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Limiting behavior of the Alizadeh–Haeberly–Overton weighted paths in semidefinite programming

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This paper studies the limiting behavior of weighted infeasible central paths for semidefinite programming obtained from centrality equations of the form $XS + SX = 2\nu W$, where W is a fixed positive definite matrix and $\nu > 0$ is a parameter, under the assumption that the problem has a strictly complementary primal–dual optimal solution. We present a different and simpler proof than the one given by Preiß and Stoer [Preiß, M. and Stoer, J., 2004, Analysis of infeasible-interior-point paths arising with semidefinite linear complementarity problems. *Mathematical Programming* **99**, 499–520.] that a weighted central path as a function of ν can be extended analytically beyond 0. In addition, the characterization of the limit points of the path and its normalized first-order derivatives is also provided. We also derive an error bound on the distance between a point lying in a certain neighborhood of the central path and the set of primal–dual optimal solutions. Finally, we make some observation for the superlinear convergence of some primal–dual interior point SDP algorithms using AHO neighborhood.

Keywords: Limiting behavior; Weighted central path; Error bound; Superlinear convergence; Semidefinite programming

AMS Subject Classification: 90C22; 90C25; 90C30; 65K05

1. Introduction

Let $\mathcal{S}^n$ denote the space of $n \times n$ real symmetric matrices. We consider the semidefinite programming (SDP) problem

$$
\begin{aligned}
&\text{minimize } C \bullet X \\
(p) \quad &\text{subject to } \mathcal{A}X = b, \\
&\qquad X \succeq 0,
\end{aligned}
\tag{1}
$$

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and its associated dual SDP problem

$$\text{maximize } b^T y$$
$$(D) \quad \text{subject to } \mathcal{A}^* y + S = C,$$
$$S \succeq 0, \tag{2}$$

where the data consists of $C \in \mathcal{S}^n$, $b \in \Re^m$ and a linear operator $\mathcal{A} : \mathcal{S}^n \to \Re^m$, the primal variable is $X \in \mathcal{S}^n$, and the dual variable consists of $(S, y) \in \mathcal{S}^n \times \Re^m$. For a matrix $V \in \mathcal{S}^n$, the notation $V \succeq 0$ means that V is positive semidefinite. Given a fixed positive definite matrix $W \in \mathcal{S}^n$, $\Delta b \in \Re^m$ and $\Delta C \in \mathcal{S}^n$, our interest in this paper is to study the set of solutions of the following system of nonlinear equations parametrized by the parameter $\nu > 0$:

$$\mathcal{A}X = b + \nu \Delta b, \quad X \succ 0, \tag{3}$$

$$\mathcal{A}^* y + S = C + \nu \Delta c, \quad S \succ 0, \tag{4}$$

$$XS + SX = 2\nu W. \tag{5}$$

Under suitable conditions on $(W, \Delta C, \Delta b)$, it has been shown in Monteiro and Zanjácomo [1] that the above system has a unique solution, denoted by $p(\nu) \equiv (X(\nu), S(\nu), y(\nu))$, for every $\nu \in (0, 1]$. We refer to the path $\nu \in (0, 1] \to p(\nu)$ as $(W, \Delta C, \Delta b)$-weighted central path associated with (P) and (D).

The main goal of this paper is to present a different and simpler analysis than Preiß and Stoer [2] on the limiting behavior of this path as $\nu \downarrow 0$. Preiß and Stoer [2] have proved the main result of this paper that the weighted central path is analytically extendible as functions of $\nu \in (0, 1]$ to $\nu = 0$. Their approach includes estimating the order of the blocks of the path, and then changing the variables twice and applying the implicit function theorem to the resulted systems twice. Our approach is to firstly derive a stronger estimate on the order of the blocks of the path by applying Hoffman Lemma [3], and then reformulate the system of (3)–(5) into a simpler system, and finally apply the implicit function theorem to the resulted system only once. In addition, the characterization of the limit points of the path and its normalized first-order derivatives is also provided. We also derive an error bound on the distance between a point lying in a certain neighborhood of the central path and the set of primal–dual optimal solutions. We also make some observation for the superlinear convergence of some primal–dual interior point SDP algorithms using AHO neighborhood.

When $(W, \Delta C, \Delta b) = (I, 0, 0)$, the path $\nu \in (0, 1] \to p(\nu)$, as shown in Alizadeh *et al.* [4], is a part of the central path associated with (P) and (D). Properties of the central path have been extensively studied on several papers due to the important role it plays in the development of interior-points algorithms for cone programming, nonlinear programming and complementarity problems. Early works dealing with the well-definedness, differentiability and limiting behavior of weighted central paths in the context of the linear programming and the monotone complementarity problem include [5–24]

Using the fact that every real algebraic variety has a triangulation, Kojima *et al.* showed in ref. [25] that the central path associated with a monotone linear complementarity problem converges to a solution. In ref. [26], Kojima *et al.* claims that similar arguments as the ones used in ref. [25] can also be used to show that the central path of a monotone linear semidefinite complementarity problem (which is equivalent to SDP) converges to a solution of the problem. More generally, Drummond and Peterzil [8] established convergence of the central path for analytic convex nonlinear SDP problems. An alternative proof based on a deep result from algebraic geometry (see for example Lemma 3.1 of Milnor [27]) of the convergence of the

central path for an SDP problem was given by Halická *et al.* [28]. Characterization of the limit point of the central path has been obtained by De Klerk *et al.* [29] and Luo *et al.* [30] for SDP problems possessing strictly complementary primal–dual optimal solutions. Using an approach based on the implicit function theorem described in Stoer and Wechs [21,22], Halická [31] showed that the central path of SDP problems possessing strictly complementary primal–dual optimal solutions can be extended analytically as a function of $\nu > 0$ to $\nu = 0$. For more general SDP problems, the above issues regarding the central path still remain open although some progress has been made on a few papers. These include De Klerk *et al.* [32] and Goldfarb and Scheinberg [33] who proved that any cluster point of the central path must be a maximally complementary optimal solution. Also, Halická *et al.* [34] and Sporre and Forsgren [20] provided partial characterizations of the limit point of the central path as being the analytic center of some convex subset of the optimal solution set and the unique solution of a perturbed log barrier problem over the optimal solution set, respectively. Finally, the recent paper by Cruz Neto *et al.* [35] establishes the convergence of the central path for a special class of SDPs which do not satisfy the strict complementarity condition.

Generalization of the notion of weighted central paths from linear programming to SDP problems is a delicate issue. While for a linear programming a weighted central path can be characterized as optimal solutions of certain weighted logarithmic barrier problems, this characterization does not seem to be a good source to obtain a suitable notion of weighted central paths for SDP. Instead, Monteiro and Zanjácomo [1] (see also Monteiro and Pang [36]) work directly with a system consisting of (3), (4) and an equation of the form $\Psi(X, S) = \nu W$, for some suitable map $\Psi : D \subseteq \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n$, and show that this system has a unique solution for every $\nu \in (0, 1]$. Special instances of the map Ψ for which the above result applies include the map $(X, S) \to (XS + SX)/2$ and $(X, S) \to X^{1/2}SX^{1/2}$.

Lu and Monteiro [37] have investigated the limiting behavior of the weighted central paths associated with the map $(X, S) \to X^{1/2}SX^{1/2}$ and their derivatives for the SDPs possessing strictly complementary primal–dual optimal solutions. They have showed that the weighted central path as a function of $\sqrt{\nu}$ can be extended analytically beyond 0. Sim and Zhao [38] studied a class of off-central paths defined as the unique solutions of a system of ordinary differential equations. Through an example, they showed that some off-central paths as a function of $\sqrt{\nu}$ cannot be analytically extended to $\mu = 0$ (see section Theorem 3.2 of [38]). However, we should notice that the paths defined in [38] and [37], though related, are defined differently.

In this paper, we will be interested in the first map above and its corresponding weighted central path, i.e. the path of solutions of systems of the form (3)–(5). More specifically, we will investigate the asymptotic properties of the weighted central paths $\nu \in (0, 1] \to p(\nu)$ for the special class of SDPs possessing strictly complementary primal–dual optimal solutions. Using a suitable change of variables together with the technique described in refs. [21,22] based on the implicit function theorem, we prove in section 4 that the path $\nu \in (0, 1] \to p(\nu)$ can be extended analytically beyond 0. As a consequence, we see that a weighted central path converges as $\Theta(\nu)$. We also characterize the limit point and the first-order derivative of the normalized weighted central path as $\nu \downarrow 0$. Using these results, we derive in section 5 an error bound on the distance between a point lying in a certain neighborhood of the central path and the set of primal–dual optimal solutions. Finally, in section 6, we apply this bound to make some observation for the superlinear convergence of some primal–dual interior point SDP algorithms using AHO neighborhood.

The organization of this paper is as follows. Section 2 introduces the assumptions made throughout the paper. We discusses some properties about the weighted central paths in section 3. Sections 4–6 establish the results mentioned in the previous paragraph.

1.1 *Notation*

The space of symmetric $n \times n$ matrices will be denoted by S^n. Given matrices X and Y in $\Re^{p \times q}$, the standard inner product is defined by $X \bullet Y \equiv \mathrm{tr}(X^T Y)$, where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. The Euclidean norm and its associated operator norm, i.e., the spectral norm, are both denoted by $\| \cdot \|$. The Frobenius norm of a $p \times q$-matrix X is defined as $\|X\|_F \equiv \sqrt{X \bullet X}$. Given a point f and a set F in a finite dimensional normed vector space, the distance from f to F is defined as $\mathrm{dist}(f, F) \equiv \inf_{\tilde{f} \in F} \|f - \tilde{f}\|$. If $X \in \mathcal{S}^n$ is positive semidefinite (resp., definite), we write $X \succeq 0$ (resp., $X \succ 0$). The cone of positive semidefinite (resp., definite) matrices is denoted by $\mathcal{S}^n_+$ (resp., $\mathcal{S}^n_{++}$). Either the identity matrix or operator will be denoted by I. The image (or range) space of a linear operator $\mathcal{A}$ will be denoted by $\mathrm{Im}(\mathcal{A})$, and its kernel (or null) space will be denoted by $N(\mathcal{A})$; the dimension of the subspace $\mathrm{Im}(\mathcal{A})$, referred to as the rank of $\mathcal{A}$, will be denoted by $\mathrm{rank}(\mathcal{A})$. Given a linear operator $\mathcal{F} : E \rightarrow F$ between two finite dimensional inner product spaces $(E, \langle \cdot, \cdot \rangle_E)$ and $(F, \langle \cdot, \cdot \rangle_F)$, its adjoint is the unique operator $\mathcal{F}^* : F \rightarrow E$ satisfying $\langle \mathcal{F}(u), v \rangle_F = \langle u, \mathcal{F}^*(v) \rangle_E$ for all $u \in E$ and $v \in F$.

