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A POTENTIAL REDUCTION NEWTON METHOD FOR CONSTRAINED EQUATIONS*

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Abstract. Extending our previous work [T. Wang, R. D. C. Monteiro, and J.-S. Pang, *Math. Programming*, 74 (1996), pp. 159–195], this paper presents a general potential reduction Newton method for solving a constrained system of nonlinear equations. A major convergence result for the method is established. Specializations of the method to a convex semidefinite program and a monotone complementarity problem in symmetric matrices are discussed. Strengthened convergence results are established in the context of these specializations.

Key words. potential reduction algorithm, constrained equation, Newton method, interior point methods, global convergence, potential function, complementarity problems, variational inequality, semidefinite programming, primal-dual methods

AMS subject classifications. 65K05, 90C25, 90C33

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1. Introduction.

In the paper [36], we have introduced the problem of solving a system of nonlinear equations subject to additional constraints on the variables, i.e., a *constrained* system of equations. We have demonstrated that constrained equations (CEs) provide a unifying framework for the study of complementarity problems of various types, including the standard nonlinear complementarity problem and the Karush–Kuhn–Tucker system of a variational inequality. Postulating a partitioning property of the CE, we have introduced an interior point potential reduction algorithm for solving the CE and have applied this method to convex programs and monotone complementarity problems of different kinds. The goal of this paper is to present a potential reduction Newton method for solving a CE, without assuming the existence of the partitioning property that is key to the previous work.

The central problem studied in section 2 of this paper is as follows. Let $H : \Re^n \to \Re^n$ be a given mapping from the real Euclidean space $\Re^n$ into itself and let Ω be a given closed subset of $\Re^n$. The constrained equation defined by the pair (Ω, H) is to find a vector $x \in \Re^n$ such that

$$H(x) = 0, \quad x \in \Omega.$$

We refer the reader to [36] for the initial motivation to study the CE. The method proposed in this paper for solving the CE (Ω, H) combines ideas from the classical damped Newton method for solving the unconstrained system of equations $H(x) = 0$, $x \in \Re^n$, and the family of interior point methods for solving constrained optimization and complementarity problems. A general convergence theory for the proposed

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method is presented in section 2.4. Unlike the previous study [36], where we assume that the function $H(x)$ has a certain partition conformal to the set Ω, we make no such assumption herein. Instead, the present work is based on a set of broad hypotheses on the pair (Ω, H).

In sections 3 and 4, we consider applications of our results to a monotone complementarity problem and a semidefinite convex program on the cone of positive semidefinite matrices. These applications yield new interior point methods for solving these problems whose convergence can be established under some mild assumptions. It should be noted that many interior point methods for the linear version of these problems have been proposed in the literature (e.g., see [1, 2, 3, 4, 6, 9, 10, 11, 12, 15, 16, 19, 20, 22, 23, 24, 25, 26, 27, 29, 32, 34, 35, 37]).

We explain some terminology and fix the notation used throughout the paper. For a given subset S of $\Re^n$, we let int S, cl S, and bd S denote, respectively, the interior, closure, and boundary of S. If the mapping H is (Fréchet) differentiable at a point x in its domain, the Jacobian matrix of H at x is denoted $H'(x)$; thus the (i,j)-entry of $H'(x)$ is equal to $\partial H_i(x)/\partial x_j$ for $i,j = 1,\dots,n$. We write $H'(x;v) \equiv H'(x)v$ for any vector $v \in \Re^n$; thus $H'(x;v)$ is the Fréchet derivative of H at x along the direction v. If $H(x,y)$ is a function of two arguments $(x,y) \in \Re^{n+m}$, then H'_x denotes the partial Jacobian matrix of H with respect to the variable x. For a real-valued function $\phi : \Re^n \to \Re$, we write $\nabla\phi(x)$ for the gradient vector of ϕ at the vector $x \in \Re^n$. The p-norm of a vector x is denoted by $\|x\|_p$; in particular, its 2-norm or Euclidean norm is denoted by $\|x\|$. For a vector $a \in \Re^n$, we let $[0,a]$ denote the line segment joining the origin and a. For a positive vector u, we let u^{-1} denote the vector whose components are the reciprocals of the corresponding components of u. For a mapping $G : M \to N$ with domain M, range N, and subsets $D \subset M$ and $E \subset N$, we let

$$G(D) \equiv \{ G(u) : u \in D \} \quad \text{and} \quad G^{-1}(E) \equiv \{ u \in M : G(u) \in E \}.$$

The set of real matrices of order n is denoted by $\mathcal{M}^n$; the subset of symmetric matrices in $\mathcal{M}^n$ is denoted by $\mathcal{S}^n$. The set $\mathcal{M}^n$ forms a finite-dimensional inner-product vector space with the inner product given by

$$X \bullet Y \equiv \mathrm{tr}(X^T Y), \quad (X,Y) \in \mathcal{M}^n,$$

where "tr" denotes the trace of a matrix. This inner product induces the Frobenius norm for matrices given by

$$\| X \|_F \equiv \sqrt{\mathrm{tr}(X^T X)}, \quad X \in \mathcal{M}^n.$$

The subsets of $\mathcal{S}^n$ consisting of the positive semidefinite and positive definite matrices are denoted by $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$, respectively. For two matrices A and B in $\mathcal{S}^n$, we write $A \preceq B$ if $B - A \in \mathcal{S}^n_+$; similarly, $A \prec B$ means $B - A \in \mathcal{S}^n_{++}$. For any matrix $A \in \mathcal{S}^n_+$, $A^{1/2}$ denotes the square root of A; i.e., $A^{1/2}$ is the unique matrix in $\mathcal{S}^n_+$ such that $(A^{1/2})^2 = A$.

2. Description and analysis of the algorithm. In this section, we describe the potential reduction Newton algorithm for solving the CE (Ω, H), where Ω is a closed subset of $\Re^n$ and H is a continuous mapping from $\Re^n$ into itself. This section is divided into four subsections as follows: in the first subsection, we lay down the basic assumptions on the pair (Ω, H); in the second subsection, we give some results which guarantee the existence of a solution for the CE (Ω, H); in the third subsection, we present the detailed statement of the algorithm; in the fourth subsection, we establish a convergence theorem for the algorithm.

2.1. Basic assumptions. We introduce several key assumptions on the pair (Ω, H). Subsequently, these assumptions will be verified in the context of several applications of the CE. Among these assumptions, we postulate the existence of a closed convex subset S that relates to the range of H and possesses certain special properties. Based on such a set S and a corresponding potential function p, an algorithm for solving the CE is developed. Part of the generality of the present framework stems from the freedom in the choice of S. There are two immediate benefits of this generality. One is that our framework provides a unified basis for the study of many iterative algorithms for solving nonlinear equations and mathematical programs. More importantly, the other benefit is that new algorithms can be constructed with novel choices of S. Of particular interest is the construction of sets S and associated potential functions that depend on given starting points. These details will appear in subsequent sections. The blanket assumptions are as follows.

(A1) The closed set Ω has a nonempty interior.

(A2) There exists a closed convex set $S \subset \Re^n$ such that
- (a) $0 \in S$;
- (b) the (open) set $\Omega_{\mathcal{I}} \equiv H^{-1}(\text{int } S) \cap \text{int } \Omega$ is nonempty;
- (c) the set $H^{-1}(\text{int } S) \cap \text{bd } \Omega$ is empty.

(A3) H is continuously differentiable on $\Omega_{\mathcal{I}}$, and $H'(x)$ is nonsingular for all $x \in \Omega_{\mathcal{I}}$.

Assumption (A1) is needed for the applicability of an interior point method. The sets S and $\Omega_{\mathcal{I}}$ in assumption (A2) contain the key elements of the proposed algorithm. (As noted by a referee, if H is considered to be a mapping with domain Ω, conditions (b) and (c) in (A2) are equivalent to the condition that $\emptyset \neq H^{-1}(\text{int } S) \subset \text{int } \Omega$.) Whereas S pertains to the range of H, $\Omega_{\mathcal{I}}$ pertains to the domain. Initiated at a vector x^0 in $\Omega_{\mathcal{I}}$, the algorithm generates a sequence of iterates $\{x^k\} \subset \Omega_{\mathcal{I}}$ so that the sequence $\{H(x^k)\} \subset \text{int } S$ will eventually converge to zero, thus accomplishing the goal of solving the CE (Ω, H), at least approximately. Assumption (A3) facilitates the application of a Newton scheme for the generation of $\{x^k\}$; this scheme relies on a potential function for the set $\Omega_{\mathcal{I}}$ that is induced by such a function for int S. Specifically, we postulate the existence of a potential function $p : \text{int } S \to \Re$ satisfying the following properties:

(A4) for every sequence $\{u^k\} \subset \text{int } S$ such that

$$\text{either } \lim_{k\to\infty} \|u^k\| = \infty \text{ or } \lim_{k\to\infty} u^k = \bar{u} \in \text{bd } S \setminus \{0\},$$

we have

$$\lim_{k\to\infty} p(u^k) = \infty. \tag{1}$$

(A5) p is continuously differentiable on its domain and $u^T \nabla p(u) > 0$ for all nonzero $u \in \text{int } S$.

A condition equivalent to (A4) is stated in the following straightforward result.

LEMMA 1. *Condition* (A4) *holds if and only if for all* $\gamma \in \Re$ *and* $\varepsilon > 0$, *the set*

$$\Lambda(\varepsilon, \gamma) \equiv \{ u \in \text{int } S : p(u) \leq \gamma, \|u\| \geq \varepsilon \}$$

is compact.

The notion of the central path has played a fundamental role in all interior point methods for solving optimization and complementarity problems [7, 13, 14]. Inspired by this notion, we introduce an important vector a that will be used to define a modified Newton direction that is key to the generation of the iterates for solving the CE (Ω, H). Although the vector a is inspired by the central vector of all ones in the case where S is the nonnegative orthant, since our present setting is very broad, the vector a should not be thought of as just a "central vector" for int S; instead, a is closely linked with the potential function p, which itself is fairly loosely restricted.

(A6) There exists a pair $(a, \bar{\sigma}) \in \Re^n \times (0,1]$ such that

$$\|a\|^2 \left(u^T \nabla p(u)\right) \;\geq\; \bar{\sigma}\,(a^T u)(a^T \nabla p(u)) \quad \forall\, u \in \operatorname{int} S.$$

Trivially, (A6) holds with $a = 0$ and any $\bar{\sigma} \in (0,1]$. It follows that the entire development in this paper holds with $a = 0$. Nevertheless, the interesting case is when $a \neq 0$. The purpose of (A6) is to identify a broad class of such vectors a for which one can establish the convergence of the potential reduction algorithm of section 2.3. For many problems (such as those described in this paper), a nonzero vector a satisfying (A6) can be identified easily; for others, we could always resort to the zero vector.

The basic role of the potential function p is to keep the sequence $\{H(x^k)\}$ away from the set bd $S \backslash \{0\}$ while leading it toward the zero vector. Hence, its role is slightly different from that of a standard barrier function used in nonlinear programming, which in contrast penalizes an iterate when it gets close to *any* boundary point of S.

Our framework includes the most basic case of solving a smooth system of unconstrained equations. This case corresponds to $\Omega = \Re^n$. In this case, we may simply take S to be the entire space $\Re^n$ (so that bd $S = \emptyset$), $p(u)$ to be the function $\|u\|^2$, a to be any vector, and $\bar{\sigma} = 1$. It is then clear that (A2) and (A4)–(A6) all hold easily.

