\mathbb{E}}_{\sigma}[u_i(\sigma(t))] \geq \mathop{\mathbb{E}}_{\sigma}[u_i(a_i, \sigma_{-i}(t_{-i}))] - \eta$$

This means that a player $i$ would not (approximately up to $\eta$) improve his utility by changing his action to other action $a_i$ from this equilibrium, given types $t$.

**Definition 4** (Dominant Strategy)**.** *A strategy profile $\sigma$ forms an $\eta$-dominant strategy equilibrium for the game defined by types $t$ if for all $i \in [n], a_i \in A_i, \sigma_{-i}$,*

$$\mathop{\mathbb{E}}_{\sigma}[u_i(\sigma(t))] \geq \mathop{\mathbb{E}}_{\sigma}[u_i(a_i, \sigma_{-i}(t_{-i}))] - \eta$$

The difference is the added $\sigma_{-i}$. This means a player is doing its best for any actions other players may take, whereas Nash equilibrium means a player is doing the best **given** other players' actions.
Note that this implies that Nash equilibrium requires players to know all other players' types. This is called *complete information*. We can think of games with *incomplete information* as well with $t$ sampled from some prior distribution, and we then have to redefine Nash equilibrium accordingly.

**Definition 5** (BNE)**.** *A strategy profile $\sigma$ forms an $\eta$-BaynesNash Equilibrium (BNE) if for all $i \in [n], t_i, a_i \in A_i$,*

$$\mathop{\mathbb{E}}_{t_{-i},\sigma}[u_i(\sigma(t))|t_i] \geq \mathop{\mathbb{E}}_{t_{-i},\sigma}[u_i(a_i, \sigma_{-i}(t_{-i}))|t_i] - \eta$$

Think of $t_{-i}$ as my belief about other player's type. This means I can't gain more than $\eta$ (in expectation over my belief of other player's type) by deviating my action from this equilibrium. A stringer notion can be defined if this property is true over all types.

**Definition 6** (Ex-post Nash Equilibrium)**.** *A strategy profile $\sigma$ forms an $\eta$-ex-post Nash Equilibrium if for all $i \in [n], t \in T^n, a_i \in A_i$,*

$$\mathop{\mathbb{E}}_{,\sigma}[u_i(\sigma(t))|t_i] \geq \mathop{\mathbb{E}}_{\sigma}[u_i(a_i, \sigma_{-i}(t_{-i}))|t_i] - \eta$$

Note that the famous Nash's Theorem is that for all finite games ($n$ and $|A_i|$ are finite), there exists a Nash equilibrium. But dominant strategy equilibrium is a much stronger notion and may not always exist.

# 4 Mediated Games

Now let's thinking of augmenting a game by including a mediator that will help players coordinate on an equilibrium.

| | All strategy | Not all strategy |
|---|---|---|
| All types | Ex-post Dominant Strategy | Ex-post Nash Equilibrium |
| Not all types | Dominant Strategy | BNE (for random $t$) or Nash Equilibrium (for a fixed $t$) |

Table 1: Summary of Types of Equilibrium

## 4.1 Mechanism

A *direct revelation mechanism* is of the form $M : T^n \to \mathcal{R}$, where each player's eaction is to report a type In this setting, $A_i = T$ are the same for all $i$.

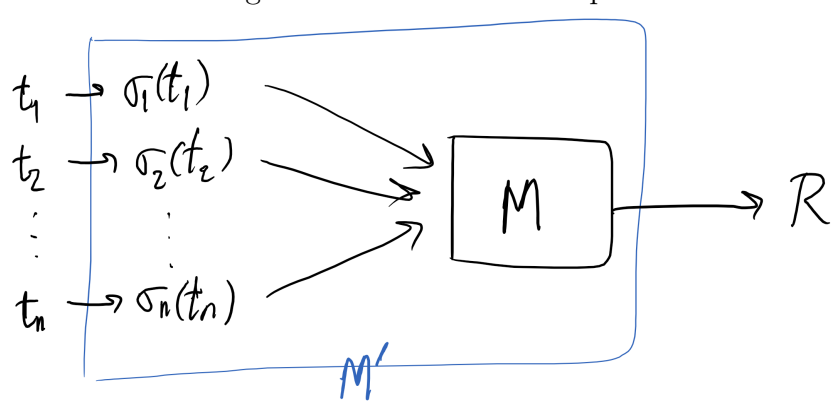
**Definition 7** (IC)**.** *A mechanism is (Bayesian) incentive compatible (BIC) (or truthful) is for all $i$ and misreport $t_i' \in T$ by a player $i$,*

$$\mathop{\mathbb{E}}_{M,t_{-i}}[u_i(t)|t_i] \geq \mathop{\mathbb{E}}_{M,t_{-i}}[u_i(t_i', t_{-i})|t_i]$$

**Theorem 8** (Revelation Principle, Myerson 81)**.** *Any mechanism $M : A^n \to \mathcal{R}$ can be implemented as a direct revelation mechanism $M' : T^n \to \mathcal{R}$ where truthtelling is BNE.*

The idea is to construct $M'$ as a composition of $M$ together with input $(\sigma_i(t_i))_{i=1}^n$.