If $\{u(\nu) : \nu > 0\}$ and $\{v(\nu) : \nu > 0\}$ are real sequences with $v(\nu) > 0$, then $u(\nu) = o(v(\nu))$ means that $\lim_{\nu \to 0} u(\nu)/v(\nu) = 0$. Given functions $f : \Omega \rightarrow E$ and $g : \Omega \rightarrow \Re_{++}$, where Ω is an arbitrary set and E is a normed vector space, and a subset $\tilde{\Omega} \subset \Omega$, we write $f(w) = \mathcal{O}(g(w))$ for all $w \in \tilde{\Omega}$ to mean that there exists $M > 0$ such that $\|f(w)\| \leq Mg(w)$ for all $w \in \tilde{\Omega}$; moreover, for a function $U : \Omega \rightarrow \mathcal{S}^n_{++}$, we write $U(w) = \Theta(g(w))$ for all $w \in \tilde{\Omega}$ if $U(w) = \mathcal{O}(g(w))$ and $U(w)^{-1} = \mathcal{O}(1/g(w))$ for all $w \in \tilde{\Omega}$. The latter condition is equivalent to the existence of a constant $M > 0$ such that

$$\frac{1}{M} I \preceq \frac{1}{g(w)} U(w) \preceq MI, \quad \forall w \in \Omega.$$

2. Preliminaries

In this section, we describe our assumptions that will be used in our presentation. We also describe the weighted central path that will be the subject of our study in this paper. The conditions for its well-definedness are also stated.

Throughout this paper we will be dealing with the pair of dual SDPs (P) and (D) (see (1) and (1), respectively). Denote the feasible sets of (P) and (D) by $\mathcal{F}_P$ and $\mathcal{F}_D$, respectively. Throughout our presentation, we make the following assumptions on the pair of problems (P) and (D).

A.1 $\mathcal{A} : \mathcal{S}^n \rightarrow \Re^m$ is an onto linear operator;
A.2 There exists a pair of strictly complementary primal–dual optimal solution for (P) and (D), that is a triple $(X^*, S^*, y^*) \in \mathcal{F}_P \times \mathcal{F}_D$ satisfying $X^* S^* = 0$ and $X^* + S^* \succ 0$.

We will assume that Assumptions **A.1** and **A.2** are in force throughout our presentation. Hence, we will state our results without explicitly mentioning them.

Assumption **A.1** is not really crucial for our analysis but it is convenient to ensure that the variables S and y are in one-to-one correspondence. We will see that the dual weighted central path can always be defined in the S-space. The goal of Assumption **A.1** is just to ensure that this path can also be extended to the y-space.

Assumption **A.2** is the one that is commonly used in the analysis of superlinear convergence of interior-point algorithms and it plays an important role in our analysis. In fact, it is a very

challenging problem to generalize the analysis of this paper to the case where Assumption **A.2** is dropped or simply relaxed.

By assumption **A.2**, since $X^* S^* = S^* X^* = 0$, we can diagonalize X^* and S^* simultaneously, i.e. find an orthonormal $P \in \Re^{n \times n}$ such that $P^T X^* P$ and $P^T S^* P$ are both diagonal. Performing the change of variables $\hat{X} = P^T X P$ and $(\hat{S}, \hat{y}) = (P^T S P, y)$ on problems (P) and (D) yield another pair of primal and dual SDPs which has a primal–dual optimal solution $(\hat{X}^*, \hat{S}^*, \hat{y}^*)$ such that $\hat{X}^*$ and $\hat{S}^*$ are both diagonal. To simplify our notation, we will assume without loss of generality that the original (P) and (D) already have a primal–dual optimal solution (X^*, S^*, y^*) such that

$$X^* = \begin{bmatrix} \Lambda_B & 0 \\ 0 & 0 \end{bmatrix}, \quad S^* = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_N \end{bmatrix}, \tag{6}$$

where $\Lambda_B \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_K)$, $\Lambda_N \equiv \mathrm{diag}(\lambda_{K+1}, \ldots, \lambda_n)$ for some integer $0 \leq K \leq n$ and some scalars $\lambda_i > 0$, $i = 1, 2, \ldots, n$. Here the subscripts B and N signify the 'basic' and 'nonbasic' subspaces (following the terminology of linear programming). Throughout this paper, the decomposition of any $n \times n$ matrix V is always made with respect to the above partition B and N, namely:

$$V = \begin{bmatrix} V_B & V_{BN} \\ V_{NB} & V_N \end{bmatrix}.$$

Notice that $X \in \mathcal{F}_P$ is an optimal solution of (P) if and only if $XS^* = 0$. Hence, by assumption **A.2**, the primal–optimal solution set $\mathcal{F}_P^*$ is given by

$$\mathcal{F}_P^* \equiv \{X \in \mathcal{F}_P : X_{BN} = 0, X_{NB} = 0 \quad \text{and} \quad X_N = 0\}.$$

Analogously, the dual optimal solution set $\mathcal{F}_D^*$ is given by

$$\mathcal{F}_D^* \equiv \{(S, y) \in \mathcal{F}_D : S_{BN} = 0, S_{NB} = 0 \quad \text{and} \quad S_B = 0\}.$$

Define the linear map $\mathcal{G} : \mathcal{S}^n \times \mathcal{S}^n \times \Re^m \to \mathcal{S}^n \times \Re^m$ by

$$\mathcal{G}(X, S, y) \equiv (\mathcal{A}^* y + S - C, \mathcal{A}X - b) \tag{7}$$

and the set $\mathcal{G}_{++}$ by

$$\mathcal{G}_{++} \equiv \mathcal{G}(\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \Re^m). \tag{8}$$

Given $(W, \Delta C, \Delta b) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$, in this paper we are interested in the solutions of the system of nonlinear equations (3)–(5) parametrized by the parameter $\nu > 0$. The following result gives condition on $(W, \Delta C, \Delta b)$ for system (3)–(5) to have a unique solution for each $\nu \in (0, 1]$.

PROPOSITION 2.1 *Assume that* $(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$. *Then, for any* $\nu \in (0, 1]$, *the system* (3)−(5) *has a unique solution, denoted by* $(X(\nu), S(\nu), y(\nu))$. *Moreover, the path* $\nu \in (0, 1] \to (X(\nu), S(\nu), y(\nu))$ *is analytic.*

Proof By **A.2** and the assumption that $(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$, we easily see that $\nu(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$ for all $\nu \in (0, 1]$. The first conclusion of the proposition now follows from Theorem 1(b) of Monteiro and Zanjácomo [1] by letting F, Φ and $\mathcal{V}$ in that theorem be defined as $F = \mathcal{G}$, $\Phi(X, S) = (XS + SX)/2$ for all $(X, S) \in \mathcal{S}_+^n \times \mathcal{S}_+^n$ and $\mathcal{V} = \mathcal{W}$. The second conclusion follows by applying the analytic version of the implicit function theorem to system (3)–(5) viewed as a function of (X, S, y, ν) and using the fact that the

assumption $(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$ implies that the Jacobian of this system with respect to (X, S, y) is nonsingular at $(X(\nu), S(\nu), y(\nu), \nu)$ for every $\nu \in (0, 1]$. (See Lemma 4 of [36] and the paragraph following it.) ∎

Given a $(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$, the path $\nu \in (0, 1] \to (X(\nu), S(\nu), y(\nu))$ will be referred to as the $(W, \Delta C, \Delta b)$-weighted central path. In view of the above proposition, we will assume throughout Sections 3 and 4 that the following condition is true, without explicitly mentioning it in the statements of the results.

A.3 $(W, \Delta C, \Delta b) \in \mathcal{S}_{++}^n \times \mathcal{G}_{++}$.

3. Properties of the weighted central path

In this section, we will introduce some properties of the weighted central path. The estimates on the sizes of some blocks of the weighted path are given. Although Kojima *et al.* have essentially developed them in section 5 of ref. [39], we derive them in a different and simpler approach with the aid of Hoffman Lemma [3]. The analysis of the limiting behavior of the weighted central path will strongly rely on these estimates.

The following result gives some estimates on the sizes of the blocks of $X(\nu)$ and $S(\nu)$.

LEMMA 3.1 *For all $\nu > 0$ sufficiently small, we have*:

$$X_B(\nu) = \Theta(1), \quad X_N(\nu) = \Theta(\nu), \tag{9}$$

$$S_B(\nu) = \Theta(\nu), \quad S_N(\nu) = \Theta(1), \tag{10}$$

$$X_{BN}(\nu) = \mathcal{O}(\sqrt{\nu}), \quad S_{BN}(\nu) = \mathcal{O}(\sqrt{\nu}). \tag{11}$$

As a consequence, the weighted central path $\{(X(\nu), y(\nu), S(\nu))\}$ is bounded and any its accumulation point as $\nu \downarrow 0$ is a strictly complementary primal–dual optimal solution of (P) and (D).

Proof From (5), we have $X \bullet S = \nu \operatorname{tr}(W)$. Following the same proof as Lemma 2.2 of Lu and Monteiro [37], we obtain that

$$X_B(\nu) = \mathcal{O}(1), \quad S_N(\nu) = \mathcal{O}(1), \quad X_N(\nu) = \mathcal{O}(\nu), \quad S_B(\nu) = \mathcal{O}(\nu) \tag{12}$$

and (11) holds. By Lemma 3.3 of Monteiro [40], we have

$$\lambda_{\min}(X^{1/3} S X^{1/2}) \geq \lambda_{\min} \frac{(SX + XS)}{2}$$

where $\lambda_{\min}(\cdot)$ is the minimal eigenvalue of the associated matrix. Noting that $\lambda_{\min}(SX + XS)/2 = \nu \lambda_{\min}(W)$ by (5), we have $X^{1/2} S X^{1/2} \succeq \nu \lambda_{\min}(W) I$. Using this fact together with (12), we can easily follow the proof of Lemma 3.2 of Luo *et al.* [30] to derive (9) and (10), except that the identity matrix I should be replaced by $\lambda_{\min}(W) I$ throughout their proof. ∎

The following result establishes the relationship between $X_{BN}(\nu)$ and $S_{BN}(\nu)$.