Another simple case to illustrate the above assumptions (with an unspecified Ω) is when S is the nonnegative orthant $\Re^n_+$. In what follows, we establish the validity of conditions (A4)–(A6) for the function

$$p(u) \;=\; \zeta\, \log u^T u - \sum_{i=1}^n \log u_i, \qquad u \;>\; 0$$

and the pair $(a, \bar{\sigma}) = (e, 1)$, where $\zeta > n/2$ is an arbitrary scalar and e is the n-dimensional vector of all ones. (Note: The ℓ_1-norm of u, instead of $u^T u$, could also be used in the first logarithmic term. The analysis remains the same with the constant ζ properly adjusted.) Clearly, p is norm-coercive on $\Re^n_{++}$; i.e.,

$$\lim_{\substack{u>0 \\ \|u\| \to \infty}} \; p(u) \;=\; \infty,$$

because for $u > 0$,

$$\begin{aligned} p(u) &\geq \zeta \left(2 \log \left(\sum_{i=1}^n u_i \right) - \log n \right) - \sum_{i=1}^n \log u_i \\ &> (2\,\zeta - n)\, \log \left(\sum_{i=1}^n u_i \right) - (\zeta - n)\, \log n, \end{aligned}$$

where the first and second inequalities follow from the fact that $\|u\|_1 \leq \sqrt{n}\|u\|$ and $n \log(\sum_{i=1}^n u_i) - \sum_{i=1}^n \log u_i \;\geq\; n \log n$, respectively. Moreover, for any positive sequence $\{u^k\}$ converging to a nonzero nonnegative vector with at least one zero component, the limit (1) clearly holds. Thus (A4) follows. Since

$$u^T \nabla p(u) \;=\; u^T \left(\frac{2\,\zeta}{\|\,u\,\|^2}\, u - u^{-1} \right) \;=\; 2\,\zeta - n \;>\; 0,$$

(A5) holds. Moreover, with $(a, \bar{\sigma}) = (e, 1)$, we now show that (A6) also holds. Indeed, we have for $u > 0$,

$$a^T \nabla p(u) \;=\; \frac{2\,\zeta \sum_{i=1}^n u_i}{\sum_{i=1}^n u_i^2} - \sum_{i=1}^n u_i^{-1}\,;$$

thus

$$\begin{aligned} \frac{(a^T \nabla p(u))\,(a^T u)}{\|a\|^2} &= n^{-1} \left[\frac{2\,\zeta\,\|u\|_1^2}{\|u\|^2} - \left(\sum_{i=1}^n u_i^{-1} \right) \left(\sum_{i=1}^n u_i \right) \right] \\ &\le 2\,\zeta - n = u^T \nabla p(u), \end{aligned}$$

where the last inequality follows from the fact that $\|u\|_1 \le \sqrt{n}\,\|u\|$ and from the arithmetic-geometric mean inequality.

Other choices for the function p exist for $S = \Re^n_+$. The above choice will be generalized to the case where S involves the cone of symmetric positive semidefinite matrices.

Admittedly, the set S, function p, and vector a as stated in the general assumptions (A2) and (A4)–(A6) are somewhat abstract. In particular, a question raised by a referee is whether our framework is applicable to a linear program over a general convex cone, the latter being an elegant problem that has received substantial interest in the optimization community in recent years. Needless to say, to be amenable to our framework, the cone linear program has to be written in the form of a CE. (We are convinced that this can be done via duality theory.) After this conversion, the ability to identify S, p, and a depends on how much we know about the given cone. We believe that for cones arising most frequently in applications (such as the well-known quadratic cone), this set, function, and vector can be identified (although the identification could entail considerable additional efforts). For general cones without additional properties, the applicability of our approach is not clear. A careful investigation may reveal some interesting connection between S, p, and a and certain intrinsic conic properties; nevertheless, such an investigation is clearly beyond the scope of this paper.

2.2. Existence of solutions. In this subsection, we study conditions that guarantee the existence of solutions of the CE (Ω, H). We start by giving a few definitions. Assume that M and N are two metric spaces and that $G : M \to N$ is a map between these two spaces. The map G is said to be *proper* with respect to a set $E \subset N$ if $G^{-1}(K) \subset M$ is compact for every compact set $K \subset E$. If G is proper with respect to N, we will simply say that G is proper. For $D \subset M$, and $E \subset N$ such that $G(D) \subset E$, the restricted map $\tilde{G} : D \to E$ defined by $\tilde{G}(u) \equiv G(u)$ for all $u \in D$ is denoted by $G|_{(D,E)}$; if $E = N$, then we write this $\tilde{G}$ simply as $G|_D$. We will also refer to $G|_{(D,E)}$ as "G restricted to the pair (D, E)," and to $G|_D$ as "G restricted to D." We say that (V_1, V_2) forms a *partition* of the set V if $V_1 \subset V$, $V_2 \subset V$, $V_1 \cup V_2 = V$, and $V_1 \cap V_2 = \emptyset$. A metric space M is said to be *connected* if there exists no partition

$(\mathcal{O}_1, \mathcal{O}_2)$ for which both $\mathcal{O}_1$ and $\mathcal{O}_2$ are nonempty and open. A metric space M is said to be *path-connected* if for any two points $u_0, u_1 \in M$, there exists a continuous $p : [0,1] \to M$ such that $p(0) = u_0$ and $p(1) = u_1$.

The following result and its proof can be found in Monteiro and Pang [17] (see Corollary 1 of this reference).

PROPOSITION 1. *Let M and N be two metric spaces and $F : M \to N$ be a continuous map. Let $M_0 \subset M$ and $N_0 \subset N$ be given sets satisfying the following conditions: $F|_{M_0}$ is a local homeomorphism and $\emptyset \neq F^{-1}(N_0) \subset M_0$. Assume that F is proper with respect to some set E such that $N_0 \subset E \subset N$. Then F restricted to the pair $(F^{-1}(N_0), N_0)$ is a proper local homeomorphism. If, in addition, N_0 is connected, then $F(M_0) \supseteq N_0$ and $F(\text{cl } M_0) \supseteq E \cap \text{cl } N_0$.*

Using Proposition 1, we now derive two existence results for the CE (Ω, H).

THEOREM 1. *Assume that conditions* (A1)–(A3) *hold and that there exists a convex set $E \subset S$ such that $0 \in E$, $E \cap H(\Omega_{\mathcal{I}})$ is nonempty, and $H : \Re^n \to \Re^n$ is proper with respect to E. Then*

(a) *$E \subset H(\Omega)$; in particular, CE (Ω, H) has a solution;*

(b) *H restricted to the pair $(\,\Omega_{\mathcal{I}} \cap H^{-1}(E),\ E \cap \text{int } S\,)$ is a proper local homeomorphism.*

Proof. To apply Proposition 1, let $M \equiv \Omega$, $N \equiv \Re^n$, $M_0 \equiv \Omega_{\mathcal{I}}$, $N_0 \equiv E \cap \text{int } S$, and $F \equiv H|_{\Omega}$. Using (A2) and the assumption that $E \cap H(\Omega_{\mathcal{I}}) \neq \emptyset$, we easily see that $\emptyset \neq F^{-1}(N_0) \subset M_0$. Moreover, by (A3) and the inverse function theorem, it follows that $F|_{M_0}$ is a local homeomorphism. Since F is proper with respect to E by assumption, it follows from Proposition 1 that

$$H(\Omega) \supseteq H(\text{cl } \Omega_{\mathcal{I}}) = F(\text{cl } M_0) \supseteq E \cap \text{cl } N_0 = E \cap \text{cl } (E \cap \text{int } S) = E,$$

where the last equality follows from the fact that cl $(E \cap \text{int } S) = (\text{cl } E) \cap \text{cl } (\text{int } S) = (\text{cl } E) \cap S$, by elementary properties of convex sets (see section 2.1 in Chapter 3 of [5]). Hence, (a) holds. It also follows from Proposition 1 that F restricted to the pair $(F^{-1}(N_0),\ N_0)$ is a proper local homeomorphism. Since by (A2) and the definition of F, we have

$$F^{-1}(N_0) \;=\; \Omega \cap H^{-1}(E \cap \text{int } S) \;=\; \Omega_{\mathcal{I}} \,\cap\, H^{-1}(E),$$

we conclude that (b) holds. □

THEOREM 2. *Assume that conditions* (A1)–(A3) *hold and that H is proper with respect to S. Then* (i) *$S \subset H(\Omega)$ and* (ii) *H restricted to $\Omega_{\mathcal{I}}$ maps each path-connected component of $\Omega_{\mathcal{I}}$ homeomorphically onto* int S. *In particular, CE (Ω, H) has a solution.*

Proof. Conclusion (i) follows immediately from Theorem 1(a) with $E = S$. Using Theorem 1(b) with $E = S$, we conclude that H restricted to the pair $(\Omega_{\mathcal{I}}, \text{int } S)$ is a proper local homeomorphism. If $\mathcal{T} \subset \Omega_{\mathcal{I}}$ is a path-connected component of $\Omega_{\mathcal{I}}$, then H restricted to the pair $(\mathcal{T},\ \text{int } S)$ is a proper local homeomorphism since $\mathcal{T}$ is both open and closed with respect to $\Omega_{\mathcal{I}}$. Since every proper local homeomorphism from a path-connected set into a convex set is a homeomorphism (see, for example, Theorem 1 of [17]), (ii) follows. □

2.3. The algorithm. The algorithm for solving the CE (Ω, H) is a modified, damped Newton method applied to the equation $H(x) = 0$. Referring the reader to [28] for the basic family of Newton methods for solving this unconstrained equation, we highlight the modifications to deal with the presence of the constraint set Ω. In essence, there are two major modifications. One, the Newton equation to compute the search directions is modified using the (central) vector a in assumption (A6). Two, the merit function for the line searches is based on the merit function:

$$\psi(x) \equiv p(H(x)), \quad x \in \Omega_{\mathcal{I}}. \tag{2}$$

This is different from the norm functions of H that are the common merit functions used in a classical damped Newton method. Note that by (A3) and (A5) the function ψ is continuously differentiable on $\Omega_{\mathcal{I}}$.

With the above explanation, we now give the full details of the Newton method for solving the CE (Ω, H) under the setting given in the last subsection.

Step 0. (Initialization) Let a vector $x^0 \in \Omega_{\mathcal{I}}$ and scalars $\rho \in (0,1)$ and $\alpha \in (0,1)$ be given. Let a sequence of scalars $\{\sigma_k\} \subset [0, \bar{\sigma})$ also be given. (The scalar $\bar{\sigma}$ is as given in assumption (A6).) Set the iteration counter $k = 0$.

Step 1. (Computing the modified Newton direction) Solve the system of linear equations

$$H(x^k) + H'(x^k; d) = \sigma_k \frac{a^T H(x^k)}{\|a\|^2} a \tag{3}$$

to obtain the search direction d^k. (The right-hand side of the above equation is assumed to be zero if $a = 0$. This convention will be assumed throughout our presentation.)

Step 2. (Armijo line search) Let m_k be the smallest nonnegative integer m such that $x^k + \rho^m d^k \in \Omega_{\mathcal{I}}$ and

$$\psi(x^k + \rho^m d^k) - \psi(x^k) \le \alpha \rho^m \nabla\psi(x^k)^T d^k.$$

Set $x^{k+1} \equiv x^k + \rho^{m_k} d^k$.

Step 3. (Termination test) If

$$\| H(x^{k+1}) \| \le \text{ prescribed tolerance},$$

stop; accept x^{k+1} as an approximate solution of the CE (Ω, H). Otherwise, return to Step 1 with k replaced by $k+1$.

By (A3) and the fact that $x^k \in \Omega_{\mathcal{I}}$, the Newton equation (3) has a unique solution which we have denoted by d^k. The following lemma guarantees that d^k is a descent direction for the function ψ at x^k. This property, along with the openness of $\Omega_{\mathcal{I}}$, ensures that the integer m_k can be determined in a finite number of trials (starting with $m_k = 0$ and increasing it by one at each trial), thus guaranteeing the well-definedness of the next iterate x^{k+1}.

LEMMA 2. *Suppose that conditions* (A5) *and* (A6) *hold. Assume also that* $x \in \Omega_{\mathcal{I}}$, $d \in \Re^n$, *and* $\sigma \in \Re$ *are such that*

$$H(x) \ne 0, \qquad 0 \le \sigma < \bar{\sigma}, \tag{4}$$

$$H'(x; d) = -H(x) + \sigma \frac{a^T H(x)}{\|a\|^2} a, \tag{5}$$

where $a \in \Re^n$ *and* $\bar{\sigma} \in [0,1]$ *are as in condition* (A6). *Then,* $\nabla\psi(x)^T d < 0$.

