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A note on the existence of the Alizadeh–Haeberly–Overton direction for semidefinite programming [1]

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Abstract

This note establishes a new sufficient condition for the existence and uniqueness of the Alizadeh–Haeberly–Overton direction for semidefinite programming. © 1997 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

Keywords: Semidefinite programming; Primal-dual search directions; Interior-point methods

1. Introduction

Let $\mathbb{R}^p$, $\mathbb{R}^{p\times q}$ and $\mathcal{S}^p$ denote the p-dimensional Euclidean space, the set of all $p \times q$ real matrices and the set all $p \times p$ real symmetric matrices, respectively. Let $\mathcal{S}^p_+$ and $\mathcal{S}^p_{++}$ denote the set of all matrices in $\mathcal{S}^p$ which are positive semidefinite and positive definite, respectively. Given P and Q in $\mathbb{R}^{p\times q}$, let $P \bullet Q \equiv \sum_{i,j} P_{ij}Q_{ij}$.

Consider the *semidefinite programming* (SDP) problem

$$(\mathrm{P}) \qquad \min\{C \bullet X : A_i \bullet X = b_i,\ i = 1,\ldots,m,\ X \in \mathcal{S}^n_+\}, \tag{1}$$

and its associated dual SDP problem

$$(\mathrm{D}) \qquad \max\left\{b^{\mathrm{T}}y : \sum_{i=1}^{m} y_iA_i + S = C,\ S \in \mathcal{S}^n_+\right\}, \tag{2}$$

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where $C \in \mathcal{S}^n$, $A_i \in \mathcal{S}^n$, $i = 1, \dots, m$, and $b = (b_1, \dots, b_m)^{\mathrm{T}} \in \mathbb{R}^m$ are the data, and $X \in \mathcal{S}^n_+$ and $(S, y) \in \mathcal{S}^n_+ \times \mathbb{R}^m$ are the primal and dual variables, respectively. Throughout this note we assume that the matrices $A_1, \dots, A_m$ are linearly independent.

Shida, Shindoh and Kojima [4] show that the Alizadeh–Haeberly–Overton (AHO) search direction (see Ref. [1]) at a possibly infeasible point $(X, S, y) \in \mathcal{S}_{++} \times \mathcal{S}_{++} \times \mathbb{R}^m$ exists and is unique whenever the matrix $XS + SX$ is positive semidefinite. This note establishes an alternative sufficient condition for the existence of the AHO search direction, namely: $\|S^{1/2}XS^{1/2} - \nu I\| \leqslant \nu/2$ for some scalar $\nu > 0$.

In addition to the notation introduced above, we also use the following one throughout this note. The superscript $^{\mathrm{T}}$ denotes transpose. The trace of a matrix $Q \in \mathbb{R}^{p\times p}$ is denoted by $\operatorname{Tr} Q \equiv \sum_{i=1}^n Q_{ii}$. For a matrix $Q \in \mathbb{R}^{p\times p}$ with all real eigenvalues, we denote its smallest and largest eigenvalues by $\lambda_{\min}[Q]$ and $\lambda_{\max}[Q]$, respectively. The Euclidean norm and its associated operator norm are both denoted by $\|\cdot\|$; hence, $\|Q\| \equiv \max_{\|u\|=1} \|Qu\|$ for any $Q \in \mathbb{R}^{p\times p}$. The Frobenius norm of $Q \in \mathbb{R}^{p\times p}$ is $\|Q\|_F \equiv (Q \bullet Q)^{1/2}$.