LEMMA 3.2 *For all $\nu > 0$ sufficiently small, we have*:

$$\|S_{BN}(\nu)\| = \Theta(\|X_{BN}(\nu)\|) + \mathcal{O}(\nu),$$

$$-X_{BN}(\nu) \bullet S_{BN}(\nu) = \Theta(\|X_{BN}(\nu)\|^2) + \mathcal{O}(\nu \|X_{BN}(\nu)\|).$$

Proof For notational convenience, let $X \equiv X(\nu)$, $S \equiv S(\nu)$ and $V \equiv XS$. Using Lemma 3.1, we obtain that

$$\|V_{NB}\| = \|X_{NB}S_B + X_N S_{NB}\| = \mathcal{O}(\nu^{3/2}). \tag{13}$$

Using (5) and the fact that $V = XS$, we have

$$X_B S_{BN} + X_{BN} S_N + V_{NB}^T = 2\nu W_{BN},$$

which implies that

$$S_{BN} = -X_B^{-1}(X_{BN} S_N + V_{NB}^T - 2\nu W_{BN}).$$

Using this identity, (13) and Lemma 3.1, we obtain that the first conclusion holds, and also

$$X_{BN} \bullet S_{BN} = -\text{tr}(X_{BN}^T X_B^{-1} X_{BN} S_N) + \mathcal{O}(\nu\|X_{BN}\|),$$

$$= -\|X_B^{-1/2} X_{BN} S_N^{1/2}\|_F^2 + \mathcal{O}(\nu\|X_{BN}\|),$$

which together with Lemma 3.1 implies the second conclusion. ∎

Next we state Hoffman Lemma [3] (see also Lemma A.3 of Wright [41]), which will be used subsequently.

LEMMA 3.3 *Let $G \in \Re^{r \times q}$ and $H \in \Re^{p \times q}$ be given matrices. Then there exists a nonnegative constant L depending only on G and H such that, for all vectors*

$$\begin{bmatrix} g \\ h \end{bmatrix} \in \text{Im}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right),$$

there is a solution to the system

$$Gw \le g, \quad Hw = h$$

that satisfies $\|w\| \le L\|(g, h)\|$.

The following result improves the estimates on the sizes of $X_{BN}(\nu)$ and $S_{BN}(\nu)$.

LEMMA 3.4 *For all $\nu > 0$ sufficiently small, we have*:

$$X_{BN}(\nu) = \mathcal{O}(\nu), \quad S_{BN}(\nu) = \mathcal{O}(\nu).$$

Proof Suppose that $X_{BN}(\nu) = \mathcal{O}(\nu)$ does not hold. Then there exists a sequence $\nu_k \downarrow 0$ as $k \to \infty$ such that $\nu_k = o(\|X_{BN}(\nu_k)\|)$. For convenience, we omit the index k from ν_k throughout the remaining proof. Then the above identity can be written as $\nu = o(\|X_{BN}(\nu)\|)$, which together with Lemma 3.2 implies that

$$\|S_{BN}(\nu)\| = \Theta(\|X_{BN}(\nu)\|), \tag{14}$$

$$-X_{BN}(\nu) \bullet S_{BN}(\nu) = \Theta(\|X_{BN}(\nu)\|^2). \tag{15}$$

For any $\nu \in (0, 1]$, consider the linear system

$$\mathcal{A}(X - X(\nu)) = -\nu \Delta b,$$

$$X_{BN} - X_{BN}(\nu) = -X_{BN}(\nu), \tag{16}$$

$$X_N - X_N(\nu) = -X_N(\nu). \tag{17}$$

We see that any $X^* \in \mathcal{F}_P^*$ is a feasible solution to this system. Hence, by Lemma 3.3, there exists a sufficiently large constant $\hat{C}$ (independent on ν) such that for any $\nu \in (0, 1]$, this

system has a solution $\bar{X} \in \mathcal{S}^n$, which satisfies

$$\|\bar{X} - X(\nu)\| \leq \hat{C}(\nu\|\Delta b\| + \|X_N(\nu)\| + \|X_{BN}(\nu)\|). \tag{18}$$

Analogously, for any $\nu \in (0, 1]$, there exists $(\bar{S}, \bar{y}) \in \mathcal{S}^n \times \Re^m$ which satisfies

$$\mathcal{A}^*(y - y(\nu)) + S - S(\nu) = -\nu\Delta C,$$
$$S_B - S_B(\nu) = -S_B(\nu), \tag{19}$$
$$S_{BN} - S_{BN}(\nu) = -S_{BN}(\nu), \tag{20}$$
$$\hat{C}(\nu\|\Delta C\| + \|S_B(\nu)\| + \|S_{BN}(\nu)\|) \geq \|S - S(\nu)\|. \tag{21}$$

Noting that $(\Delta C, \Delta b) \in \mathcal{G}_{++}$, we have $\Delta C = \mathcal{A}^* y^0 + S^0 - C$ and $\Delta b = \mathcal{A}X^0 - b$ for some $(X^0, S^0, y^0) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \Re^m$. We easily see that, for any given $(X^*, S^*, y^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$,

$$\mathcal{A}(\bar{X} - X(\nu) + \nu(X^0 - X^*)) = 0, \quad \bar{S} - S(\nu) + \nu(S^0 - S^*) \in \text{Im}(\mathcal{A}^*).$$

Hence, we obtain that

$$(\bar{X} - X(\nu) + \nu(X^0 - X^*)) \bullet (\bar{S} - S(\nu) + \nu(S^0 - S^*)) = 0. \tag{22}$$

Since $(\bar{X}, \bar{S})$ satisfies (16), (17), (19) and (20), we immediately have

$$\bar{X}_{BN} = \bar{S}_{BN} = 0, \quad \bar{X}_N = 0, \quad \bar{S}_B = 0.$$

This together with (22) implies that

$$X_{BN}(\nu) \bullet S_{BN}(\nu) = (\bar{X}_B - X_B(\nu)) \bullet S_B(\nu) + X_N(\nu) \bullet (\bar{S}_N - S_N(\nu))$$
$$- \nu(\bar{X} - X(\nu)) \bullet (S^0 - S^*) - \nu(X^0 - X^*) \bullet (\bar{S} - S(\nu))$$
$$- \nu^2(X^0 - X^*) \bullet (S^0 - S^*). \tag{23}$$

Note that $\|\bar{X}B - X_B(\nu)\| \leq \|\bar{X} - X(\nu)\|$ and $\|\bar{S}N - S_N(\nu)\| \leq \|\bar{S} - S(\nu)\|$. Using this relation together with (23) and Lemma 3.1, we obtain that, for all $\nu > 0$ sufficiently small,

$$|X_{BN}(\nu) \bullet S_{BN}(\nu)| \leq \check{C}\nu(\nu + \|\bar{X} - X(\nu)\| + \|\bar{S} - S(\nu)\|) \tag{24}$$

for some constant $\check{C}$. In view of (14), (15), (18), (21), (24) and Lemma 3.1, one has that, for all $\nu > 0$ sufficiently small,

$$|X_{BN}(\nu) \bullet S_{BN}(\nu)| \leq \bar{C}\nu(\nu + \|X_{BN}(\nu)\| + \|S_{BN}(\nu)\|),$$
$$\leq \tilde{C}\nu(\nu + |X_{BN}(\nu) \bullet S_{BN}(\nu)|^{1/2}),$$

where $\bar{C}$ and $\tilde{C}$ are some constants (independent on ν) and the last inequality follows from (14) and (15). Let $\xi = |X_{BN}(\nu) \bullet S_{BN}(\nu)|^{1/2}$. From the latter inequality above, we have $\xi^2 \leq \tilde{C}\nu(\nu + \xi)$, which together with the fact $\xi > 0$, implies that $\xi \leq (\tilde{C} + (5\tilde{C})^{1/2})\nu/2$. Hence, $\xi = \mathcal{O}(\nu)$. Using this result and (15), we obtain that $\|X_{BN}(\nu)\| = \mathcal{O}(\nu)$, which contradicts with the early assumption of the proof. Therefore, $X_{BN}(\nu) = \mathcal{O}(\nu)$ holds. The proof of $S_{BN}(\nu) = \mathcal{O}(\nu)$ directly follows from Lemma 3.2. ∎

We end this section by stating a convergence result of the $(W, \Delta C, \Delta b)$-weighted central path to a primal–dual optimal solution of (1) and (2). We do not provide a proof for it since it is similar to the one given in the Appendix of Halická *et al.* [28].

PROPOSITION 3.5　*There exists some $\epsilon > 0$ and an analytic function $v : [0, \epsilon) \to (0, 1)$ such that $v(0) = 0$ and the path $t \in (0, \epsilon) \to (X(v(t)), S(v(t)), y(v(t)))$ is analytic at $t = 0$. In particular, $(X(v(t)), S(v(t)), y(v(t)))$ converges to some primal–dual optimal solution (X^*, S^*, y^*) as $t \downarrow 0$.*

We observe that Proposition 3.5 holds even without requiring Assumption **A.2**. As a consequence, its main advantage is that it holds for any SDP problem provided that it satisfies strong duality and Assumption **A.3**. Its main drawbacks are that it neither gives a characterization of the limit point (X^*, S^*, y^*) nor describes how fast $v(t)$ converges to 0. These issues and others will be addressed in the remaining sections of this paper in the context of SDPs satisfying Assumption **A.2**.

4.　Analyticity of the weighted central path

In this section, we will show that the weighted central path can be extended analytically to $v = 0$. We also characterize the limit point and the first-order derivative of the normalized weighted central path as $v \downarrow 0$.

For the sake of brevity, it is convenient to introduce the following definition.

DEFINITION 4.1　*Let $w : (0, \delta) \to E$ be a given function where $\delta > 0$ and E is a finite dimensional normed vector space. The function w is said to be* analytic at 0 *if there exist $\epsilon > 0$ and an analytic function $\psi : (-\epsilon, \epsilon) \to E$ such that $w(t) = \psi(t)$ for all $t \in (0, \epsilon)$.*

The following theorem is one of the main results of this section. Its proof will be given at the end of this section.