Proof. Let $u \equiv H(x)$. Then, $0 \neq u \in \operatorname{int} S$ due to (4) and the assumption that $x \in \Omega_{\mathcal{I}}$. This together with (2), (5), (4), (A5), and (A6) imply

$$\nabla\psi(x)^T d = \nabla p(H(x))^T H'(x;d) \;=\; \nabla p(u)^T \left(-u + \sigma\, \frac{a^T u}{\|a\|^2}\, a\right)$$
$$\leq -\nabla p(u)^T u \left(1 - \frac{\sigma}{\bar\sigma}\right) < 0,$$

as claimed. □

2.4. A convergence result. In what follows, we state and prove a limiting property of an infinite sequence of iterates $\{x^k\}$ generated by the algorithm. Before stating the theorem, we observe that such a sequence necessarily belongs to the set $\Omega_{\mathcal{I}}$; thus $\{H(x^k)\} \subset \operatorname{int} S$. Since the sequence $\{x^k\}$ is infinite, we have $H(x^k) \neq 0$ for all k. Theorem 3 below contains four conclusions, (a)–(d). The first three of these do not assert the boundedness of the sequence $\{x^k\}$; this boundedness is established under the assumptions of statement (d), which implies the existence of a solution of the CE (Ω, H). A consequence of statement (c) in the theorem is

$$\inf\{\, \|H(x)\| \,:\, x \in \Omega\,\} \;=\; 0;$$

consequently, CE (Ω, H) has "ε-solutions" for every $\varepsilon > 0$ in the sense that for any such ε, there exists a vector $x^\varepsilon \in \Omega$ satisfying $\|H(x^\varepsilon)\| \leq \varepsilon$; moreover x^ε can be computed by the potential reduction Newton method starting at the given vector x^0.

THEOREM 3. *Assume conditions* (A1)–(A6) *hold and that* $\limsup_k \sigma_k < \bar\sigma$. *Let* $\{x^k\}$ *be any infinite sequence produced by the potential reduction Newton algorithm. Then, the following statements hold:*

(a) *the sequence* $\{H(x^k)\}$ *is bounded;*

(b) *any accumulation point of* $\{x^k\}$*, if it exists, solves the CE* (Ω, H)*; in particular, if* $\{x^k\}$ *is bounded, then the CE* (Ω, H) *has a solution.*

Moreover, for any closed subset E *of* S *containing the sequence* $\{H(x^k)\}$,

(c) *if* H *is proper with respect to* $E \cap \operatorname{int} S$*, then* $\lim_{k\to\infty} H(x^k) = 0$*;*

(d) *if* H *is proper with respect to* E*, then* $\{x^k\}$ *is bounded.*

Proof. Let $\gamma \equiv \psi(x^0)$ and $u^k \equiv H(x^k) \in \operatorname{int} S$ for all k. Clearly, $p(u^k) = \psi(x^k) \leq \psi(x^0) = \gamma$ for all k. Hence, for any $\varepsilon > 0$ we have $\{u^k\} \subset \Lambda(\varepsilon,\gamma) \cup \{u \in \Re^n : \|u\| \leq \varepsilon\}$. Since by Lemma 1 the set $\Lambda(\varepsilon,\gamma)$ is compact, and hence bounded, we conclude that $\{u^k\}$ is bounded. Hence, (a) follows.

To show (b), let x^∞ be an accumulation point of $\{x^k\}$. Clearly $x^\infty \in \Omega$ because Ω is a closed set. Assume for contradiction that $u^\infty \equiv H(x^\infty) \neq 0$. Let $\{x^k : k \in \kappa\}$ be a subsequence converging to x^∞ and assume without loss of generality that $\{\sigma_k : k \in \sigma\}$ converges to some scalar σ_∞. Since $\sigma_k \geq 0$ for all k and $\limsup_k \sigma_k < \bar\sigma$, we must have $\sigma_\infty \in [0, \bar\sigma)$. Since $p(u^k) \leq p(u^0) = \gamma$ for all k and

$$\lim_{k(\in\kappa)\to\infty} u^k \;=\; u^\infty \;\neq\; 0,$$

there exists $\varepsilon > 0$ such that the subsequence $\{u^k : k \in \kappa\} \subset \Lambda(\varepsilon,\gamma)$. Since by Lemma 1 the set $\Lambda(\varepsilon,\gamma)$ is compact, we conclude that $u^\infty = H(x^\infty) \in \Lambda(\varepsilon,\gamma) \subset \operatorname{int} S$, and hence that $x^\infty \in H^{-1}(\operatorname{int} S)$. By assumption (A2), it follows that $x^\infty \in \Omega_{\mathcal{I}}$. Hence, by assumption (A3), $H'(x^\infty)^{-1}$ exists. This implies that the sequence $\{d^k : k \in \kappa\}$ converges to a vector d^∞ satisfying

$$H(x^\infty) + H'(x^\infty; d^\infty) \;=\; \sigma_\infty\, \frac{a^T H(x^\infty)}{\|a\|^2}\, a.$$

Hence, it follows from Lemma 2 that $\nabla\psi(x^\infty)^T d^\infty < 0$.

Since $\{x^k : k \in \kappa\}$ converges to $x^\infty \in \Omega_{\mathcal{I}}$ where ψ is continuous, it follows that $\{\psi(x^k) : k \in \kappa\}$ converges. This implies that the whole sequence $\{\psi(x^k)\}$ converges due to the fact that it is monotonically decreasing. Using the relation

$$\psi(x^{k+1}) - \psi(x^k) \,=\, \psi(x^k + \rho^{m_k} d^k) - \psi(x^k) \,\leq\, \alpha\, \rho^{m_k}\, \nabla\psi(x^k)^T d^k \,<\, 0$$

for all k, we conclude that

$$\lim_{k\to\infty} \rho^{m_k}\, \nabla\psi(x^k)^T d^k \,=\, 0$$

and hence that

$$\lim_{k(\in\kappa)\to\infty} \rho^{m_k} \,=\, 0$$

because

$$\lim_{k(\in\kappa)\to\infty} \nabla\psi(x^k)^T d^k \,=\, \nabla\psi(x^\infty)^T d^\infty \,<\, 0.$$

Thus

$$\lim_{k(\in\kappa)\to\infty} m_k \,=\, \infty,$$

which implies that $m_k \geq 2$ for all $k \in \kappa$ sufficiently large. Consequently, by the definition of m_k, we deduce that

$$\frac{\psi(x^k + \rho^{m_k - 1} d^k) - \psi(x^k)}{\rho^{m_k - 1}} \,>\, \alpha\, \nabla\psi(x^k)^T d^k$$

for all $k \in \kappa$ sufficiently large. Letting $k \in \kappa$ tend to infinity in the above expression, we obtain

$$\nabla\psi(x^\infty)^T d^\infty \,\geq\, \alpha\nabla\psi(x^\infty)^T d^\infty,$$

which contradicts the fact that $\alpha < 1$ and $\nabla\psi(x^\infty)^T d^\infty < 0$. Consequently, we must have $H(x^\infty) = 0$, and hence (b) follows.

Assume now that E is a closed subset of S containing the sequence $\{H(x^k)\}$. To prove (c), assume for contradiction that for an infinite subset $\kappa \subset \{0, 1, 2, \ldots\}$, we have

$$\liminf_{k(\in\kappa)\to\infty} \|\, u^k \,\| \,>\, 0.$$

By an argument similar to that employed above, we conclude that for some $\varepsilon > 0$ we have $\{u^k : k \in \kappa\} \subset \Lambda(\varepsilon, \gamma) \cap E$. By Lemma 1 and the fact that E is closed, we conclude that $\Lambda(\varepsilon, \gamma) \cap E$ is a compact subset of $\operatorname{int} S \cap E$. Since H is proper with respect to $\operatorname{int} S \cap E$, the inverse image of $\Lambda(\varepsilon, \gamma) \cap E$ under H is compact, and hence bounded. This implies that $\{x^k : k \in \kappa\}$ is bounded. By (b), every accumulation point of the latter subsequence is a zero of H. This contradiction establishes (c).

Finally, using (a) and the fact that E is closed, we conclude that $\{u^k\}$ is contained in a compact subset E_1 of E. Since H is proper with respect to E, it follows that the set $H^{-1}(E_1) \supset \{x^k\}$ is bounded. Hence, (d) follows. $\square$

The framework of the CE (Ω, H) that we have set forth so far is very broad. In addition to not assuming any sign restriction on the components of H (like we did in [36]; see Assumption 1 therein), as we have mentioned before, the freedom in the choice of the set S and the associated potential function p and vector a adds to the versatility of the framework. The results in the next two sections will demonstrate how S, p, and a can easily be constructed in important cases under very mild assumptions.

3. Monotone complementarity problems in symmetric matrices. We consider a mixed complementarity problem defined on the cone of symmetric positive semidefinite matrices. The linear version of this problem was introduced by Kojima, Shindoh, and Hara [10] and has received a great deal of research attention recently. In what follows, we consider a nonlinear version of this problem defined in [18]. This reference contains a fairly extensive bibliography on interior point methods for solving optimization and complementarity problems defined on the cone of semidefinite matrices; it will be the source for several results that will be used freely in the subsequent development.

3.1. Implicit mixed complementarity problems. We recall the framework considered in [18]. Let $F : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m \to \mathcal{S}^n \times \Re^m$ be a given mapping. The mixed complementarity problem in symmetric matrices is to find a triple $(X, Y, z) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ satisfying

$$F(X,Y,z) = 0, \quad X \bullet Y = 0, \quad (X,Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+. \tag{6}$$

As explained in [18] and the references therein, there are several equivalent ways of stating the complementarity condition $X \bullet Y = 0$, each leading to a different interior point method for solving the above problem. In what follows, we consider the equivalent formulation of this problem as the CE defined by the pair (Ω, H), where the set Ω and the map $H : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m \to \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ are defined by

$$\Omega \equiv \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m, \tag{7}$$

$$H(X,Y,z) \equiv \begin{pmatrix} (XY+YX)/2 \\ F(X,Y,z) \end{pmatrix}, \quad (X,Y,z) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m. \tag{8}$$

Similar treatment can be applied to other equivalent formulations and to generalizations of the basic problem (6). Throughout the following discussion, F is assumed to be continuous on its domain and continuously differentiable on $\mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \Re^m$.

Associated with the above mapping H, define the set

$$\mathcal{U} \equiv \{ (X,Y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} : XY + YX \in \mathcal{S}^n_{++} \}. \tag{9}$$

The set $\mathcal{U}$ was introduced in [31] and subsequently used in the papers [8, 33] for the analysis of primal-dual semidefinite programming algorithms based on the Alizadeh–Haeberly–Overton (AHO) direction [2]. It has also been used in [18] for the study of the fundamental properties of the interior point map (8). The fundamental role of the set $\mathcal{U}$ in the study of the problem (6) is well explained in the above-cited references. It has been shown in Lemma 1 of [18] that

$$\mathcal{U} = \{ (X,Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ : XY + YX \in \mathcal{S}^n_{++} \}. \tag{10}$$

We introduce an important assumption on the mapping F that will be used to verify the nonsingularity of the Jacobian matrix $H'(X,Y,z)$.

(B1) The mapping F is (X,Y)-*differentiably-monotone* at every triple $(X,Y,z) \in \mathcal{U} \times \Re^m$; i.e., for any such triple,

$$\left.\begin{array}{r} F'((X,Y,z); (dX,dY,dz)) = 0 \\ (dX,dY,dz) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m \end{array}\right\} \Longrightarrow dX \bullet dY \geq 0. \tag{11}$$

(B2) The mapping F is *z-differentiably-injective* at every triple $(X, Y, z) \in \mathcal{U} \times \Re^m$; i.e., for any such triple,

$$F'((X, Y, z); (0, 0, dz)) = 0 \implies dz = 0. \tag{12}$$

The following lemma asserts that the basic assumptions (A1)–(A3) in section 2.1 are valid under the above hypotheses.

LEMMA 3. *Consider the CE* (Ω, H) *with* Ω *and* H *defined by* (7) *and* (8)*, and let* $S \equiv \mathcal{S}^n_+ \times \mathcal{S}^n \times \Re^m$*. If conditions* (B1) *and* (B2) *hold, then*

$$\Omega_{\mathcal{I}} \equiv H^{-1}(\text{int } S) \cap \text{int } \Omega = \mathcal{U} \times \Re^m;$$

moreover, the pair (Ω, H) *and the set* S *satisfy conditions* (A1), (A2), *and* (A3).