2. A sufficient condition

The AHO search direction at a point $(X, S, y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \mathbb{R}^m$ with centrality parameter $\sigma \in \mathbb{R}$ is by definition the solution $(\Delta X, \Delta S, \Delta y) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ of the linear system

$$
\begin{aligned}
&X\Delta S + \Delta S X + S\Delta X + \Delta X S = 2\sigma\mu I - XS - SX,\\
&\sum_{i=1}^m \Delta y_i A_i + \Delta S = C - \sum_{i=1}^m y_i A_i - S,\\
&A_i \bullet \Delta X = b_i - A_i \bullet X, \quad \forall i = 1, \dots, m,
\end{aligned}
\tag{3}
$$

where $\mu \equiv (X \bullet S)/n$. It has been shown by Shida, Shindoh and Kojima [4] (see also Todd, Toh and Tütüncü [5]) that system (3) has a unique solution, and hence that the AHO search direction is well-defined, whenever the condition $XS + SX \in \mathcal{S}^n_+$ holds.

The following theorem provides an alternative sufficient condition for the above system to have a unique solution.

Theorem 2.1. *If $(X, S, y) \in \mathcal{S}^n_{++} \times \mathcal{S}^n_{++} \times \mathbb{R}^m$ is such that*

$$\|S^{1/2}XS^{1/2} - \nu I\| \leqslant \frac{\nu}{2} \tag{4}$$

for some scalar $\nu > 0$, then system (3) has a unique solution.

Proof. The left-hand side of system (3) can be viewed as a linear map of $(\Delta X, \Delta S, \Delta y)$ from the space $\mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ into itself. Hence, to show that (3) has a unique solution it suffices to prove that the unique solution $(\widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ of the homogeneous system

$$
\begin{aligned}
&X\widetilde{\Delta S} + \widetilde{\Delta S}X + S\widetilde{\Delta X} + \widetilde{\Delta X}S = 0,\\
&\sum_{i=1}^{m} \widetilde{\Delta y}_i A_i + \widetilde{\Delta S} = 0, \qquad (5)\\
&A_i \bullet \widetilde{\Delta X} = 0, \quad \forall i = 1,\dots,m,
\end{aligned}
$$

is $(\widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0,0,0)$. Using the two last equations of (5), we easily see that

$$\widetilde{\Delta X} \bullet \widetilde{\Delta S} = \mathrm{Tr}\ (\widetilde{\Delta X}\widetilde{\Delta S}) = 0. \qquad (6)$$

If $V \equiv X\widetilde{\Delta S} + \widetilde{\Delta X}S = 0$ then it is easy to verify with the aid of (6) that $\widetilde{\Delta X} = \widetilde{\Delta S} = 0$. Since the matrices $A_1, \dots, A_m$ are linearly independent, it follows from the second equation of (5) that $\widetilde{\Delta y} = 0$. Hence, $V = 0$ implies that $(\widetilde{\Delta X}, \widetilde{\Delta S}, \widetilde{\Delta y}) = (0,0,0)$. To complete the proof, it remains to show that $V = 0$. Assume for contradiction that $V \neq 0$. Using this assumption and the first equation of (5), we obtain

$$0 = \|V + V^{\mathrm{T}}\|_F^2 = 2\|V\|_F^2 + 2\,\mathrm{Tr}\ V^2 > 2\,\mathrm{Tr}\ V^2 = 2\,\mathrm{Tr}\ (X\widetilde{\Delta S} + \widetilde{\Delta X}S)^2.$$

On the other hand, letting $\widehat{\Delta X} \equiv X^{-1/2}\widetilde{\Delta X}X^{-1/2}$ and $\widehat{\Delta S} \equiv X^{1/2}\widetilde{\Delta S}X^{1/2}$, we have by (6) that $\mathrm{Tr}\ (\widehat{\Delta X}\widehat{\Delta S}) = 0$. This identity together with (4), the fact that $\|AB\|_F \leqslant \|A\|_F\|B\|$ and $\|B^{\mathrm{T}}AB\|_F \geqslant \lambda_{\min}[B^{\mathrm{T}}B]\ \|A\|_F$ for every $A \in \mathcal{S}^n$ and $B \in \mathbb{R}^{n\times n}$ (see exercise 20 of Section 5.6 and Corollary 4.5.11 of Ref. [2], respectively) and the Cauchy–Schwarz inequality for the Frobenius norm imply that