THEOREM 4.2　*The $(W, \Delta C, \Delta b)$-weighted central path $v \in (0, 1] \to \big(X(v), S(v), y(v)\big)$ is analytic and also analytic at $v = 0$. As a consequence, the $(W, \Delta C, \Delta b)$-weighted path and all its k-th order derivatives, $k \geq 1$, converge as $v \downarrow 0$.*

A key step towards showing the above result is a reformulation of the weighted central path system (3)–(5) as we now discuss. Now, let

$$\mathcal{U}^n = \{U \in \Re^{n \times n} : U_B \in \mathcal{S}^{|B|}, U_N \in \mathcal{S}^{|N|}, U_{NB} = 0\},$$
$$\mathcal{U}^n_{++} = \{U \in \mathcal{U}^n : U_B \succ 0, \ U_N \succ 0\}.$$

and define $\mathcal{L} : \mathcal{U}^n \to \Re^{n \times n}$ as

$$\mathcal{L}(U) = \begin{bmatrix} 0 & 0 \\ U_{BN}^T & 0 \end{bmatrix}, \quad \forall \, U \in \mathcal{U}^n.$$

Given any $(X, S) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n$, we define $(\tilde{U}, \tilde{V}, \tilde{X}, \tilde{S}) \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n$ as follows

$$\tilde{X} = \begin{bmatrix} X_B & \dfrac{X_{BN}}{\nu} \\ \dfrac{X_{NB}}{\nu} & \dfrac{X_N}{\nu} \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} \dfrac{S_B}{\nu} & \dfrac{S_{BN}}{\nu} \\ \dfrac{S_{NB}}{\nu} & S_N \end{bmatrix}, \tag{25}$$

$$\tilde{U} = \begin{bmatrix} \tilde{X}_B & \tilde{X_{BN}} \\ 0 & \tilde{X}_N \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} \tilde{S}_B & \tilde{S}_{BN} \\ 0 & \tilde{S}_N \end{bmatrix}. \tag{26}$$

Letting

$$D_N = \begin{bmatrix} I & 0 \\ 0 & \dfrac{I}{\nu} \end{bmatrix},$$

we immediately obtain that

$$X D_N = \begin{bmatrix} \tilde{X}_B & \tilde{X}_{BN} \\ \nu \tilde{X}_{NB} & \tilde{X}_N \end{bmatrix} = \tilde{U} + \nu \mathcal{L}(\tilde{U}) = \mathcal{U}_\nu(\tilde{U}),$$

$$\frac{D_N^{-1} S}{\nu} = \begin{bmatrix} \tilde{S}B & \tilde{S}_{BN} \\ \nu \tilde{S}_{NB} & \tilde{S}_N \end{bmatrix} = \tilde{V} + \nu \mathcal{L}(\tilde{V}) = \mathcal{U}_\nu(\tilde{V})$$

where $\mathcal{U}_\nu \equiv I + \nu \mathcal{L}$. Using the above identities, we easily see that, for $\nu > 0$, (5) is equivalent to

$$\mathcal{U}_\nu(\tilde{U})\mathcal{U}_\nu(\tilde{V}) + (\mathcal{U}_\nu(\tilde{U})\mathcal{U}_\nu(\tilde{V}))^T = 2W. \tag{27}$$

Accordingly, we define $(\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ by replacing $(\tilde{X}, \tilde{S}, X, S)$ by $(\tilde{X}(\nu), \tilde{S}(\nu), X(\nu), S(\nu))$ in (26) and (25), respectively. Proposition 2.1 and the above arguments establish the following key result.

PROPOSITION 4.3 Let $(X^*, S^*, y^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$ be given. Then, for every $\nu \in (0, 1]$, $(\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ is a solution of the system defined by (26), (27) and the linear equations

$$\mathcal{A} \bullet \begin{bmatrix} \tilde{X}_B - X_B^* & \nu \tilde{X}_{BN} \\ \nu \tilde{X}_{NB} & \nu \tilde{X}_N \end{bmatrix} = \nu \Delta b, \tag{28}$$

$$\begin{bmatrix} \nu \tilde{S}_B & \nu \tilde{S}_{BN} \\ \nu \tilde{S}_{NB} & \tilde{S}_N - S_N^* \end{bmatrix} \in \nu \Delta C + \mathrm{Im}(\mathcal{A}^*). \tag{29}$$

Moreover, the path $\nu \in (0, 1] \to (\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ is analytic.

The next result states some basic properties about the accumulation points of $(\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ as ν approaches 0.

LEMMA 4.4 The path $\nu \in (0, 1] \to (\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ remains bounded as ν approaches 0 and any accumulation point $(\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^*)$ of this path as ν approaches

0 *is in* $\mathcal{U}^n_{++} \times \mathcal{U}^n_{++} \times \mathcal{S}^n \times \mathcal{S}^n$ *and satisfies*

$$\tilde{U}^* = \begin{bmatrix} \tilde{X}^*_B & \tilde{X}^*_{BN} \\ 0 & \tilde{X}^*_N \end{bmatrix}, \quad \tilde{V}^* = \begin{bmatrix} \tilde{S}^*_B & \tilde{S}^*_{BN} \\ 0 & \tilde{S}^*_N \end{bmatrix}, \tag{30}$$

$$\tilde{U}^* \tilde{V}^* + (\tilde{U}^* \tilde{V}^*)^T = 2W. \tag{31}$$

Proof Relation (25), Lemma 3.1 and Lemma 3.4 imply that $(\tilde{X}(\nu), \tilde{S}(\nu))$ remains bounded as ν approaches 0. So does $(\tilde{U}(\nu), \tilde{V}(\nu))$ according to this fact and relation (26). Using (25) and Lemma 3.1, we see that $(\tilde{U}^*, \tilde{V}^*) \in \mathcal{U}^n_{++} \times \mathcal{U}^n_{++}$. The remaining proof follows directly from (26) and (27). ■

Our next goal is to show that the path $\nu \in (0, 1] \to (\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu))$ is analytic at $\nu = 0$. The basic tool we use to establish this fact is the implicit function theorem applied to a specific system of equations. A first natural candidate for such a system seems to be the one given by (26)–(29). However, the main drawback of this system is that its derivative with respect to $(\tilde{U}, \tilde{V}, \tilde{X}, \tilde{S})$ is generally singular for $\nu = 0$ (even though for $\nu \in (0, 1)$ it is always nonsingular). The main cause for this phenomenon is that the 'rank' of the linear equations (28) and (29) changes when ν becomes 0.

We will now show how the linear equations (28) and (29) can be reformulated into equivalent linear equations for every $\nu \in (0, 1]$. Moreover, the new linear equations have the property that their rank remains constant for every $\nu \in \Re$. First note that the linear operator $\mathcal{A} : \mathcal{S}^n \to \Re^m$ can be expressed as

$$\mathcal{A}(X) = \mathcal{A}_B(X_B) + \mathcal{A}_{BN}(X_{BN}) + \mathcal{A}_N(X_N) \equiv (\mathcal{A}_B \ \mathcal{A}_{BN} \ \mathcal{A}_N) \begin{pmatrix} X_B \\ X_{BN} \\ X_N \end{pmatrix}, \tag{32}$$

for some linear operators $\mathcal{A}_B : \mathcal{S}^{|B|} \to \Re^m$, $\mathcal{A}_{BN} : \Re^{|B| \times |N|} \to \Re^m$ and $\mathcal{A}_N : \mathcal{S}^{|N|} \to \Re^m$.

A well-known result from linear algebra says that any matrix can be put into row-echelon form after a sequence of elementary row operations. A similar type of argument allows one to establish the following result.

LEMMA 4.5 *Let* $\mathcal{A} : \mathcal{S}^n \to \Re^m$ *be an onto linear operator. Assume that*

$$i_1 = \text{rank}(\mathcal{A}_B), \quad i_2 = \text{rank}(\mathcal{A}_B \ \mathcal{A}_{BN}) - i_1,$$
$$i_3 = \text{rank}(\mathcal{A}) - (i_1 + i_2) = m - (i_1 + i_2).$$

Then there exists an isomorphism $T : \Re^m \to \Re^m$ *such that*

$$(T \circ \mathcal{A})(X) = \begin{pmatrix} \mathcal{A}_{11}(X_B) & +\mathcal{A}_{12}(X_{BN}) & +\mathcal{A}_{13}(X_N) \\ & \mathcal{A}_{22}(X_{BN}) & +\mathcal{A}_{23}(X_N) \\ & & \mathcal{A}_{33}(X_N) \end{pmatrix}$$

$$\equiv \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} X_B \\ X_{BN} \\ X_N \end{pmatrix},$$

for some linear operators

$$\mathcal{A}_{11} : \mathcal{S}^{|B|} \longrightarrow \Re^{i_1}, \quad \mathcal{A}_{12} : \Re^{|B| \times |N|} \longrightarrow \Re^{i_1},$$

$$\mathcal{A}_{13} : \mathcal{S}^{|N|} \longrightarrow \Re^{i_1}, \quad \mathcal{A}_{22} : \Re^{|B| \times |N|} \longrightarrow \Re^{i_2},$$

$$\mathcal{A}_{23} : \mathcal{S}^{|N|} \longrightarrow \Re^{i_2}, \quad \mathcal{A}_{33} : \mathcal{S}^{|N|} \longrightarrow \Re^{i_3}$$

such that $\mathrm{rank}(\mathcal{A}_{11}) = i_1$, $\mathrm{rank}(\mathcal{A}_{22}) = i_2$, $\mathrm{rank}(\mathcal{A}_{33}) = i_3$.

We can now reformulate the linear system (28) with the use of Lemma 4.5 as follows. Using Lemma 4.5, we easily see that (28) is equivalent to the linear system

$$\begin{pmatrix} \mathcal{A}_{11} & \nu \mathcal{A}_{12} & \nu \mathcal{A}_{13} \\ 0 & \nu \mathcal{A}_{22} & \nu \mathcal{A}_{23} \\ 0 & 0 & \nu \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} \tilde{X}_B - X_B^* \\ \tilde{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = \nu \begin{pmatrix} \widetilde{\Delta b}_1 \\ \widetilde{\Delta b}_2 \\ \widetilde{\Delta b}_3 \end{pmatrix}$$

where $(\widetilde{\Delta b}_1, \widetilde{\Delta b}_2, \widetilde{\Delta b}_3) \in \Re^{i_1} \times \Re^{i_2} \times \Re^{i_3}$ and $\widetilde{\Delta b} \equiv T(\Delta b)$. Dividing the second and third blocks of rows in the above system by ν, respectively, we obtain the following system

$$\begin{pmatrix} \mathcal{A}_{11} & \nu \mathcal{A}_{12} & \nu \mathcal{A}_{13} \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{pmatrix} \begin{pmatrix} \tilde{X}_B - X_B^* \\ \tilde{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = \begin{pmatrix} \nu \widetilde{\Delta b}_1 \\ \widetilde{\Delta b}_2 \\ \widetilde{\Delta b}_3 \end{pmatrix}. \tag{33}$$

Note that the linear system (33) is equivalent to (28) for every $\nu \in (0, 1]$. Hence, $\tilde{X}(\nu)$ satisfies (33) for every $\nu \in (0, 1]$. A nice feature of (33) is that the operator on its left hand side does not lose full rankness as ν becomes 0. We state this fact in the following proposition.