Proof. Only the second assertion requires a proof. Conditions (A1) and (A2)(a) obviously hold. Clearly $\mathcal{U}$ is an open set; since $(I, I) \in \mathcal{U}$, (A2)(b) holds. Moreover, it is easy to see that the alternative representation (10) implies (A2)(c). Next we establish that (A3) holds under (B1) and (B2). This amounts to showing that for every $(X, Y, z) \in \Omega_{\mathcal{I}} = \mathcal{U} \times \Re^m$, the following implication holds:

$$\left.\begin{array}{r} H'((X, Y, z); (dX, dY, dz)) = 0 \\ (dX, dY, dz) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m \end{array}\right\} \implies (dX, dY, dz) = 0.$$

Assume the left-hand condition holds. Then,

$$X(dY) + (dY)X + Y(dX) + (dX)Y = 0, \tag{13}$$

$$F'((X, Y, z); (dx, dy, dz)) = 0. \tag{14}$$

Condition (B1) and (14) imply that $dX \bullet dY \geq 0$. This together with (13) and the fact that $(X, Y) \in \mathcal{U}$ yield $dX = dY = 0$ (see the proof of Theorem 3.1(iii) of [31]). In turn, this together with (14) imply

$$F'((X, Y, z); (0, 0, dz)) = 0,$$

which yields $dz = 0$ due to (B2). ◻

From the above result, we see that the set $\mathcal{U}$ is naturally associated with the map H given by (8). We observe that, based on the analysis of Monteiro and Zanjácomo [21], it can be shown that $H'(X, Y, z)$ is invertible over the set $\mathcal{U}' \times \Re^m$ with $\mathcal{U}'$ given by

$$\mathcal{U}' \equiv \left\{ (X, Y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} : \left\| X^{1/2} Y X^{1/2} - \mu I \right\| \leq \frac{1}{2}\mu \right\},$$

where $\mu \equiv (X \bullet Y)/n$. However, the set $\mathcal{U}'$ does not fit well with the map H in the sense of Lemma 3 even for different choices of the set S. Instead, $\mathcal{U}'$ naturally arises in connection with the interior point map $\tilde{H}(X, Y, z) \equiv (X^{1/2} Y X^{1/2}, F(X, Y, z))$ by choosing the set S as

$$S \equiv \left\{ U \in \mathcal{S}^n_{++} : \left\| U - \left(\frac{\text{tr}\, U}{n}\right) I \right\|_F \leq \frac{1}{2} \frac{\text{tr}\, U}{n} \right\}.$$

Even though this provides a viable alternative approach, we will not pursue it any further.

Next we deal with conditions (A4)–(A6). For this purpose, consider the potential function $p : \mathcal{S}^n_{++} \times \mathcal{S}^n \times \Re^m \to \Re$ defined by

$$p(M,N,v) \equiv \zeta \log\left(\|M\|_F^2 + \|N\|_F^2 + \|v\|^2\right) - \log(\det M) \tag{15}$$

for every $(M,N,z) \in \mathcal{S}^n_{++} \times \mathcal{S}^n \times \Re^m$, where $\zeta > n/2$ is an arbitrary constant.

LEMMA 4. *The potential function* (15), *the vector* $a \equiv (I,0,0) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$, *and the scalar* $\bar\sigma \equiv 1$ *satisfy conditions* (A4), (A5), *and* (A6).

Proof. Since, for a matrix $Z \in \mathcal{S}^n$, $\|Z\|_F^2$ is equal to the sum of the squares of the n eigenvalues of Z, and $\det Z$ is equal to the product of these eigenvalues, the verification of (A4) for the function $p(M,N,v)$ is the same as in the previous case of a nonnegatively constrained equation (discussed at the end of section 2.1). Noting that

$$\nabla p(M,N,v) = \begin{pmatrix} \dfrac{2\zeta}{\|M\|_F^2 + \|N\|_F^2 + \|v\|^2} M - M^{-1} \\ \dfrac{2\zeta}{\|M\|_F^2 + \|N\|_F^2 + \|v\|^2} N \\ \dfrac{2\zeta}{\|M\|_F^2 + \|N\|_F^2 + \|v\|^2} v \end{pmatrix},$$

we have

$$(M,N,v) \bullet \nabla p(M,N,v) = 2\zeta - n > 0,$$

and thus (A5) holds. We now show that (A6) is satisfied with the given a and $\bar\sigma$. Indeed we have

$$(I,0,0) \bullet \nabla p(M,N,v) = \frac{2\zeta}{\|M\|_F^2 + \|N\|_F^2 + \|v\|^2} \operatorname{tr}(M) - \operatorname{tr}(M^{-1}),$$

which implies

$$\begin{aligned}&[\,(I,0,0) \bullet \nabla p(M,N,v)\,]\,[\,(I,0,0) \bullet (M,N,v)\,] \\ &= \frac{2\zeta}{\|M\|_F^2 + \|N\|_F^2 + \|v\|^2} (\operatorname{tr}(M))^2 - \operatorname{tr}(M^{-1}) \operatorname{tr}(M).\end{aligned}$$

Noting that (i) $\operatorname{tr}(M)$ equals the sum of the eigenvalues of M, (ii) $\operatorname{tr}(M^{-1})$ equals the sum of the inverses of the same eigenvalues, and (iii) $\|M\|_F^2 = \operatorname{tr}(M^2)$ equals the sum of these eigenvalues squared, it follows from the same derivation as at the end of section 2.1 that condition (A6) holds. ∎

According to (2), the potential function (15) induces the following merit function on the set $\Omega_{\mathcal{I}} = \mathcal{U} \times \Re^m$:

$$\begin{aligned}\psi(X,Y,z) &\equiv p(H(X,Y,z)) \\ &= \zeta \log\left(\frac{\|XY+YX\|_F^2}{4} + \|F(X,Y,z)\|_{F,2}^2\right) - \log\left(\det\left(\frac{XY+YX}{2}\right)\right),\end{aligned}$$

for any triple $(X,Y,z) \in \mathcal{U} \times \Re^m$. Here, $\|\cdot\|_{F,2}$ denotes the norm on $\mathcal{S}^n \times \Re^m$ defined by $\|(N,v)\|_{F,2}^2 \equiv \|N\|_F^2 + \|v\|^2$ for every $(N,v) \in \mathcal{S}^n \times \Re^m$.

We now give a detailed description of a specialized algorithm for solving the mixed complementarity problem in symmetric matrices (6), based on the potential reduction Newton method for solving the CE (Ω, H) with Ω, H, S, $p : \text{int}\, S \to \Re$, a and $\bar{\sigma}$ defined as in (7), (8), Lemma 3, (15), and Lemma 4, respectively.

Step 0. (Initialization) Let a pair of matrices $(X^0, Y^0) \in \mathcal{U}$, a vector $z^0 \in \Re^m$, and scalars $\rho \in (0,1)$ and $\alpha \in (0,1)$ be given. Let a sequence of scalars $\{\sigma_k\}$ also be given, where $\sigma_k \in [0,1)$ for all k. Set the iteration counter $k = 0$.

Step 1. (Computing the modified Newton direction) Solve the system of linear equations:

$$\begin{pmatrix} \left[X^k Y^k + Y^k X^k + X^k(dY) + (dY)X^k + Y^k(dX) + (dX)Y^k\right]/2 \\ F(X^k, Y^k, z^k) + F'((X^k, Y^k, z^k); (dX, dY, dz)) \end{pmatrix} = \begin{pmatrix} \sigma_k \mu_k I \\ 0 \end{pmatrix}$$

$$(dX, dY, dz) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^m,$$

where $\mu_k \equiv \text{tr}(X^k Y^k)/n$, to obtain the search triple (dX^k, dY^k, dz^k).

Step 2. (Armijo line search) Let m_k be the smallest nonnegative integer m such that

$$\begin{pmatrix} X^k + \rho^m \, dX^k \\ Y^k + \rho^m \, dY^k \end{pmatrix} \in \Omega_{\mathcal{I}}$$

and

$$\begin{aligned} &\psi(X^k + \rho^m dX^k, Y^k + \rho^m dY^k, z^k + \rho^m dz^k) - \psi(X^k, Y^k, z^k) \\ &\quad \le \alpha \, \rho^m \, \psi'(\,(X^k, Y^k, dz^k);\, (dX^k, dY^k, dz^k)\,). \end{aligned}$$

Set

$$\begin{pmatrix} X^{k+1} \\ Y^{k+1} \\ dz^{k+1} \end{pmatrix} \equiv \begin{pmatrix} X^k + \rho^{m_k} \, dX^k \\ Y^k + \rho^{m_k} \, dY^k \\ z^k + \rho^{m_k} \, dz^k \end{pmatrix}.$$

Step 3. (Termination test) If

$$\|\, H(X^{k+1}, Y^{k+1}, z^{k+1}) \,\| \le \text{ prescribed tolerance},$$

stop; accept the triple $(X^{k+1}, Y^{k+1}, z^{k+1})$ as an approximate solution of the problem (6). Otherwise, return to Step 1 with k replaced by $k+1$.

We observe that the direction obtained in Step 1 of the above algorithm is an extension of the AHO direction introduced in [2] in the context of semidefinite programming.

As an immediate consequence of Lemma 3, Lemma 4, and Theorem 3, we have the following convergence result for the above algorithm.

THEOREM 4. *Assume that conditions* (B1) *and* (B2) *hold and* $\limsup_k \sigma_k < 1$. *Let* $\{(X^k, Y^k, z^k)\}$ *be any infinite sequence produced by the above algorithm for solving problem* (6). *Then, the following statements hold:*

(a) *the sequence* $\{H(X^k, Y^k, z^k)\}$ *is bounded;*

(b) *any accumulation point of* $\{(X^k, Y^k, z^k)\}$*, if it exists, solves the problem* (6)*; in particular, if* $\{(X^k, Y^k, z^k)\}$ *is bounded, then problem* (6) *has a solution.*

We now make a few remarks. The above theorem guarantees neither that $\{(X^k, Y^k, z^k)\}$ is bounded nor that it has an accumulation point. The conclusion that $\{(X^k, Y^k, z^k)\}$ is bounded would follow from Theorem 3(d) with $E = S$ if we could prove that the map H is proper with respect to the set $S \equiv \mathcal{S}^n_+ \times \mathcal{S}^n \times \Re^m$. Unfortunately, this requirement is rather strong. For monotone mixed complementarity problems, we state in Proposition 2 below a result (from Monteiro and Pang [18, Lemma 2]) asserting that the map H is proper with respect to $\mathcal{S}^n \times F(\mathcal{U} \times \Re^m)$. Hence, if the latter set contains the set $S = \mathcal{S}^n_+ \times \mathcal{S}^n \times \Re^m$, or equivalently if the equality $F(\mathcal{U} \times \Re^m) = \mathcal{S}^n \times \Re^m$ holds, then the sequence generated by the above algorithm $\{(X^k, Y^k, z^k)\}$ is bounded. Intuitively, the equality $F(\mathcal{U} \times \Re^m) = \mathcal{S}^n \times \Re^m$ might hold for maps F satisfying some kind of strong monotonicity condition. But since this type of condition is fairly restrictive, we do not pursue this issue any further.

Another possible approach which would guarantee the boundedness of $\{(X^k, Y^k, z^k)\}$ is to reduce the set S so as to have $S \subset \mathcal{S}^n \times F(\mathcal{U} \times \Re^m)$. This approach requires some knowledge of the set $F(\mathcal{U} \times \Re^m)$. We will see that for the complementarity problems studied in sections 3.2 and 4, enough information about the set $F(\mathcal{U} \times \Re^m)$ is available to allow us to choose a set S together with a potential function $p : \text{int } S \to \Re$ satisfying the inclusion $S \subset \mathcal{S}^n \times F(\mathcal{U} \times \Re^m)$ and the conditions (A1)–(A6) of section 2.1.

Before stating the properness result mentioned above, we give a few basic definitions.