Figure 1: Revelation Principle




What is this range $\mathcal{R}$?

**Example 9** (Auction)**.** $\mathcal{R}$ =*(who wins, who pays what)* = $\{$*(do you win, what you pay)*$\}_{i=1}^n$

**Example 10** (Routing game)**.** $\mathcal{R} = \{$ *route for player* $i\}_{i=1}^n$

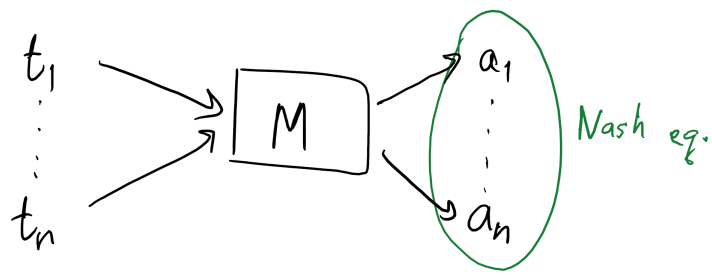
**Example 11** (Mediated game)**.** $\mathcal{R} = \{$ *action of player $i$ in equilibrium* $\}_{i=1}^n$

## 4.2 Mediator

**Definition 12.** *A mediator is a mechanism $M : (T \cup \{\perp\})^n \to (A \cup \{\perp\})^n$ mapping types (or $\perp$ if declining to give types) to suggested actions to all players (or $\perp$ if not receiving a suggested action).*

Ideally we would want to have mediator receiving true types and then giving actions $a$ that everyone follows and is an equilibrium.

Figure 2: Ideal mediator

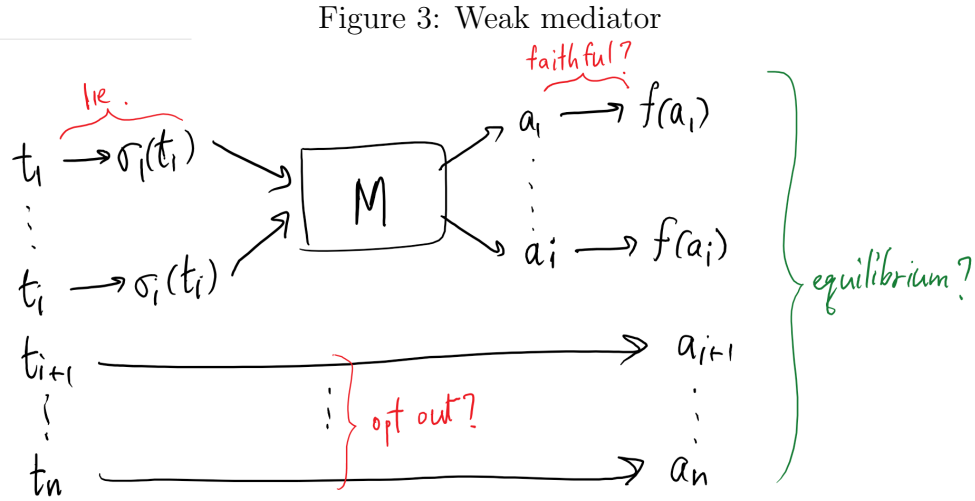



Players can deviate in several ways (mediators are weak):

1. lie about his type
2. opt out (not to report)
3. deviate from the suggested action

The weak mediator can be summarized as a picture below.

Figure 3: Weak mediator




Instead of truthfulness defined earlier, we want players to follow the *good behavior strategy*, which means:

1. truthfully report types
2. faithfully follow the suggested action

# 5 Discussion

How to make player faithfully follow the suggested action? Mediator should tell an action that is in Nash equilibrium.
Also, many of the differential privacy research applications are to move results applicable in BNE - bottom right of the table 1 - to top right or bottom left of the same table. One is to make the mediator differentially private. Then, a good behavior is dominant strategy. We can then implement an expost-Nash equilibrium of a complete information game.
More detail and rigorous proof will be coming in the next lecture.