PROPOSITION 4.6   *Let* $\mathcal{A}_\nu : \mathcal{S}^n \to \Re^m$ *be the operator defined on the left-hand side of* (33). *Then*, $\mathrm{rank}(\mathcal{A}_\nu) = m$ *for every* $\nu \in \Re$.

The linear system (29) can also be reformulated with the aid of Lemma 4.5 as follows. First note that by Lemma 4.5, we have

$$\mathrm{Im}(\mathcal{A}^*) = \mathrm{Im}[(T \circ \mathcal{A})^*] = \mathrm{Im} \left[\begin{pmatrix} \mathcal{A}_{11}^* & 0 & 0 \\ \mathcal{A}_{12}^* & \mathcal{A}_{22}^* & 0 \\ \mathcal{A}_{13}^* & \mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix} \right]$$

$$= \mathrm{Im} \left[\begin{pmatrix} \nu \mathcal{A}_{11}^* & 0 & 0 \\ \nu \mathcal{A}_{12}^* & \nu \mathcal{A}_{22}^* & 0 \\ \nu \mathcal{A}_{13}^* & \nu \mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix} \right],$$

for every $\nu \in (0, 1]$. Hence, for every $\nu \in (0, 1]$ (29) is equivalent to

$$\begin{pmatrix} \nu \tilde{S}_B \\ \nu \tilde{S}_{BN} \\ \tilde{S}_N - S_N^* \end{pmatrix} \in \nu \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \\ \Delta C_N \end{pmatrix} + \mathrm{Im} \left[\begin{pmatrix} \nu \mathcal{A}_{11}^* & 0 & 0 \\ \nu \mathcal{A}_{12}^* & \nu \mathcal{A}_{22}^* & 0 \\ \nu \mathcal{A}_{13}^* & \nu \mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix} \right].$$

Dividing the first and second block of rows in the above system by ν, respectively, we obtain the system

$$\begin{pmatrix} \tilde{S}_B \\ \tilde{S}_{BN} \\ \tilde{S}_N - S_N^* \end{pmatrix} \in \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \\ \nu \Delta C_N \end{pmatrix} + \text{Im} \left[\begin{pmatrix} \mathcal{A}_{11}^* & 0 & 0 \\ \mathcal{A}_{12}^* & \mathcal{A}_{22}^* & 0 \\ \nu \mathcal{A}_{13}^* & \nu \mathcal{A}_{23}^* & \mathcal{A}_{33}^* \end{pmatrix} \right], \tag{34}$$

which is equivalent to (29) for every $\nu \in (0, 1]$, and hence satisfied by $\tilde{S}(\nu)$ for all $\nu \in (0, 1]$.

Let the operator $\text{ß}_\nu : \Re^m \to \mathcal{S}^n$ be defined such that $\text{Im}(\text{ß}_\nu)$ is defined by the second term on the right-hand side of (34). Using the definitions of $\mathcal{A}_\nu$ and ß_ν, and the fact that $\tilde{X}(\nu)$ and $\tilde{S}(\nu)$ satisfy (33) and (34), respectively, for every $\nu \in (0, 1]$, we conclude that there exists a function $\tilde{y} : (0, 1] \to \Re^m$ such that $(\tilde{X}(\nu), \tilde{S}(\nu), \tilde{y}(\nu))$ satisfies

$$\mathcal{A}_\nu(\tilde{X} - X^*) = \begin{pmatrix} \nu \widetilde{\Delta b}_1 \\ \widetilde{\Delta b}_2 \\ \widetilde{\Delta b}_3 \end{pmatrix}, \quad \text{ß}_\nu \tilde{y} + (\tilde{S} - S^*) = \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \\ \nu \Delta C_N \end{pmatrix}. \tag{35}$$

for every $\nu \in (0, 1]$. Moreover, using Proposition 4.6 and the fact that $\{\tilde{S}(\nu) : \nu \in (0, 1]\}$ is bounded, we easily see that $\{\tilde{y}(\nu) : \nu \in (0, 1]\}$ is also bounded. We have thus established the following result.

PROPOSITION 4.7 *There exists a curve* $\tilde{y} : \Re_{++} \to \Re^m$ *such that* $(\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu), \tilde{y}(\nu))$ *is a solution of* (26), (27) *and* (35) *in* $\mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ *for every* $\nu \in (0, 1]$. *Moreover, the path* $\nu \in (0, 1] \to (\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu), \tilde{y}(\nu))$ *remains bounded as* ν *approaches* 0 *and any of its accumulation points is in* $\mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$.

The system formed by (26), (27) and (35) is the one which we will use to establish that the path $\nu \in (0, 1] \to (\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu), \tilde{y}(\nu))$ is analytic at $\nu = 0$. This will follow by the analytic version of the implicit function theorem if we can establish that the Jacobian of this system with $\nu = 0$ with respect to $(\tilde{U}, \tilde{V}, \tilde{X}, \tilde{S}, \tilde{y})$ is nonsingular as long as $(\tilde{U}, \tilde{V}) \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n$. The nonsingularity of this Jacobian can be easily seen to be equivalent to showing that $(\widetilde{\Delta U}, \widetilde{\Delta V}, \widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0, 0, 0, 0, 0) \in \mathcal{U}^n \times \mathcal{U}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ is the only solution of the following linear system:

$$\widetilde{\Delta U} = \begin{bmatrix} \widetilde{\Delta X}_B & \widetilde{\Delta X}_{BN} \\ 0 & \widetilde{\Delta X}_N \end{bmatrix}, \quad \widetilde{\Delta V} = \begin{bmatrix} \widetilde{\Delta S}_B & \widetilde{\Delta S}_{BN} \\ 0 & \widetilde{\Delta S}_N \end{bmatrix}. \tag{36}$$

$$\widetilde{\Delta U} \tilde{V} + \tilde{U} \widetilde{\Delta V} + \left(\widetilde{\Delta U} \tilde{V} + \tilde{U} \widetilde{\Delta V} \right)^T = 0, \tag{37}$$

$$\mathcal{A}_0 \widetilde{\Delta X} = 0, \quad \text{ß}_0 \widetilde{\Delta y} + \widetilde{\Delta S} = 0. \tag{38}$$

LEMMA 4.8 *Assume that* $(\tilde{U}, \tilde{V}) \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n$. *Then, the system* (36)−(38) *has* $(\widetilde{\Delta U}, \widetilde{\Delta V}, \widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0, 0, 0, 0, 0)$ *as its unique solution.*

Proof Using the definition of $\mathcal{A}_\nu$ and ß_ν, we see that the equations in (38) are, respectively,

$$\begin{pmatrix} \mathcal{A}_{11} & 0 & 0 \\ 0 & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{pmatrix} \widetilde{\Delta X} = 0 \tag{39}$$

and

$$\begin{pmatrix} \mathcal{A}_{11}^* & 0 & 0 \\ \mathcal{A}_{12}^* & \mathcal{A}_{22}^* & 0 \\ 0 & 0 & \mathcal{A}_{33}^* \end{pmatrix} \widetilde{\Delta y} + \widetilde{\Delta S} = 0. \tag{40}$$

We easily see from (39) that $\widetilde{\Delta X}_B \in N(\mathcal{A}_{11})$ and $\widetilde{\Delta X}_N \in N(\mathcal{A}_{33})$. We also see from (40) that $\widetilde{\Delta S}_B \in \mathrm{Im}(\mathcal{A}_{11}^*)$ and $\widetilde{\Delta S}_N \in \mathrm{Im}(\mathcal{A}_{33}^*)$. Hence, we conclude that

$$\widetilde{\Delta X}_B \bullet \widetilde{\Delta S}_B = 0, \quad \widetilde{\Delta X}_N \bullet \widetilde{\Delta S}_N = 0.$$

Further, in view of (36), we obtain that

$$\widetilde{\Delta U}_B \bullet \widetilde{\Delta V}_B = 0, \quad \widetilde{\Delta U}_N \bullet \widetilde{\Delta V}_N = 0. \tag{41}$$

Using the fact that $(\tilde{U}, \tilde{V}) \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n$ and $(\widetilde{\Delta U}, \widetilde{\Delta V}) \in \mathcal{U}^n \times \mathcal{U}^n$, we have $\widetilde{\Delta U}\tilde{V} + \tilde{U}\widetilde{\Delta V} \in \mathcal{U}^n$, which together with (37) implies that

$$\widetilde{\Delta U}\tilde{V} + \tilde{U}\widetilde{\Delta V} = 0.$$

This equation can be written as

$$\widetilde{\Delta U}_B \tilde{V}_B + \tilde{U}_B \widetilde{\Delta V}_B = 0, \tag{42}$$

$$\widetilde{\Delta U}_N \tilde{V}_N + \tilde{U}_N \widetilde{\Delta V}_N = 0, \tag{43}$$

$$\widetilde{\Delta U}_B \tilde{V}_{BN} + \widetilde{\Delta U}_{BN} \tilde{V}_N + \tilde{U}_B \widetilde{\Delta V}_{BN} + \tilde{U}_{BN} \widetilde{\Delta V}_N = 0. \tag{44}$$

By virtue of $(\tilde{U}, \tilde{V}) \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n$, we know that $\tilde{U}_B, \tilde{U}_N, \tilde{V}_B, \tilde{V}_N \succ 0$. Multiplying (42) on the left by $(\tilde{U}_B)^{-1/2}$ and on the right by $(\tilde{V}_B)^{-1/2}$, squaring both sides of the resulting expression and using (41), we conclude that

$$\left\| (\tilde{U}_B)^{-1/2} \widetilde{\Delta U}_B (\tilde{V}_B)^{1/2} \right\|_F = 0, \quad \left\| (\tilde{U}_B)^{1/2} \widetilde{\Delta V}_B (\tilde{V}_B)^{-1/2} \right\|_F = 0,$$

from which it follows that $\widetilde{\Delta U}_B = \widetilde{\Delta U}_B = 0$. Similarly, using (43) and the fact $\widetilde{\Delta U}_N \bullet \widetilde{\Delta V}_N = 0$, we have $\widetilde{\Delta U}_N = \widetilde{\Delta U}_N = 0$. Hence, (44) becomes

$$\widetilde{\Delta U}_{BN} \tilde{V}_N + \tilde{U}_B \widetilde{\Delta V}_{BN} = 0. \tag{45}$$