Definition 1. *A mapping* $J(X, Y, z)$ *defined on a subset* $\text{dom}(J)$ *of* $\mathcal{M}^n \times \mathcal{M}^n \times \Re^m$ *is said to be* (X, Y)-equilevel-monotone *on a subset* $\mathcal{V} \subset \text{dom}(J)$ *if for any* $(X, Y, z) \in \mathcal{V}$ *and* $(X', Y', z') \in \mathcal{V}$ *such that* $J(X, Y, z) = J(X', Y', z')$*, there holds* $(X' - X) \bullet (Y' - Y) \geq 0$. *When* $\mathcal{V} = \text{dom}(J)$*, we will simply say that* J *is* (X, Y)*-equilevel-monotone.*

In the following two definitions, we assume that W, Z, and N are three normed spaces and that $\phi(w, z)$ is a function defined on a subset of $W \times Z$ with values in N.

Definition 2. *The function* $\phi(w, z)$ *is said to be* z-bounded *on a subset* $\mathcal{V} \subset \text{dom}(\phi)$ *if for every sequence* $\{(w^k, z^k)\} \subset \mathcal{V}$ *such that* $\{w^k\}$ *and* $\{\phi(w^k, z^k)\}$ *are bounded, the sequence* $\{z^k\}$ *is also bounded. When* $\mathcal{V} = \text{dom}(\phi)$*, we will simply say that* ϕ *is* z*-bounded.*

Definition 3. *The function* $\phi(w, z)$ *is said to be* z-injective *on a subset* $\mathcal{V} \subset \text{dom}(\phi)$ *if the following implication holds:* $(w, z) \in \mathcal{V}$, $(w, z') \in \mathcal{V}$, *and* $\phi(w, z) = \phi(w, z')$ *implies* $z = z'$. *When* $\mathcal{V} = \text{dom}(\phi)$*, we will simply say that* ϕ *is* z*-injective.*

The following is the promised result from Lemma 2 of Monteiro and Pang [18].

Proposition 2. *Let* $F : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m \to \mathcal{S}^n \times \Re^m$ *be a continuous map and let* $H : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^m \to \mathcal{S}^n \times \mathcal{S}^n \times \Re^m$ *be the map defined by* (8)*. Assume that the map* F *is* (X, Y)*-equilevel-monotone and* z*-bounded on its domain. If the map* H *restricted to* $\mathcal{U} \times \Re^m$ *is a local homeomorphism, then* H *is proper with respect to* $\mathcal{S}^n \times F(\mathcal{U} \times \Re^m)$.

3.2. Standard complementarity problem. In this section, we consider the standard nonlinear complementarity problem (NCP) in symmetric matrices:

$$X \bullet f(X) = 0, \quad X \succeq 0, \quad f(X) \succeq 0, \tag{16}$$

where $f : \mathcal{S}^n_+ \to \mathcal{S}^n$ is a given continuous mapping that is continuously differentiable on $\mathcal{S}^n_{++}$. This problem is a special case of the implicit mixed complementarity problem of section 3.1, where $m = 0$ (i.e., the free variable z is not present) and $F : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \to \mathcal{S}^n$ is given by

$$F(X,Y) \equiv Y - f(X) \quad \forall (X,Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+. \tag{17}$$

We make the following assumption on the mapping f.

(C1) $f : \mathcal{S}^n_+ \to \mathcal{S}^n$ is monotone on $\mathcal{S}^n_+$; i.e., for all X and X' in $\mathcal{S}^n_+$,

$$(X - X') \bullet (f(X) - f(X')) \geq 0.$$

LEMMA 5. *If condition* (C1) *holds, then the map* $F : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \to \mathcal{S}^n$ *defined by* (17) *satisfies condition* (B1) *of section* 3.1.

Proof. By (C1), it follows that for every $X \in \mathcal{S}^n_+$, the linear map $f'(X)$ is monotone in the sense that

$$U \bullet f'(X;U) \geq 0 \quad \forall U \in \mathcal{S}^n. \tag{18}$$

To verify (B1), assume that $(dX, dY) \in \mathcal{S}^n \times \mathcal{S}^n$ satisfies $F'(X,Y)(dX,dY) = 0$, or equivalently that $dY - f'(X;dX) = 0$. Then, by (18), we have

$$dX \bullet dY = dX \bullet f'(X;dX) \geq 0.$$

This shows that implication (11) holds for $m = 0$, and since implication (12) holds vacuously for $m = 0$, (C1) follows. $\square$

It is possible to solve the NCP (16) with the use of the potential reduction algorithm described in section 3.1. However, the sequence of iterates $\{(X^k, Y^k)\}$ generated by this algorithm might not be bounded. We now develop a different potential reduction algorithm in which the set S is reduced so as to satisfy $S \subset \mathcal{S}^n_+ \times F(\mathcal{U})$, thus ensuring the boundedness of the sequence $\{(X^k, Y^k)\}$ (see the discussion at the end of the previous subsection).

To describe the alternative algorithm, it is sufficient to identify the pair (Ω, H), the set S, the potential function $p : \text{int } S \to \Re$, and the vector a and scalar $\bar\sigma$ in condition (A6). We let $\Omega \equiv \mathcal{S}^n_+ \times \mathcal{S}^n_+$ and define $H : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \to \mathcal{S}^n \times \mathcal{S}^n$ by

$$H(X,Y) \equiv \begin{pmatrix} (XY + YX)/2 \\ F(X,Y) \end{pmatrix}, \quad (X,Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+, \tag{19}$$

where F is given by (17). Moreover, we let $S \equiv \mathcal{S}^n_+ \times \mathcal{S}^n_+$ and $p : \text{int } S \to \Re$ be defined by

$$p(M,N) \equiv \zeta \log\left(\|M\|_F^2 + \|N\|_F^2\right) - \log(\det M) - \log(\det N)$$

for every $(M,N) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++}$, where $\zeta > n$ is an arbitrary constant. Finally, we let $a \equiv (I,I)$ and $\bar\sigma \equiv 1$. Clearly, the set $\Omega_{\mathcal{I}}$ and the merit function $\psi : \Omega_{\mathcal{I}} \to \Re$ become

$$\Omega_{\mathcal{I}} = \{\, (X,Y) \in \mathcal{U} : Y \succ f(X) \,\}$$

and

$$\psi(X,Y) \equiv \zeta \log\left(\frac{\|XY + YX\|_F^2}{4} + \|\, Y - f(X)\,\|^2\right)$$
$$- \log\left(\det\left(\frac{XY + YX}{2}\right)\right) - \log(\det(Y - f(X))) \quad \text{for } (X,Y) \in \Omega_{\mathcal{I}}.$$

LEMMA 6. *The pair* (Ω, H), *the set* S, *the potential function* $p : \text{int } S \to \Re$, *the vector* a, *and the scalar* $\bar{\sigma}$ *defined above satisfy conditions* (A1)–(A6) *of section* 2.1.

Proof. Condition (A2)(b) follows from the fact that $(I, \delta I) \in \Omega_{\mathcal{I}}$ for all $\delta > 0$ sufficiently large. The other conditions are either straightforward or are shown using Lemma 5 and the same arguments used in the proofs of Lemmas 3 and 4. $\square$

Before giving the convergence result for the potential reduction Newton method in the above framework, we state the following result, which will be used to establish boundedness of the iterates generated by this method.

LEMMA 7. *Suppose that* $f : \mathcal{S}^n_+ \to \mathcal{S}^n$ *is a continuous map that is continuously differentiable on* $\mathcal{S}^n_{++}$ *and satisfies condition* (C1). *Then, for the maps* F *and* H *defined by* (17) *and* (19), *respectively, we have*

(a) $F(\mathcal{U}) = F(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++})$;

(b) *if* $0 \in F(\mathcal{S}^n_+ \times \mathcal{S}^n_+)$, *then* H *is proper with respect to* $\mathcal{S}^n \times \mathcal{S}^n_{++}$;

(c) *if* $0 \in F(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++})$, *then* H *is proper with respect to* $\mathcal{S}^n \times \mathcal{S}^n_+$.

Proof. By Proposition 4(a) and Corollary 3 of [18] with $m = 0$, it follows that $\{I\} \times F(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++}) \subset H(\mathcal{S}^n_+ \times \mathcal{S}^n_+)$. Using this inclusion, we easily see that statement (a) holds.

We next show (b). By Lemma 6, $H'(X, Y)$ is invertible for all $(X, Y) \in \mathcal{U}$. Thus H restricted to $\mathcal{U}$ is a local homeomorphism. Thus it follows from Lemma 2 that H is proper with respect to $\mathcal{S}^n \times F(\mathcal{U})$. Hence, (b) follows once we prove that $\mathcal{S}^n_{++} \subset F(\mathcal{U}) = F(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++})$. Let $U \in \mathcal{S}^n_{++}$ be arbitrary. Since $0 \in F(\mathcal{S}^n_+ \times \mathcal{S}^n_+)$, there exists $(\tilde{X}, \tilde{Y}) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+$ such that $\tilde{Y} = f(\tilde{X})$. For $\epsilon > 0$, let $X_\epsilon \equiv \tilde{X} + \epsilon I$ and $Y_\epsilon \equiv U + f(X_\epsilon) = U + \tilde{Y} + f(X_\epsilon) - f(\tilde{X})$. Clearly, $X_\epsilon \succ 0$ for every $\epsilon > 0$. By the continuity of f and the fact that $U + \tilde{Y} \succ 0$, we have $Y_\epsilon \succ 0$ for $\epsilon > 0$ sufficiently small. Since $U = Y_\epsilon - f(X_\epsilon)$, it follows that U belongs to $F(\mathcal{S}^n_{++} \times \mathcal{S}^n_{++})$.

We omit the proof of (c), which is similar to that of (b). $\square$

We will skip the straightforward formulation of the potential reduction Newton method specialized to the above choices of the pair (Ω, H), set S, potential function $p : \text{int } S \to \Re$, vector a, and scalar $\bar{\sigma}$; instead, we directly give the convergence properties of the method. Among the three conclusions (a), (b), and (c) of Theorem 5, (b) provides a constructive proof that a feasible monotone complementarity problem in symmetric matrices on the positive semidefinite cone always has "ε-solutions"; (c) implies the well-known fact that for such a problem, strict feasibility yields solvability.

THEOREM 5. *Let* $f : \mathcal{S}^n_+ \to \mathcal{S}^n$ *be a continuous function which is continuously differentiable on* $\mathcal{S}^n_{++}$ *and satisfies condition* (C1). *Suppose that* $\{(X^k, Y^k)\}$ *is a sequence generated by the potential reduction Newton method with the pair* (Ω, H), *set* S, *potential function* $p : \text{int } S \to \Re$, *vector* a, *and scalar* $\bar{\sigma}$ *as specified above. Then, the following statements hold:*

(a) *every accumulation point of* $\{(X^k, Y^k)\}$ *is a solution of the NCP* (16);

(b) *if there exists* $\tilde{X} \in \mathcal{S}^n_+$ *such that* $f(\tilde{X}) \in \mathcal{S}^n_+$, *then* $\lim_{k\to\infty} H(X^k, Y^k) = 0$;

(c) *if there exists* $\hat{X} \in \mathcal{S}^n_{++}$ *such that* $f(\hat{X}) \in \mathcal{S}^n_{++}$, *then the sequence* $\{(X^k, Y^k)\}$ *is bounded.*

Proof. Statement (a) follows from Theorem 3(b). To prove statement (b), note first that the assumption implies that $0 \in F(\mathcal{S}^n_+ \times \mathcal{S}^n_+)$. Hence, by Lemma 7(b), we conclude that H is proper with respect to $\mathcal{S}^n \times \mathcal{S}^n_{++}$. It follows from Theorem 3(c) with $E = S$ that $\{H(X^k, Y^k)\}$ converges to zero. The proof of (c) follows similarly from Lemma 7(c) and Theorem 3(d) with $E = S$. $\square$

Statement (a) is within expectation; statement (b) is interesting because its assumption is the feasibility of the NCP in symmetric matrices (16). A consequence of statement (b) is that feasibility of this problem (which is also monotone by assumption (C1)) is sufficient for the sequence $\{H(X^k, Y^k)\}$ to converge to zero, although no boundedness of the sequence $\{(X^k, Y^k)\}$ is asserted. The latter assertion is established under the strict feasibility of the problem (16); this is statement (c).