$$
\begin{aligned}
\mathrm{Tr}\ (X\widetilde{\Delta S} + \widetilde{\Delta X}S)^2 &= \|X^{1/2}\widetilde{\Delta S}X^{1/2}\|_F^2 + \|S^{1/2}\widetilde{\Delta X}S^{1/2}\|_F^2 + 2\,\mathrm{Tr}\ (X\widetilde{\Delta S}\widetilde{\Delta X}S)\\
&= \|\widehat{\Delta S}\|_F^2 + \|S^{1/2}X^{1/2}\widehat{\Delta X}X^{1/2}S^{1/2}\|_F^2 + 2\,\mathrm{Tr}\ (\widehat{\Delta S}\widehat{\Delta X}X^{1/2}SX^{1/2})\\
&\geqslant \|\widehat{\Delta S}\|_F^2 + (\lambda_{\min}[S^{1/2}XS^{1/2}])^2\|\widehat{\Delta X}\|_F^2 + 2\,\mathrm{Tr}\ [\widehat{\Delta S}\widehat{\Delta X}(X^{1/2}SX^{1/2} - \nu I)]\\
&\geqslant \|\widehat{\Delta S}\|_F^2 + \frac{\nu^2}{4}\|\widehat{\Delta X}\|_F^2 - 2\,\|\widehat{\Delta S}\|_F\|\widehat{\Delta X}\|_F\|X^{1/2}SX^{1/2} - \nu I\|\\
&\geqslant \|\widehat{\Delta S}\|_F^2 + \frac{\nu^2}{4}\|\widehat{\Delta X}\|_F^2 - \nu\|\widehat{\Delta S}\|_F\|\widehat{\Delta X}\|_F\\
&= \left\|\widehat{\Delta S} - \frac{\nu}{2}\widehat{\Delta X}\right\|_F^2 \geqslant 0.
\end{aligned}
$$

Since this inequality contradicts the previous one, we conclude that $V = 0$.

It is easy to see that the set of all points $(X, S, y) \in \mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \mathbb{R}^m$ satisfying

$$\|S^{1/2}XS^{1/2} - \nu I\| \leqslant \gamma\nu, \quad \text{for some } \nu > 0,$$

coincides with the set $\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \mathbb{R}^m$ when $\gamma = 1$. Hence, for $\gamma = 1/2$ the corresponding set can be thought of as being "half" of the whole space $\mathcal{S}_{++}^n \times \mathcal{S}_{++}^n \times \mathbb{R}^m$.

Using techniques similar to those used in the proof of Theorem 2.1, it is possible to show that condition (4) implies the sufficient condition for the existence of the AHO direction proposed in Theorem 3.1 of Todd, Toh and Tütüncü [5], namely that $E^{-1}F$

is positive definite where $E \equiv S \otimes I + I \otimes S$, $F \equiv X \otimes I + I \otimes X$ and $\otimes$ denotes the Kronecker product between two matrices (see Chapter 4 of Ref. [3]). Instead of establishing the latter condition, we gave a direct proof that condition (4) implies the existence of the AHO direction.

Finally, we observe that condition (4) of Theorem 2.1 does not imply that $XS + SX \in \mathcal{S}^n_+$ as the following example of matrices X and S illustrates:

$$X \equiv \begin{bmatrix} 8 & 1 \\ 1 & 0.5 \end{bmatrix}, \qquad S \equiv \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}.$$

It is easy to see that $XS + SX$ is indefinite and that $\|S^{1/2}XS^{1/2} - \nu I\| = \nu/2$ with $\nu = 8$. Also, it is easy to see that the matrices

$$X \equiv \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \qquad S \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfy $XS + SX \in \mathcal{S}^n_+$ and $\|S^{1/2}XS^{1/2} - \nu I\| > \nu/2$ for any $\nu > 0$. Hence, the positive semidefiniteness of $XS + SX$ does not imply condition (4) either.

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