According to (36), we also have

$$\widetilde{\Delta X}_B = \widetilde{\Delta S}_B = 0, \quad \widetilde{\Delta X}_N = \widetilde{\Delta S}_N = 0.$$

In view of Lemma 4.5, $\widetilde{\Delta y}$ can be accordingly partioned into $\widetilde{\Delta y} = (\widetilde{\Delta y}_1, \widetilde{\Delta y}_2, \widetilde{\Delta y}_3) \in \Re^{i_1} \times \Re^{i_2} \times \Re^{i_3}$. This together with (40) implies that $\mathcal{A}_{11}^* \widetilde{\Delta y}_1 + \widetilde{\Delta S}_B = 0$. Using this identity and the fact that $\widetilde{\Delta S}_B = 0$ and $\mathcal{A}_{11}^*$ is one-to-one, we obtain that $\widetilde{\Delta y}_1 = 0$, which together with (40) implies that $\widetilde{\Delta S}_{BN} \in \mathrm{Im}(\mathcal{A}_{22}^*)$. Similarly, we have $\mathcal{A}_{22}(\widetilde{\Delta X}_{BN}) = 0$. Hence, we conclude that $\widetilde{\Delta X}_{BN} \bullet \widetilde{\Delta S}_{BN} = 0$, which together with (36) implies $\widetilde{\Delta U}_{BN} \bullet \widetilde{\Delta V}_{BN} = 0$. Using this identity and (45), and applying the same argument as above, we obtain that $\widetilde{\Delta U}_{BN} = \widetilde{\Delta V}_{BN} = 0$. Again, in view of (36), we have $\widetilde{\Delta X}_{BN} = \widetilde{\Delta S}_{BN} = 0$. Hence, we conclude that

$$\widetilde{\Delta U} = \widetilde{\Delta V} = \widetilde{\Delta X} = \widetilde{\Delta S} = 0.$$

Also, $\widetilde{\Delta y} = 0$ follows from (38) and the fact that $\widetilde{\Delta S} = 0$ and $\text{\ss}_0$ is one-to-one. $\blacksquare$

We are now ready to establish the analyticity of the path $v \in (0, 1] \to (\tilde{U}(v), \tilde{V}(v), \tilde{X}(v), \tilde{S}(v), \tilde{y}(v))$.

THEOREM 4.9   *Let* $(X^*, S^*, y^*) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$ *be given. There hold*:

(i) *the path* $v \in (0, 1] \to \tilde{p}(v) \equiv (\tilde{U}(v), \tilde{V}(v), \tilde{X}(v), \tilde{S}(v), \tilde{y}(v))$ *is analytic at* 0; *consequently,* $\tilde{p}(v)$ *and all its k-th order derivatives,* $k \geq 1$, *converge as* $v \downarrow 0$;
(ii) $(\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*) \equiv \lim_{v \downarrow 0}(\tilde{U}(v), \tilde{V}(v), \tilde{X}(v), \tilde{S}(v), \tilde{y}(v))$ *is the unique solution of the system defined by* (30), (31) *and*

$$\mathcal{A}_0(\tilde{X} - X^*) = \begin{pmatrix} 0 \\ \widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix}, \quad \text{ß}_0\tilde{y} + (\tilde{S} - S^*) = \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \\ 0 \end{pmatrix} \tag{46}$$

in $\mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$;
(iii) $\left(\widetilde{\delta U}^*, \widetilde{\delta V}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^*\right) \equiv \lim_{v \downarrow 0} \left(\dot{\tilde{U}}(v), \dot{\tilde{V}}(v), \dot{\tilde{X}}(v), \dot{\tilde{S}}(v), \dot{\tilde{y}}(v)\right)$ *is the unique solution of the linear system defined by*

$$\widetilde{\delta U} = \begin{bmatrix} \widetilde{\delta X_B} & \widetilde{\delta X_{BN}} \\ 0 & \widetilde{\delta X_N} \end{bmatrix}, \quad \widetilde{\delta V} = \begin{bmatrix} \widetilde{\delta S_B} & \widetilde{\delta S_{BN}} \\ 0 & \widetilde{\delta S_N} \end{bmatrix}, \tag{47}$$

$$\widetilde{\delta U}\,\tilde{V}^* + \tilde{U}^*\widetilde{\delta V} + \left(\widetilde{\delta U}\,\tilde{V}^* + \tilde{U}^*\widetilde{\delta V}\right)^T$$
$$= -\left[\mathcal{L}(\tilde{U}^*)\tilde{V}^* + \tilde{U}^*\mathcal{L}(\tilde{V}^*) + (\mathcal{L}(\tilde{U}^*)\tilde{V}^* + \tilde{U}^*\mathcal{L}(\tilde{V}^*))^T\right], \tag{48}$$

$$\mathcal{A}_0\widetilde{\delta X} = -\mathcal{C}_0\tilde{X}^* + \begin{pmatrix} \widetilde{\Delta b_1} \\ 0 \\ 0 \end{pmatrix}, \quad \beta_0\widetilde{\delta y} + \widetilde{\delta S} = -\mathcal{D}_0\tilde{y}^* + \begin{pmatrix} 0 \\ 0 \\ \Delta C_N \end{pmatrix}, \tag{49}$$

where

$$\mathcal{C}_0 \equiv \begin{pmatrix} 0 & \mathcal{A}_{12} & \mathcal{A}_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{D}_0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{A}_{13}^* & \mathcal{A}_{23}^* & 0 \end{pmatrix}.$$

Proof   Let $\mathcal{O} = \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ and $H(w, v) \equiv H(\tilde{U}, \tilde{V}, \tilde{X}, \tilde{S}, \tilde{y}, v)$ be the map defined by system (26), (27) and (35). Indeed, $H(w, v)$ is analytic of w and v. By Proposition 4.7, the path $\tilde{p}(v) = (\tilde{U}(v), \tilde{V}(v), \tilde{X}(v), \tilde{S}(v), \tilde{y}(v))$ has an accumulation point $w^* = (\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ in $\mathcal{O}$, which satisfies $H(w^*, 0) = 0$. By Lemma 4.8, it follows that $H'_w(w^*, 0)$ is nonsingular. In view of implicit function theorem, there exists a $\delta > 0$ and an analytical function $\hat{p}(v) = (\hat{U}(v), \hat{V}(v), \hat{X}(v), \hat{S}(v), \hat{y}(v)) \in \mathcal{O}$ defined on $(-\delta, \delta)$ such that $H(\hat{p}(v), v) = 0$ for every $v \in (-\delta, \delta)$ and $\hat{p}(0) = w^*$. Hence, it follows from Lemma 4.4 that $\hat{X}_B(0) = \tilde{X}_B^* = \tilde{U}_B^* \succ 0$. Similarly, we have that $\hat{X}_N(0), \hat{S}_B(0), \hat{S}_N(0) \succ 0$. Now, for $v \in (0, \delta)$, let

$$\check{x}(v) \equiv \begin{bmatrix} \hat{X}_B(v) & v\hat{X}_{BN}(v) \\ v\hat{X}_{NB}(v) & v\hat{X}_N(v) \end{bmatrix}, \quad \check{S}(v) \equiv \begin{bmatrix} v\hat{S}_B(v) & v\hat{S}_{BN}(v) \\ v\hat{S}_{NB}(v) & \hat{S}_N(v) \end{bmatrix}. \tag{50}$$

Using the fact that (28) is equivalent to (33) for $v \in (0, 1]$, we see that $\check{X}(v)$ satisfies $\mathcal{A}X = b$ for $v \in (0, \delta)$. Similarly, we have $\check{S}(v) - S^* \in \text{Im}(\mathcal{A}^*)$, which together with the

fact $S^* \in C + \mathrm{Im}(\mathcal{A}^*)$, implies $\check{S}(\nu) \in C + \mathrm{Im}(\mathcal{A}^*)$. Hence, there exists $\check{y}(\nu) \in \Re^m$ such that $(\check{s}(\nu), \check{y}(\nu))$ satisfies $\mathcal{A}^* y + S = C$ for $\nu \in (0, \delta)$. By virtue of (26), (27) and (50), we see that $(\check{X}(\nu), \check{S}(\nu))$ satisfies $XS + SX = 2\nu W$ for $\nu \in (0, \delta)$. In view of (50) and the fact that $\hat{X}_B(0), \hat{X}_N(0), \hat{S}_B(0), \hat{S}_N(0) \succ 0$, there exists an $\epsilon \in (0, \delta)$ such that $\check{X}(\nu) \succ 0$ and $\check{S}(\nu) \succ 0$ for every $\nu \in (0, \epsilon)$. Hence, for every $\nu \in (0, \epsilon)$, $(\check{X}(\nu), \check{S}(\nu), \check{y}(\nu))$ satisfies (3)–(5). By Proposition 2.1, we have $(\check{X}(\nu), \check{S}(\nu), \check{y}(\nu)) = (X(\nu), S(\nu), y(\nu))$ for every $\nu \in (0, \epsilon)$. According to (25) and (50), we obtain that $\check{X}(\nu) = \hat{X}(\nu)$ and $\check{S}(\nu) = \hat{S}(\nu)$ for all $\nu \in (0, \epsilon)$. Using (35) and the fact that $ß_\nu$ is one-to-one, we have $\tilde{y}(\nu) = \hat{y}(\nu)$ for all $\nu \in (0, \epsilon)$. Hence, we conclude that $\tilde{p}(\nu) = \hat{p}(\nu)$ for all $\nu \in (0, \epsilon)$. In term of definition 4.1, it follows that (i) holds.