4. Convex semidefinite programs. In this section we consider the convex semidefinite program studied in [18, 30], namely,

$$\begin{array}{ll} \text{minimize} & \theta(x) \\ \text{subject to} & G(x) \preceq 0, \\ & h(x) = 0, \end{array} \tag{20}$$

where $\theta : \Re^m \to \Re$, $G : \Re^m \to \mathcal{S}^n$, and $h : \Re^m \to \Re^p$ are given smooth mappings. Under a suitable constraint qualification, if x^* is a locally optimal solution of the semidefinite program, then there must exist $(\eta^*, U^*) \in \Re^p \times \mathcal{S}^n_+$ such that

$$\nabla_x L(x^*, U^*, \eta^*) = 0, \quad U^* G(x^*) = 0, \quad U^* \succeq 0, \tag{21}$$

where $L : \Re^m \times \mathcal{S}^n \times \Re^p \to \Re$ is the Lagrangian function defined by

$$L(x, U, \eta) \equiv \theta(x) + U \bullet G(x) - \eta^T h(x) \quad \text{for } (x, U, \eta) \in \Re^m \times \mathcal{S}^n \times \Re^p. \tag{22}$$

Clearly, the first-order optimality condition (21) and the feasibility of x^* is equivalent to the implicitly mixed complementarity problem (6) in which the map $F : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m} \to \mathcal{S}^n \times \Re^{p+m}$ is defined by

$$F(U, V, \eta, x) \equiv \begin{pmatrix} V + G(x) \\ h(x) \\ \nabla_x L(x, U, \eta) \end{pmatrix} \quad \forall (U, V, \eta, x) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m}, \tag{23}$$

and the following correspondence of variables are made: $(U, V) \leftrightarrow (X, Y)$ and $(\eta, x) \leftrightarrow z$. Hence, as in section 3.1, the feasibility of x^* and the first-order optimality condition (21) can be formulated as the CE (Ω, H), where the set Ω and the map $H : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m} \to \mathcal{S}^n \times \Re^{p+m}$ are defined by

$$\Omega \equiv \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m}, \tag{24}$$

$$H(U, V, \eta, x) \equiv \begin{pmatrix} (UV + VU)/2 \\ F(U, V, \eta, x) \end{pmatrix} \quad \text{for } (U, V, \eta, x) \in \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m}. \tag{25}$$

Our goal is to solve the CE (Ω, H) by the potential reduction Newton method. For this purpose, we make several blanket assumptions on problem (20). These are all fairly standard assumptions; in particular, (D4) is a second-order sufficiency condition. The assumptions are as follows.

(D1) The objective function $\theta : \Re^m \to \Re$ is twice continuously differentiable and convex.

(D2) The map $G : \Re^m \to \mathcal{S}^n$ is twice continuously differentiable and *positive semidefinite convex* (psd-convex); that is,

$$G(tx + (1-t)y) \preceq tG(x) + (1-t)G(y) \quad \forall x, y \in \Re^m \ \forall t \in (0, 1).$$

(D3) The map $h : \Re^m \to \Re^p$ is affine, and the (constant) gradients $\{\nabla h_j(x)\}_{j=1}^p$ are linearly independent.

(D4) For every $(x, U, \eta) \in \Re^m \times \mathcal{S}^n_{++} \times \Re^p$, the following implication holds:

$$\left.\begin{array}{c} h'(x; v) = 0 \\ G'(x; v) = 0 \\ v \neq 0 \end{array}\right\} \Longrightarrow v^T L''_{xx}(x, U, \eta) v > 0.$$

(D5) The feasible set

$$X \equiv \{ x \in \Re^m : G(x) \preceq 0;\ h(x) = 0 \}$$

is nonempty and bounded.

We propose below a new interior point method for solving the convex semidefinite program (20) based on the potential reduction Newton algorithm of section 2.3. This method not only generalizes the algorithm developed in section 4.2 of [36] to the context of the nonlinear semidefinite programming problem, but it also allows for a more general choice of starting points. The new algorithm uses a novel potential function ψ which depends on the starting point. A key advantage of the new algorithm is that good convergence properties can be established for arbitrary starting points. This differs from the results in [36], which either require the starting point to satisfy the linear equality constraint $h(x) = 0$ (Theorem 5 in the reference) or do not guarantee the boundedness of the sequence of multipliers (Theorem 4 in the reference).

Let $(U^0, V^0, \eta^0, x^0) \in \mathcal{U} \times \Re^{p+m}$ denote an arbitrary starting point and let $c^0 \equiv h(x^0)$ and $G^0 \in \mathcal{S}^n$ be any matrix such that

$$G(x^0) \prec G^0 \prec G(x^0) + V^0 \quad \text{if } c^0 \neq 0;$$

$$G^0 \succ 0 \quad \text{if } c^0 = 0.$$

Define

$$(26)\quad S \equiv \begin{cases} \left\{ (A, B, c, d) \in \mathcal{S}^n_+ \times \mathcal{S}^n \times \Re^{p+m} : B \succeq \dfrac{c^T c^0}{\|c^0\|^2} G^0 \right\} & \text{if } c^0 \neq 0, \\ \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m} \quad \text{if } c^0 = 0. \end{cases}$$

Note that S depends on the starting point when $h(x^0) \neq 0$.

The following technical lemma is a partial restatement of Lemma 6 of [18] and is used in the subsequent Lemma 9 to establish that the CE (Ω, H) and the set S defined above satisfy conditions (A1)–(A3) of section 2.1.

LEMMA 8. *Assume that $G : \Re^m \to \mathcal{S}^n$ is psd-convex and $h : \Re^m \to \Re^p$ is an affine function. Then the following statements hold:*

(a) *for every $U \in \mathcal{S}^n_+$, the function $x \in \Re^m \mapsto U \bullet G(x)$ is convex;*

(b) *if condition* (D5) *holds then, for every $\bar{B} \in \mathcal{S}^n$ and $\bar{\gamma} \in \Re$, the set*

$$\{ x \in \Re^m : G(x) \preceq \bar{B},\ \| h(x) \| \leq \bar{\gamma} \}$$

is bounded.

LEMMA 9. *Assume that problem* (20) *satisfies conditions* (D1)–(D4). *The following three statements hold:*

(a) *the map F defined by* (23) *satisfies* (B1) *and* (B2) *of section* 3.1;

(b) *the pair* (Ω, H) *with* Ω *and* H *defined by* (25) *and* (24), *respectively, and the set* S *defined by* (26) *satisfy conditions* (A1), (A2), *and* (A3) *of section* 2.1; *and*

(c) *the map* H *restricted to the set* $\mathcal{U} \times \Re^{p+m}$ *is a local homeomorphism.*

Proof. Since the case where $c^0 = 0$ is easy to deal with, the proof below focuses on the case where $c^0 \neq 0$. Conditions (A1) and (A2)(a) are obvious. Clearly, we have

$$\Omega_{\mathcal{I}} = \left\{ (U, V, \eta, x) \in \mathcal{U} \times \Re^{p+m} \,:\, V + G(x) \succ \frac{h(x^0)^T h(x)}{\|h(x^0)\|^2} G^0 \right\}, \tag{27}$$

which is nonempty because it contains the tuple (U^0, V^0, η^0, x^0). Moreover, using (10) we easily see that the set $H^{-1}(\text{int } S) \cap \text{bd } \Omega$ is empty. We have thus proved that condition (A2) holds. Using the same arguments as in the proof of Lemma 3, we can show that if statement (a) holds, then $H'(U, V, \eta, x)$ is nonsingular for every $(U, V, \eta, x) \in \mathcal{U} \times \Re^{p+m}$; in particular, we can conclude that (A3) holds due to (27), and that H restricted to the set $\mathcal{U} \times \Re^{p+m}$ is a local homeomorphism by the inverse function theorem. Thus the remaining proof is to show that F satisfies (B1) and (B2). For this purpose, assume that $(U, V, x, \eta) \in \mathcal{U} \times \Re^{p+m}$ satisfies

$$F'((U, V, x, \eta); (dU, dV, dx, d\eta)) = 0$$

for some $(dU, dV, dx, d\eta) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^{p+m}$ or, equivalently,

$$dV + G'(x; dx) = 0, \tag{28}$$

$$L''_{xx}(x, U, \eta)dx + \sum_{i,j=1}^{n} dU_{ij} \nabla G_{ij}(x) - \sum_{k=1}^{\ell} d\eta_k \nabla h_k(x) = 0, \tag{29}$$

$$h'(x; dx) = 0. \tag{30}$$

Lemma 8(a) together with conditions (D1), (D2), and (D3) and the fact that $U \succeq 0$ imply that $L(x, U, \eta)$ is a convex function of x. Hence, we have $dx^T L''_{xx}(x, U, \eta)dx \geq 0$. Multiplying (29) on the left by dx^T and using this last observation together with (28) and (30), we obtain

$$dU \bullet dV = -dU \bullet G'(x; dx) + d\eta^T h'(x; dx) = dx^T L''_{xx}(x, U, \eta)dx \geq 0. \tag{31}$$

Thus F satisfies (B1). Assume now that

$$F'((U, V, x, \eta); (0, 0, dx, d\eta)) = 0.$$

Then all the relations above hold with $(dU, dV) = (0, 0)$. In particular, (28), (30), and (31) imply that $h'(x; dx) = 0$, $G'(x; dx) = 0$, and $dx^T L''_{xx}(x, U, \eta)dx = 0$. Hence, we conclude that $dx = 0$ due to (D4). Using this and the fact that relation (29) holds with $dU = 0$, we obtain

$$\sum_{k=1}^{\ell} d\eta_k \nabla h_k(x) = 0,$$

which in turn implies that $d\eta = 0$ due to (D3). We have thus shown that F satisfies (B2). $\square$

Associated with the set S, we now introduce the following potential function $p : \text{int } S \to \Re$ defined for any tuple $(A, B, c, d) \in \text{int } S$ by

$$p(A,B,c,d) \equiv \zeta \log \left(\| A \|_F^2 + \left\| B - \frac{c^T c^0}{\| c^0 \|} G^0 \right\|_F^2 + \| c \|^2 + \| d \|^2 \right)$$

$$- \log(\det A) - \log \left(\det \left(B - \frac{c^T c^0}{\| c^0 \|^2} G^0 \right) \right), \tag{32}$$

where ζ is a suitable constant.

We establish in the next result that if $\zeta \geq 3n/2$, then the above potential function satisfies conditions (A4), (A5), and (A6) of section 2.1.

LEMMA 10. *If $\zeta \geq 3n/2$, then the potential function* (32)*, the tuple $a \equiv (I, 0, 0, 0) \in \mathcal{S}^n \times \mathcal{S}^n \times \Re^{p+m}$, and the constant $\bar{\sigma} \equiv 1/2$ satisfy conditions* (A4), (A5), *and* (A6) *of section* 2.1.

Proof. The verification of (A4) is similar to that of Lemma 4. Define

$$\tau \equiv \| A \|_F^2 + \left\| B - \frac{c^T c^0}{\| c^0 \|} G^0 \right\|_F^2 + \| c \|^2 + \| d \|^2,$$

$$\tilde{B} \equiv B - \frac{c^T c^0}{\| c^0 \|} G^0.$$

It is easy to see that

$$\nabla p(A,B,c,d) = \begin{pmatrix} \dfrac{2\zeta}{\tau} A - A^{-1} \\[2ex] \dfrac{2\zeta}{\tau} \tilde{B} - \tilde{B}^{-1} \\[2ex] \dfrac{2\zeta}{\tau} \left(c - \dfrac{\tilde{B} \bullet G^0}{\|c^0\|^2} c^0 \right) + \dfrac{\tilde{B}^{-1} \bullet G^0}{\|c^0\|^2} c^0 \\[2ex] \dfrac{2\zeta}{\tau} d \end{pmatrix}.$$

The definition of τ and $\tilde{B}$ together with a simple algebraic manipulation reveals that

$$\nabla p(A,B,c,d) \bullet (A,B,c,d) = 2(\zeta - n) > 0 \quad \text{for all } (A,B,c,d) \in \text{int } S,$$

and hence that (A5) holds. Moreover, using the fact that

$$(\text{tr} P)^2 \leq n \| P \|_F^2 \quad \text{and} \quad (\text{tr} P^{-1})(\text{tr} P) \geq n^2$$

for every $P \in \mathcal{S}^n$ and $\zeta \geq 3n/2$, we obtain for every $(A, B, c, d) \in \text{int } S$,

$$\begin{aligned} &\frac{[\nabla p(A,B,c,d) \bullet (I,0,0,0)][(A,B,c,d) \bullet (I,0,0,0)]}{\| (I,0,0,0) \|_F^2} \\ &= \frac{1}{n} \left[\frac{2\zeta}{\tau} (\text{tr} A)^2 - (\text{tr} A^{-1})(\text{tr} A) \right] \leq \frac{2\zeta (\text{tr} A)^2}{n \| A \|_F^2} - \frac{(\text{tr} A^{-1})(\text{tr} A)}{n} \\ &\leq 2\zeta - n < 4\zeta - 4n = 2[p(A,B,c,d) \bullet (A,B,c,d)]. \end{aligned}$$

Hence (A6) holds with $a = (I, 0, 0, 0)$ and $\bar{\sigma} = 1/2$. $\square$

The next two results will be used in Theorem 3 to establish the boundedness of the sequence of iterates generated by the potential reduction Newton method under the framework of this section.