Upon letting $\nu \downarrow 0$ on $H(\tilde{p}(\nu), \nu) = 0$, we easily see that w^* satisfies (30), (31) and (46). The proof of uniqueness follows from the similar argument as above. Indeed, if the system $H(w, 0) = 0$ has another solution $\hat{w}^* \in \mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$. By Lemma 4.8, it follows that $H_w'(\hat{w}^*, 0)$ is nonsingular. The implicit function theorem implies that the system $H(w, \nu) = 0$ has a different solution from $\tilde{p}(\nu)$ in a small neighborhood of $\nu = 0$. By the similar argument as above, then there exists two distinct weighted paths in a small neighborhood of $\nu = 0$. It contradicts with Proposition 2.1. Differentiating the identity $H(w(\nu), \nu) = 0$ with respect to ν and letting $\nu \downarrow 0$, we conclude that $\delta w = \delta w^* \equiv \left(\widetilde{\delta U}^*, \widetilde{\delta V}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^* \right)$ satisfies

$$H_w'(w^*, 0)\delta w = -H_\nu'(w^*, 0).$$

Statement (iii) now follows from the fact that $H_w'(w^*, 0)$ is nonsingular and the latter system is equivalent to (47)–(49). ∎

The proof of Theorem 4.2 is now obvious. Indeed, the analyticity of the map $\nu \to (X(\nu), S(\nu))$ follows from (25) and the analyticity of $\nu \to (\tilde{X}(\nu), \tilde{S}(\nu))$. The analyticity of $\nu \to y(\nu)$ follows from the analyticity of $\nu \to S(\nu)$ and Assumption **A.1**. The last statement of the theorem is obvious.

In the remainder of this paper, we will let $\left(\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^* \right)$ and $\left(\widetilde{\delta U}^*, \widetilde{\delta V}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*, \widetilde{\delta y}^* \right)$ denote the limits of $(\tilde{U}(\nu), \tilde{V}(\nu), \tilde{X}(\nu), \tilde{S}(\nu), \tilde{y}(\nu))$ and $(\dot{\tilde{U}}(\nu), \dot{\tilde{V}}(\nu), \dot{\tilde{X}}(\nu), \dot{\tilde{S}}(\nu), \dot{\tilde{y}}(\nu))$, respectively, as $\nu \downarrow 0$ (as in Theorem 4.9 above). Observe that Theorem 4.9 provides a characterization of $(\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^*, \tilde{y}^*)$ as being the unique solution of a certain system of equations, which arises by first performing some transformations to the original weighted central path system, and then setting $\nu = 0$ in the resulting system. Hence, it is reasonable to expect that the linear equations (46) can be entirely described in terms of the original data $(W, \mathcal{A}, C, \Delta C, b, \Delta b)$. Indeed, the following result gives this alternative description of (46).

THEOREM 4.10 $\left(\tilde{U}^*, \tilde{V}^*, \tilde{X}^*, \tilde{S}^* \right)$ *is the unique solution of the system given by* (30), (31) *and the linear equations*

$$\mathcal{A}_B(\tilde{X}_B) = b, \quad \begin{bmatrix} \mathcal{A}_{BN} & \mathcal{A}_N \end{bmatrix} \begin{bmatrix} \tilde{X}_{BN} \\ \tilde{X}_N \end{bmatrix} \in \Delta b + \mathrm{Im}(\mathcal{A}_B), \tag{51}$$

$$\begin{pmatrix} \tilde{S}_B \\ \tilde{S}_{BN} \end{pmatrix} \in \begin{pmatrix} \Delta C_B \\ \Delta C_{BN} \end{pmatrix} + \mathrm{Im}\left(\begin{bmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \end{bmatrix} \right), \quad \begin{pmatrix} 0 \\ 0 \\ \tilde{S}_N \end{pmatrix} \in C + \mathrm{Im}\left(\begin{bmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \\ \mathcal{A}_N^* \end{bmatrix} \right) \tag{52}$$

in $\mathcal{U}_{++}^n \times \mathcal{U}_{++}^n \times \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n$.

Proof From Theorem 4.9(ii), it suffices to show that (46) is equivalent to (51) and (52). Since the first equation of (46) is the same as (33) with $\nu = 0$, we have that the first equation of (46) holds if and only if

$$\mathcal{A}_{11}(\tilde{X}_B) = \mathcal{A}_{11}(X_B^*), \quad \mathcal{A}_{22}(\tilde{X}_{BN}) + \mathcal{A}_{23}(\tilde{X}_N) = \widetilde{\Delta b}_2, \quad \mathcal{A}_{33}(\tilde{X}_N) = \widetilde{\Delta b}_3. \tag{53}$$

By Lemma 4.5, the first identity in (53) can be written as

$$(T \circ \mathcal{A}) \begin{pmatrix} \tilde{X}_B \\ 0 \\ 0 \end{pmatrix} = (T \circ \mathcal{A}) \begin{pmatrix} X_B^* \\ 0 \\ 0 \end{pmatrix},$$

and hence it is equivalent to $\mathcal{A}_B(\tilde{X}_B) = \mathcal{A}_B(X_B^*) = b$, in view of relation (32) and the fact that T is an isomorphism. By Lemma 4.5 and the fact that $\mathcal{A}_{11}$ is onto, the second and third identities in (53) hold if and only if

$$(T \circ \mathcal{A}) \begin{pmatrix} \check{X}_B \\ \tilde{X}_{BN} \\ \tilde{X}_N \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta b}_1 \\ \widetilde{\Delta b}_2 \\ \widetilde{\Delta b}_3 \end{pmatrix} = T(\Delta b)$$

for some $\check{X}_B \in \mathcal{S}^{|B|}$, and hence it is equivalent to $\mathcal{A}_{BN}(\tilde{X}_{BN}) + \mathcal{A}_N(\tilde{X}_N) \in \Delta b + \mathrm{Im}(\mathcal{A}_B)$, in view of (32) and the fact that T is an isomorphism. We have thus shown that the first equation of (46) is equivalent to (51).

The fact that the second equation of (46) holds if and only if (52) holds can be proved in a similar way as above. ∎

The following result gives an alternative characterization of $(\widetilde{\delta U}^*, \widetilde{\delta V}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*)$ involving the original data $(W, \mathcal{A}, C, \Delta C, b, \Delta b)$.

THEOREM 4.11 $\left(\widetilde{\delta U}^*, \widetilde{\delta V}^*, \widetilde{\delta X}^*, \widetilde{\delta S}^*\right)$ *is the unique solution of the linear system of equations* (47), (48) *and*

$$\begin{bmatrix} \mathcal{A}_B & \mathcal{A}_{BN} & \mathcal{A}_N \end{bmatrix} \begin{bmatrix} \widetilde{\delta X}_B \\ \tilde{X}_{BN}^* \\ \tilde{X}_N^* \end{bmatrix} = \Delta b, \quad \begin{bmatrix} \mathcal{A}_{BN} & \mathcal{A}_N \end{bmatrix} \begin{bmatrix} \widetilde{\delta X}_{BN} \\ \widetilde{\delta X}_N \end{bmatrix} \in \mathrm{Im}(\mathcal{A}_B), \tag{54}$$

$$\begin{pmatrix} \widetilde{\delta S}_B \\ \widetilde{\delta S}_{BN} \end{pmatrix} \in \mathrm{Im}\left(\begin{bmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \end{bmatrix}\right), \quad \begin{pmatrix} \tilde{S}_B^* \\ \tilde{S}_{BN}^* \\ \widetilde{\delta S}_N \end{pmatrix} \in \Delta C + \mathrm{Im}\left(\begin{bmatrix} \mathcal{A}_B^* \\ \mathcal{A}_{BN}^* \\ \mathcal{A}_N^* \end{bmatrix}\right). \tag{55}$$

Proof From Theorem 4.9(iii), it suffices to show that (49) is equivalent to (54) and (55). Observe that the first equation of (49) can be written as

$$\begin{aligned} \mathcal{A}_{11}(\widetilde{\delta S}_B) + \mathcal{A}_{12}(\tilde{X}_{BN}^*) + \mathcal{A}_{13}(\tilde{X}_N^*) &= \widetilde{\Delta b}_1, \\ \mathcal{A}_{22}(\widetilde{\delta X}_{BN}) + \mathcal{A}_{23}(\widetilde{\delta X}_N) &= 0, \\ \mathcal{A}_{33}(\widetilde{\delta X}_N) &= 0. \end{aligned} \tag{56}$$

Using Lemma 4.5, the fact that $\mathcal{A}_{11}$ is onto and the identities $\mathcal{A}_{22}(\tilde{X}_{BN}^*) + \mathcal{A}_{23}(\tilde{X}_N^*) = \widetilde{\Delta b}_2$ and $\mathcal{A}_{33}\tilde{X}_N^* = \widetilde{\Delta b}_3$ that hold in view of (46), we easily see that the first and last two equations

above are respectively equivalent to

$$(T \circ \mathcal{A}) \begin{pmatrix} \widetilde{\delta S_B} \\ \tilde{X}^*_{BN} \\ \tilde{X}^*_N \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta b_1} \\ \widetilde{\Delta b_2} \\ \widetilde{\Delta b_3} \end{pmatrix} = T(\Delta b), \ (T \circ \mathcal{A}) \begin{pmatrix} \check{X}_B \\ \widetilde{\delta X}_{BN} \\ \delta b_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for some $\check{X}_B \in \mathcal{S}^{|B|}$. The latter conditions in turn are equivalent to (54) in view of (32) and the fact that T is an isomorphism.

Using similar arguments as to ones used above, it can be shown that the second equation of (49) holds if and only if (55) holds. $\blacksquare$

5. Error bound analysis

By strengthening some of the results of the previous sections, in this section we derive an error bound on the distance of a point lying in a certain neighborhood of the central path to the primal–dual optimal set by means of a standard technique (e.g., see Stoer *et al.* [42], and Preiß and Stoer [43].

For any given nonempty compact set $\mathcal{K} \subset \mathcal{G}_{++}$ and constants $\gamma_1, \gamma_2, \tau > 0$ with $\gamma_2 \geq \gamma_1$, define

$$\mathcal{N}(\gamma_1, \gamma_2, \tau, \mathcal{K}) \equiv \left\{ (X, S, y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \mathfrak{R}^m : \begin{array}{c} \mathcal{G}(X, S, y) \in \tau\mathcal{K}, \\ \gamma_1 \tau I \preceq \dfrac{XS + SX}{2} \preceq \gamma_2 \tau I \end{array} \right\},$$

where the map $\mathcal{G}$ and the set $\mathcal{G}_{++}$ are defined in (7) and (8), respectively.

Observe that the set $\cup_{\tau > 0} \mathcal{N}(\gamma_1, \gamma_2, \tau, \mathcal{K})$ forms a neighborhood of the primal–dual central path. This neighborhood and related ones have once been used in the development of primal–dual interior point algorithms for SDP. For example, see Kojima *et al.* [39].

The following result gives an error bound on the distance of a point lying in $\mathcal{N}(\gamma_1, \gamma_2, \tau, \mathcal{K})$ to the primal–dual optimal set $\mathcal{F}^*_P \times \mathcal{F}^*_D$. Its proof will be given at the end of this section after we have derived stronger versions of the results of the previous sections.

THEOREM 5.1   *Let $\gamma_2 \geq \gamma_1 > 0$ and any nonempty compact set $\mathcal{K} \subset \mathcal{G}_{++}$ be given. Then, there exists a constant $M = M(\gamma_1, \gamma_2, \mathcal{K}) > 0$ such that*

$$\text{dist}((X, S, y), \mathcal{F}^*_P \times \mathcal{F}^*_D) \leq M\tau, \tag{57}$$

for every $\tau \in (0, 1]$ and $(X, S, y) \in \mathcal{N}(\gamma_1, \gamma_2, \tau, \mathcal{K})$.

In view of Proposition 2.1, for each $(\nu, W, \Delta C, \Delta b) \in (0, 1] \times \mathcal{S}^n_{++} \times \mathcal{G}_{++}$, the system of nonlinear equations (3)–(5) has a unique solution, which in this section we denote by $(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b), y(\nu, W, \Delta C, \Delta b))$ in order to emphasize and study its dependence on $(W, \Delta C, \Delta b)$. Moreover, in view of Theorem 4.2, the limit

$$\lim_{\nu \downarrow 0} (X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b), y(\nu, W, \Delta C, \Delta b)),$$

denoted by $(X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b), y(0, W, \Delta C, \Delta b))$, exists for every $(W, \Delta C, \Delta b) \in \mathcal{S}^n_{++} \times \mathcal{G}_{++}$. Hence, the functions $X(\cdot, \cdot, \cdot, \cdot)$, $S(\cdot, \cdot, \cdot, \cdot)$ and $y(\cdot, \cdot, \cdot, \cdot)$ are well-defined over the set $[0, 1] \times \mathcal{S}^n_{++} \times \mathcal{G}_{++}$. In an obvious way, we can also define the

functions $\tilde{X}(\nu, W, \Delta C, \Delta b)$, $\tilde{S}(\nu, W, \Delta C, \Delta b)$ and $\tilde{y}(\nu, W, \Delta C, \Delta b)$ over the set $[0, 1] \times \mathcal{S}_{++}^n \times \mathcal{G}_{++}$.

It turns out that the above functions are analytic according to the following definition. We say that a function $f : \Omega \subseteq E \to F$, where E, F are two finite dimensional normed vector spaces, is analytic if there exists an open set $\mathcal{O} \subseteq E$ containing Ω and an analytic function $\tilde{f} : \mathcal{O} \to F$ such that $\tilde{f}$ restricted to Ω is equal to f.

THEOREM 5.2 *There hold*:

(i) *the map* $(\nu, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{S}_{++}^n \times \mathcal{G}_{++} \to (\tilde{X}(\nu, W, \Delta C, \Delta b)$, $\tilde{S}(\nu, W, \Delta C, \Delta b)$, $\tilde{y}(\nu, W, \Delta C, \Delta b))$ *is analytic*;

(ii) *the map* $(\nu, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{S}_{++}^n \times \mathcal{G}_{++} \to (X(\nu, W, \Delta C, \Delta b)$, $S(\nu, W, \Delta C, \Delta b)$, $y(\nu, W, \Delta C, \Delta b))$ *is analytic*.

Proof The proof of the theorem is identical to the proof of Theorem 4.9 and Theorem 4.2, except that when invoking the implicit function theorem, we should view $(\nu, W, \Delta C, \Delta b)$ as the parameter vector. ∎

THEOREM 5.3 *Let* $\gamma_2 \geq \gamma_1 > 0$ *be given. Then, for all* $(\nu, W, \Delta C, \Delta b) \in [0, 1] \times \mathcal{W}(\gamma_1, \gamma_2) \times \mathcal{K}$, *there exists a constant* $M = M(\gamma_1, \gamma_2, \mathcal{K}) > 0$ *such that*

$$\|(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b)) - (X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b))\| \leq M\nu,$$

where $\mathcal{W}(\gamma_1, \gamma_2) \equiv \{W \in \mathcal{S}^n : \gamma_1 I \preceq W \preceq \gamma_2 I\}$.

Proof By the mean value theorem, we have

$$\|(X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b)) - (X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b))\|$$
$$\leq \sup_{\theta \in [0,1]} \|(X'(\theta\nu, W, \Delta C, \Delta b), S'(\theta\nu, W, \Delta C, \Delta b))\|\nu.$$

By Theorem 5.2(ii) and the fact that $\mathcal{W}(\gamma_1, \gamma_2) \times \mathcal{K}$ is compact, there exists a constant $M = M(\gamma, \mathcal{K}) > 0$ such that

$$\|(X'(\theta\nu, W, \Delta C, \Delta b), S'(\theta\nu, W, \Delta C, \Delta b))\| \leq M$$

for all $(\theta, \nu, W, \Delta C, \Delta b) \in [0, 1] \times [0, 1] \times \mathcal{W}(\gamma_1, \gamma_2) \times \mathcal{K}$. Hence, the conclusion follows. ∎

The proof of Theorem 5.1 now follows from Assumption **A.1** and Theorem 5.3 with $\nu = \tau$, $W = (XS + SX)/2\tau$, $(X, S) = (X(\nu, W, \Delta C, \Delta b), S(\nu, W, \Delta C, \Delta b))$ and the fact $(X(0, W, \Delta C, \Delta b), S(0, W, \Delta C, \Delta b), y(0, W, \Delta C, \Delta b)) \in \mathcal{F}_P^* \times \mathcal{F}_D^*$.

6. Some observation for superlinear convergence

In this section, we will use the error bound derived in Theorem 5.3 to make some observation for the superlinear convergence of some primal–dual interior point algorithms for SDP using AHO neighborhood, for example, Kojima *et al.* [39].

Potra and Sheng [44] have developed a primal–dual infeasible-interior-point algorithm which, for some $\alpha \in (0, 1/2]$, generates a sequence of iterates $\{(X^k, S^k, y^k)\} \subseteq \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$ satisfying

$$\|W^k - I\|_F \leq \alpha, \quad r_p^k = \frac{\tau_k}{\tau_0} r_p^0, \quad r_d^k = \frac{\tau_k}{\tau_0} r_d^0, \tag{58}$$

for some sequence $\{\tau_k\} \subset \Re_{++}$ converging to 0 at least Q-linearly, where

$$r_p^k \equiv \mathcal{A}X^k - b,$$
$$r_d^k \equiv \mathcal{A}^* y^k + S^k - C,$$
$$W^k \equiv \frac{(X^k)^{1/2} S^k (X^k)^{1/2}}{\tau_k},$$

for all $k \geq 0$. The derived linear rate of convergence of the sequence $\{\tau_k\}$ is sufficient to guarantee polynomial convergence of their method under some suitable conditions on the initial point (X^0, S^0, y^0). However, some sufficient conditions are needed to guarantee the Q-superlinear convergence of $\{\tau_k\}$ to zero. One such condition is the tangential condition proposed by Kojima *et al.* [45], namely

$$\lim_{k \to \infty} W^k = I. \tag{59}$$

Another such condition is the one that has been proposed by Potra and Sheng [46], namely

$$\lim_{k \to \infty} \frac{X^k S^k}{\sqrt{\tau_k}} = 0. \tag{60}$$

We remark that Potra and Sheng [46] have shown that the tangential condition (59) implies their condition (60). Recently, Lu and Monteiro [37] have shown that the condition (60) is equivalent to a natural condition

$$\lim_{k \to \infty} W^k_{BN} = 0.$$

The following result shows that the condition (60) automatically holds when the iterates $\{(X^k, S^k, y^k)\}$ are in an AHO neighborhood of the central path, namely,

$$\gamma_1 I \preceq W^k \preceq \gamma_2 I, \quad r_p^k = \frac{\tau_k}{\tau_0} r_p^0, \quad r_d^k = \frac{\tau_k}{\tau_0} r_d^0, \tag{61}$$

where $\gamma_2 \geq \gamma_1 > 0$ are given and $W^k = (X^k S^k + S^k X^k)/(2\tau_k)$ for $k \geq 0$.

THEOREM 6.1 *Assume that the iterates $\{(X^k, S^k, y^k)\} \subseteq \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$ satisfy* (61) *for all $k \geq 0$. Then, $X^k S^k = \mathcal{O}(\tau_k)$ holds.*

Proof We easily see that the set $\mathcal{K} = \{(r_p^0/\tau_0, r_p^0/\tau_0)\} \subset \mathcal{G}_{++}$ is nonempty compact. Moreover, we know that for some constants $\gamma_2 \geq \gamma_1 > 0$, $\gamma_1 I \preceq W^k \preceq \gamma_2 I$ for all $k \geq 0$. Hence, noting that $X^k S^k + S^k X^k = 2\tau_k W^k$, it follows from Theorem 5.2(ii) and 5.3 that

$$X^k = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\tau_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(\tau_k) \end{pmatrix}, \quad S^k = \begin{pmatrix} \mathcal{O}(\tau_k) & \mathcal{O}(\tau_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(1) \end{pmatrix},$$

and hence

$$X^k S^k = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\tau_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(\tau_k) \end{pmatrix} \begin{pmatrix} \mathcal{O}(\tau_k) & \mathcal{O}(\tau_k) \\ \mathcal{O}(\tau_k) & \mathcal{O}(1) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\tau_k) & \mathcal{O}(\tau_k) \\ \mathcal{O}(\tau_k^2) & \mathcal{O}(\tau_k) \end{pmatrix} = \mathcal{O}(\tau_k). \qquad \blacksquare$$

Theorem 6.1 implies that a primal–dual interior point algorithm for SDPs using AHO neighborhood automatically satisfies the condition (60) with no need to perform multiple centrality steps (e.g., see Luo *et al.* [30]) between two consecutive standard steps. This indirectly explains why it is more likely to be naturally superlinearly convergent for this type of algorithm. Actually, Kojima *et al.* [39], and Lu and Monteiro [47] have shown the local quadratic convergence for a predictor-corrector infeasible-interior-point algorithm using AHO neighborhood, which was proposed by Kojima *et al.* [39] for the monotone semidefinite linear complementarity problem (which is equivalent to an SDP).

Acknowledgements

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