LEMMA 11. *Assume that problem* (20) *satisfies conditions* (D1)–(D5). *Then the map* $H : \mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m} \to \mathcal{S}^n \times \Re^{p+m}$ *defined in* (25) *is proper with respect to the set* $\mathcal{S}^n \times F(\mathcal{U} \times \Re^{p+m})$.

Proof. Using Proposition 4(a) and Lemma 7 of [18], we conclude that the map F defined in (23) is (U, V)-equilevel monotone on $\mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m}$. Moreover, by Proposition 4(c) and Lemma 9 of [18], it follows that F is (η, x)-bounded on $\mathcal{S}^n_+ \times \mathcal{S}^n_+ \times \Re^{p+m}$. Since, by Lemma 9, the map H restricted to $\mathcal{U} \times \Re^{m+p}$ is a local homeomorphism, we conclude from Proposition 2 that H is proper with respect to $\mathcal{S}^n \times F(\mathcal{U} \times \Re^{p+m})$. $\square$

In the next result we describe in more detail the set $F(\mathcal{U} \times \Re^{p+m})$ for the map F given by (23).

LEMMA 12. *Assume that problem* (20) *satisfies conditions* (D1)–(D5). *Then* $F(\mathcal{U} \times \Re^{p+m}) = \mathcal{F} \times \Re^m$, *where* F *is the map given by* (23) *and*

$$\mathcal{F} \equiv \{ (B, c) \in \mathcal{S}^n \times \Re^p : \exists x \in \Re^m \text{ such that } G(x) \prec B \text{ and } h(x) = c \}.$$

Moreover, $\mathcal{F}$ *is a convex set.*

Proof. The inclusion $F(\mathcal{U} \times \Re^{p+m}) \subset \mathcal{F} \times \Re^m$ follows straightforwardly from the definition of the map F and the set $\mathcal{U}$. Assume now that $(B, c, d) \in \mathcal{F} \times \Re^m$. We have proved in Lemma 10 of [18] that if conditions (D1)–(D5) hold and $(0, 0) \in \mathcal{F}$, then $(0, 0, 0) \in F(\mathcal{U} \times \Re^{p+m})$. Consider now the problem

$$\begin{aligned} &\text{minimize} \quad \tilde{\theta}(x) \\ &\text{subject to} \quad \tilde{G}(x) \preceq 0, \quad \tilde{h}(x) = 0, \end{aligned}$$

where $\tilde{\theta}(x) \equiv \theta(x) - d^T x$, $\tilde{G}(x) \equiv G(x) - B$, and $\tilde{h}(x) \equiv h(x) - c$ for all $x \in \Re^m$. It is easy to see that the functions $\tilde{\theta}$, $\tilde{G}$, and $\tilde{h}$ also satisfy conditions (D1)–(D5). Hence, applying Lemma 10 of [18] to this new problem, we conclude that $(0, 0, 0) \in \tilde{F}(\mathcal{U} \times \Re^{p+m})$, where $\tilde{F}$ is defined like the function F in (23) with θ, G, and h replaced by $\tilde{\theta}$, $\tilde{G}$, and $\tilde{h}$, respectively. A simple verification shows that $(0, 0, 0) \in \tilde{F}(\mathcal{U} \times \Re^{p+m})$ is equivalent to $(B, c, d) \in F(\mathcal{U} \times \Re^{p+m})$. We have thus shown that $F(\mathcal{U} \times \Re^{p+m}) \supseteq \mathcal{F} \times \Re^m$. Using conditions (D2) and (D3), and some standard arguments, we can easily show that $\mathcal{F}$ is a convex set. $\square$

We establish one technical lemma, which will be used to prove an important conclusion of the main result of this section, Theorem 6.

LEMMA 13. *Let* $\{U^k\}$ *and* $\{V^k\}$ *be two sequences in* $\mathcal{S}^n_{++}$ *such that*

$$\lim_{k \to \infty} (U^k V^k + V^k U^k) = 0.$$

Then

$$\lim_{k \to \infty} (U^k)^{1/2} V^k (U^k)^{1/2} = 0. \tag{33}$$

Proof. Since $(U^k)^{1/2} V^k (U^k)^{1/2}$ is a symmetric matrix, its eigenvalues are all real. Since

$$(U^k)^{-1/2} (U^k V^k) (U^k)^{1/2} = (U^k)^{1/2} V^k (U^k)^{1/2},$$

it follows that all the eigenvalues of U^kV^k are real too. This implies that the eigenvalues of $(U^kV^k)^2$ are all positive. Therefore,

$$2\,\|\,U^kV^k\,\|_F^2 \;\leq\; 2\,\|\,U^kV^k\,\|_F^2 + 2\,\mathrm{tr}\left(U^kV^k\right)^2 \;=\; \|\,U^kV^k + V^kU^k\,\|_F^2.$$

Since the right-hand norm converges to zero as $k \to \infty$, the same holds for the left-hand norm. Thus the spectrum of U^kV^k converges to the single element $\{0\}$. Since this spectrum is the same as that of $(U^k)^{1/2}V^k(U^k)^{1/2}$, the desired limit (33) follows. $\square$

The following is the main convergence result of the potential reduction Newton method specialized to the convex semidefinite program (20). A noteworthy remark about this result is that part (d) does not require the sequence of multipliers $\{(U^k,\eta^k)\}$ to be bounded.

THEOREM 6. *Suppose that problem* (20) *satisfies conditions* (D1)–(D5), *and that* $\{(U^k,V^k,\eta^k,x^k)\}$ *is a sequence generated by the potential reduction Newton method of section* 2.3 *initialized at an arbitrary tuple* $(U^0,V^0,\eta^0,x^0)\} \in \mathcal{U}\times\Re^{p+m}$, *and with* (Ω,H), S, $p : \mathrm{int}\, S \to \Re$ *given by* (24), (25), (26), and (32), *respectively,* $a \equiv (I,0,0,0) \in \mathcal{S}^n\times\mathcal{S}^n\times\Re^{p+m}$, *and* $\bar\sigma \equiv 1/2$. *Assume also that* $\zeta \geq 3/2$ *and* $\limsup_k \sigma_k < 1/2$. *Then, the following statements hold:*

(a) *every accumulation point of* $\{(U^k,V^k,\eta^k,x^k)\}$ *is a solution of the CE* (Ω,H);
(b) *the sequence* $\{(V^k,x^k)\}$ *is bounded; thus* $\{x^k\}$ *has at least one accumulation point;*
(c) $\lim_{k\to\infty} H(U^k,V^k,\eta^k,x^k) = 0$;
(d) *every accumulation point of the sequence* $\{x^k\}$ *is an optimal solution of problem* (20);
(e) *if there exists* $\bar x \in \Re^m$ *such that* $h(\bar x) = 0$ *and* $G(\bar x) \prec 0$ *(that is, problem* (20) *has a Slater point), then the whole sequence* $\{(U^k,V^k,\eta^k,x^k)\}$ *is bounded.*

Proof. By Lemmas 9 and 10, the assumptions of the theorem guarantee that (Ω,H), S, $p : \mathrm{int}\, S \to \Re$, $a = (I,0,0,0)$, and $\bar\sigma = 1/2$ satisfy conditions (A1)–(A6) of section 2.1. Hence, by Theorem 3, we conclude that statement (a) holds and that the sequence $\{H(U^k,V^k,\eta^k,x^k)\}$ is bounded. By the definition of H, this implies that $\{V^k + G(x^k)\}$ and $\{h(x^k)\}$ are bounded, and hence $\{x^k\} \subset \{x \in \Re^m : G(x) \preceq \bar B,\ \|h(x)\| \leq \bar\gamma\}$ for some $(\bar B,\bar\gamma) \in \mathcal{S}^n\times\Re$. Since by Lemma 8(b) the latter set is bounded, we conclude that $\{x^k\}$ is bounded. Clearly, this and the fact that $\{V^k + G(x^k)\}$ is bounded imply that $\{V^k\}$ is also bounded. Hence, statement (b) follows.

The proofs of statements (c) and (e) are based on statements (c) and (d) of Theorem 3. For simplicity, we assume in the remaining proof that $c^0 \equiv h(x^0) \neq 0$; the proof when $c^0 = 0$ is analogous. Define

$$E \equiv \mathcal{S}^n_+ \times \left\{ (B,c) \in \mathcal{S}^n\times[0,c^0] \,:\, B \succeq \frac{c^Tc^0}{\|c^0\|^2}\,G^0 \right\} \times \Re^m.$$

Note that E is a closed subset of S. Moreover, using (D3) and the fact that the third component of a is zero, we easily see that $\{h(x^k)\} \subset [0,c^0]$. Clearly, this implies that $\{H(U^k,V^k,\eta^k,x^k)\} \subset E$. In view of (c) and (d) of Theorem 3, statements (c) and (e) follow once we establish that the map H is proper with respect to

$$E \cap \mathrm{int}\, S \;=\; \mathcal{S}^n_{++} \times \left\{ (B,c) \in \mathcal{S}^n\times[0,c^0] \,:\, B \succ \frac{c^Tc^0}{\|\,c^0\,\|^2}\,G^0 \right\} \times \Re^m$$

and also proper with respect to E under the assumption that $(0,0) \in \mathcal{F}$. We prove first the properness assertion with respect to int $S \cap E$. By Lemmas 11 and 12, we know that H is proper with respect to $\mathcal{S}^n \times F(\mathcal{U} \times \Re^{p+m}) = \mathcal{S}^n \times \mathcal{F} \times \Re^m$. Hence, it suffices to show that int $S \cap E$ is contained in $\mathcal{S}^n \times \mathcal{F} \times \Re^m$, or equivalently that

$$\left\{ (B,c) \in \mathcal{S}^n \times [0,c^0] : B \succ \frac{c^T c^0}{\|c^0\|^2} G^0 \right\} \subset \mathcal{F}. \tag{34}$$

Using the definition of $\mathcal{F}$ and Lemma 8(b), it is easy to see that

$$\text{cl } \mathcal{F} = \{ (B,c) \in \mathcal{S}^n \times \Re^p : \exists x \in \Re^m \text{ such that } G(x) \preceq B \text{ and } h(x) = c \}. \tag{35}$$

Moreover, it follows immediately from the definition of $\mathcal{F}$ and (35) that

$$(B,c) \in \mathcal{F} \Rightarrow (B',c) \in \mathcal{F} \quad \forall B' \succeq B, \tag{36}$$

$$(B,c) \in \text{cl } \mathcal{F} \Rightarrow (B',c) \in \mathcal{F} \quad \forall B' \succ B. \tag{37}$$

Let (B,c) be an arbitrary element of the left-hand set in (34). Since $c \in [0,c^0]$, we have $c = tc^0$ for some $t \in [0,1]$. Hence,

$$B \succ \frac{c^T c^0}{\|c^0\|^2} G^0 = tG^0. \tag{38}$$

Since $(0,0) \in \text{cl } \mathcal{F}$ by (D5), $(G^0,c^0) \in \mathcal{F}$ by (26), and cl $\mathcal{F}$ is a convex set due to Lemma 12 and Proposition III.1.2.7 of [5], we conclude that $(tG^0,tc^0) = t(G^0,c^0) + (1-t)(0,0) \in \text{cl } \mathcal{F}$. Hence, by (37) and (38), we have $(B,c) = (B,tc^0) \in \mathcal{F}$. Hence, (34) holds.

Assume now that $(0,0) \in \mathcal{F}$. To prove the properness assertion with respect to E, it suffices to show that $E \subset \mathcal{S}^n \times \mathcal{F} \times \Re^m$ or, equivalently, that

$$\left\{ (B,c) \in \mathcal{S}^n \times [0,c^0] : B \succeq \frac{c^T c^0}{\| c^0 \|^2} G^0 \right\} \subset \mathcal{F}. \tag{39}$$

If (B,c) is in the left-hand set, then we have $c = tc^0$ and $B \succeq tG^0$ for some $t \in [0,1]$. Since $(0,0) \in \mathcal{F}$ by assumption, $(G^0,c^0) \in \mathcal{F}$ by (26), and $\mathcal{F}$ is convex by Lemma 12, we conclude that $(tG^0,tc^0) \in \mathcal{F}$. Hence, by (36) and the fact that $B \succeq tG^0$, we have $(B,c) = (B,tc^0) \in \mathcal{F}$. Hence, (39) holds.

Finally, we prove statement (d). For each k, let $B^k \equiv G(x^k) + V^k$, $\tilde{B}^k \equiv G(x^k) + (U^k)^{-1}$, and $d^k \equiv \nabla_x L(x^k,U^k,\eta^k)$. It follows that x^k is an optimal solution of the convex program

$$\min \left\{ f(x) - (d^k)^T x - \log\det\left(\tilde{B}^k - G(x)\right) \ : \ h(x) = h(x^k) \right\}, \tag{40}$$

due to the fact that x^k together with the multiplier pair (U^k,η^k) satisfy the optimality condition for this problem. Now let x^∞ be an arbitrary accumulation point of $\{x^k\}$. Clearly, x^∞ is a feasible solution of (20) due to Theorem 6(c). To show the global optimality of x^∞, assume that $\tilde{x}$ is an arbitrary feasible solution of (20). Let $t_k \in [0,1]$ be such that $h(x^k) = t_k h(x^0)$ and define $\tilde{x}^k \equiv t_k x^0 + (1-t_k)\tilde{x}$. Clearly, $\tilde{x}^k$ is feasible to (40). Since $\{t_k\}$ converges to zero, it follows that $\{\tilde{x}^k\}$ converges to $\tilde{x}$. Moreover, since $H(U^k,V^k,\eta^k,x^k) \in S$, by the definition of S (26), we have for each k (cf. (38)),

$$B^k \succ t_k G^0.$$

Hence, it follows that

$$\begin{aligned}
&f(\tilde{x}^k) - (d^k)^T\tilde{x}^k - \log\det\left(\tilde{B}^k - G(\tilde{x}^k)\right)\\
&\geq f(x^k) - (d^k)^T x^k - \log\det\left(\tilde{B}^k - G(x^k)\right)\\
&= f(x^k) - (d^k)^T x^k + \log\det\left(U^k\right)
\end{aligned}$$

for all k. Rearranging this inequality, we obtain

$$\begin{aligned}
&f(\tilde{x}^k) - f(x^k) - (d^k)^T(\tilde{x}^k - x^k)\\
&\qquad \geq \log\det\left((U^k)^{1/2}\left[\tilde{B}^k - G(\tilde{x}^k)\right](U^k)^{1/2}\right)\\
&\qquad = \log\det\left(I + (U^k)^{1/2}\left[G(x^k) - G(\tilde{x}^k)\right](U^k)^{1/2}\right)\\
&\qquad = \log\det\left(I - (U^k)^{1/2}V^k(U^k)^{1/2} + (U^k)^{1/2}\left[B^k - G(\tilde{x}^k)\right](U^k)^{1/2}\right)\\
&\qquad \geq \log\det\left(I - (U^k)^{1/2}V^k(U^k)^{1/2}\right),
\end{aligned}$$

where the last inequality follows from the fact that

$$B^k - G(\tilde{x}^k) \succeq B^k - t_k G(x^0) - (1-t_k)G(\tilde{x}) \succeq B^k - t_k G(x^0) \succ B^k - t_k G^0 \succ 0.$$

Hence, as k goes to ∞, we may invoke Lemma 13 to conclude that $f(\tilde{x}) - f(x^\infty) \geq 0$. We have thus proved that x^∞ is an optimal solution of (20). ∎

Assuming that $G^0 \succ 0$, it is possible to show that the potential function (32), $a \equiv (I, I, 0, 0)$, and $\bar{\sigma} = 1$ satisfy the inequality in condition (A6) for every (A, B, c, d) in the set $E \cap \operatorname{int} S$, where E is defined as in the proof of Theorem 6. Using this fact, it is possible to establish a convergence result similar to Theorem 6 for $a \equiv (I, I, 0, 0)$ and $\bar{\sigma} = 1$. The interesting point to note is that Theorem 3 still holds if we assume the inequality in condition (A6) to be valid only for points in the sequence $\{H(x^k)\}$. Details are omitted.

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REFERENCES

[1] F. Alizadeh, *Interior point methods in semidefinite programming with applications to combinatorial optimization*, SIAM J. Optim., 5 (1995), pp. 13–51.

[2] F. Alizadeh, J.-P. Haeberly, and M. Overton, *Primal-dual interior-point methods for semidefinite programming: Convergence rates, stability and numerical results*, SIAM J. Optim., 8 (1998), pp. 746–768.

[3] R. M. Freund, *Complexity of an Algorithm for Finding an Approximate Solution of a Semidefinite Program with No Regularity Condition*, Working Paper OR 302-94, Operations Research Center, Massachusetts Institute of Technology, Cambridge, December 1994.

[4] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz, *An interior-point method for semidefinite programming*, SIAM J. Optim., 6 (1996), pp. 342–361.

[5] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms* I, Comprehensive Study in Mathematics 305, Springer-Verlag, New York, 1993.

[6] F. Jarre, *An interior-point method for minimizing the maximum eigenvalue of a linear combination of matrices*, SIAM J. Control and Optim., 31 (1993), pp. 1360–1377.

[7] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Lecture Notes in Comput. Sci. 538, Springer-Verlag, Berlin, 1991.

[8] M. Kojima, M. Shida, and S. Shindoh, *A predictor-corrector interior-point algorithm for the semidefinite linear complementarity problem using the Alizadeh-Haeberly-Overton search direction*, SIAM J. Optim, 9 (1999), pp. 444–465.

[9] M. Kojima, M. Shida, and S. Shindoh, *Local convergence of predictor-corrector infeasible-interior-point algorithms for SDPs and SDLCPs*, Math. Programming, 80 (1998), pp. 129–160.

[10] M. Kojima, S. Shindoh, and S. Hara, *Interior-point methods for the monotone semidefinite linear complementarity problem in symmetric matrices*, SIAM J. Optim., 7 (1997), pp. 86–125.

[11] C.-J. Lin and R. Saigal, *A Predictor-Corrector Method for Semi-Definite Programming*, Working Paper, Dept. of Industrial and Operations Engineering, The University of Michigan, Ann Arbor, 1995.

[12] Z.-Q. Luo, J. F. Sturm, and S. Zhang, *Superlinear convergence of a symmetric primal-dual path following algorithm for semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 59–81.

[13] L. McLinden, *The complementarity problem for maximal monotone multifunctions*, in Variational Inequalities and Complementarity Problems, R. Cottle, F. Giannessi, and J.-L. Lions, eds., John Wiley, New York, 1980, pp. 251–270.

[14] N. Megiddo, *Pathways to the optimal set in linear programming*, in Progress in Mathematical Programming: Interior Point and Related Methods, N. Megiddo, ed., Springer-Verlag, New York, 1989, pp. 131–158.

[15] R. D. C. Monteiro, *Primal-dual path-following algorithms for semidefinite programming*, SIAM J. Optim., 7 (1997), pp. 663–678.

[16] R. D. C. Monteiro, *Polynomial convergence of primal-dual algorithms for semidefinite programming based on the Monteiro and Zhang family of directions*, SIAM J. Optim., 8 (1998), pp. 797–812.

[17] R. D. C. Monteiro and J.-S. Pang, *Properties of an interior-point mapping for mixed complementarity problems*, Math. Oper. Res., 21 (1996), pp. 629–654.

[18] R. D. C. Monteiro and J.-S. Pang, *On two interior-point mappings for nonlinear semidefinite complementarity problems*, Math. Oper. Res., 23 (1998), pp. 39–60.

[19] R. D. C. Monteiro and T. Tsuchiya, *Polynomial convergence of a new family of primal-dual algorithms for semidefinite programming*, SIAM J. Optim., 9 (1999), pp. 551–577.

[20] R. D. C. Monteiro and T. Tsuchiya, *Polynomiality of Primal-Dual Algorithms for Semidefinite Linear Complementarity Problems Based on the Kojima-Shindoh-Hara Family of Directions*, Manuscript, School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, August 1996; Math. Programming, to appear.

[21] R. D. C. Monteiro and P. Zanjácomo, *A note on the existence of the Alizadeh-Haeberly-Overton direction for semidefinite programming*, Math. Programming, 78 (1997), pp. 393–396.

[22] R. D. C. Monteiro and Y. Zhang, *A unified analysis for a class of path-following primal-dual interior-point algorithms for semidefinite programming*, Math. Programming, 81 (1998), pp. 281–299.

[23] Yu. E. Nesterov and A. S. Nemirovskii, *Polynomial barrier methods in convex programming*, Ekonomika i Mat. Metody, 24 (1988), pp. 1084–1091 (in Russian).

[24] Yu. E. Nesterov and A. S. Nemirovskii, *Self-Concordant Functions and Polynomial Time Methods in Convex Programming*, Preprint, Central Economic & Mathematical Institute, USSR Acad. Sci. Moscow, 1989.

[25] Yu. E. Nesterov and A. S. Nemirovskii, *Interior Point Methods in Convex Programming: Theory and Applications*, SIAM, Philadelphia, PA, 1994.

[26] Yu. E. Nesterov and M. Todd, *Self-scaled barriers and interior-point methods for convex programming*, Math. Oper. Res., 22 (1997), pp. 1–42.

[27] Yu. E. Nesterov and M. Todd, *Primal-dual interior-point methods for self-scaled cones*, SIAM J. Optim., 8 (1998), pp. 324–364.

[28] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, San Diego, 1970.

[29] F. A. Potra and R. Sheng, *A superlinearly convergent primal-dual infeasible-interior-point algorithm for semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 1007–1028.

[30] A. Shapiro, *First and second order analysis of nonlinear semidefinite programs*, Math. Programming, 77 (1997), pp. 301–320.

[31] M. Shida, S. Shindoh, and M. Kojima, *Existence and uniqueness of search directions in interior-point algorithms for the SDP and the monotone SDLCP*, SIAM J. Optim., 8 (1998), pp. 387–396.

[32] J. F. Sturm and S. Zhang, *Symmetric Primal-Dual Path-Following Algorithms for Semidefinite Programming*, Report 9554/A, Econometric Institute, Erasmus University, Rotterdam, The Netherlands, November 1995.
[33] M. J. Todd, K. C. Toh, and R. H. Tütüncü, *On the Nesterov–Todd direction in semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 769–796.
[34] P. Tseng, *Search directions and convergence analysis of some infeasible path-following methods for the monotone semi-definite LCP*, Optim. Methods. Softw., 9 (1998), pp. 245–268.
[35] L. Vandenberghe and S. Boyd, *A primal-dual potential reduction method for problems involving matrix inequalities*, Math. Programming, 69 (1995), pp. 205–236.
[36] T. Wang, R. D. C. Monteiro, and J.-S. Pang, *An interior point potential reduction method for constrained equations*, Math. Programming, 74 (1996), pp. 159–195.
[37] Y. Zhang, *On extending some primal-dual interior-point algorithms from linear programming to semidefinite programming*, SIAM J. Optim., 8 (1998), pp. 365–386.