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CONVERGENCE AND BOUNDARY BEHAVIOR OF THE PROJECTIVE SCALING TRAJECTORIES FOR LINEAR PROGRAMMING*†

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We analyze the convergence and boundary behavior of the continuous trajectories of the vector field induced by the projective scaling algorithm as applied to (possibly degenerate) linear programming problems in Karmarkar's standard form. We show that a projective scaling trajectory tends to an optimal solution which in general depends on the starting point. When the optimal solution is unique, we show that all projective scaling trajectories approach the optimal solution through the same asymptotic direction. Our analysis is based on the affine scaling trajectories for the homogeneous standard form linear program that arises from Karmarkar's standard form linear program by removing the unique nonhomogeneous constraint.

1. Introduction. Consider the linear programming problem in Karmarkar's standard form as follows:

$$\text{(P)} \qquad \min\{c^T w \mid Aw = 0,\ e^T w = 1,\ w \geqslant 0\},$$

where A is an $(m \times n)$-matrix $(m \leqslant n - 1)$ of full rank, c and w are n-vectors, and e denotes the n-vector of all ones. Assume that (P) has a feasible solution $w > 0$. The projective scaling algorithm designed for solving problem (P), under the assumption that the optimal value of (P) is 0, was first presented by Karmarkar in his seminal paper [11]. Variants of the projective scaling algorithm have been presented by many authors including Anstreicher [2], de Ghellinck and Vial [5], Gay [8], Gonzaga [9], Todd and Burrell [18], and Ye and Kojima [21]. Another algorithm designed for solving linear programming problems in standard form, namely, the affine scaling algorithm, was presented by Dikin [6] and was later (independently) reintroduced by Barnes [3] and Vanderbei, Meketon, and Freedman [19] as a variant of the projective scaling algorithm. The continuous trajectories generated by the vector fields induced by the projective and affine scaling algorithms are commonly referred to as projective scaling trajectories and affine scaling trajectories respectively.

Continuous trajectories in the context of nonlinear programming have been discussed long ago within the nonlinear programming community. For instance, in the book of Fiacco and McCormick [7], continuous trajectories that arise as sets of minimizers for parametrized families of weighted logarithmic barrier problems, usually called weighted barrier trajectories, are systematically studied.

With the discovery of Karmarkar's algorithm and other related interior point methods, a new effort arose towards studying their underlying continuous trajectories.

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Because of the special structure exhibited by linear programming problems, these continuous trajectories turn out to be more amenable to a deeper analysis than the trajectories associated with algorithms for general nonlinear optimization problems.

This paper is concerned with two issues regarding the projective scaling trajectories. The first issue, which we always refer to as convergence behavior, is whether a projective scaling trajectory converges to an optimal solution, and if so, there is also the problem of characterizing this optimal solution when more than one optimal solution exist. The second issue, which we always refer to as boundary behavior, is about the existence and characterization of the asymptotic directions of these trajectories when they approach the optimal face.

Before describing our approach in this paper, we give some historic review on the analysis of continuous trajectories associated with interior point algorithms. The importance of analyzing the projective and affine scaling trajectories was recently pointed out by Karmarkar [12]. Bayer and Lagarias [4] present a systematic study of some properties and mathematical structures of these trajectories. Megiddo and Shub [14] and Shub [17] analyze the boundary behavior of these trajectories near the optimal vertex under nondegeneracy assumptions. It should be pointed out that [14] and [17] assume that these trajectories converge to the unique optimal vertex without providing any rigorous proof of this fact. The complete analysis of the convergence and boundary behavior of the affine scaling trajectories without imposing any nondegeneracy assumptions was given for the first time in Adler and Monteiro [1]. The analysis in [1] is based on characterizing the affine scaling trajectories as solutions of certain parametrized logarithmic barrier families of problems. Using a different approach, Witzgall, Boggs, and Domich [20] also investigated the convergence and boundary of the affine scaling trajectories without nondegeneracy assumptions. The latter also contains a good historic review on the development and relations of several works which discuss continuous trajectories for linear and nonlinear programming problems.

Another effort towards analyzing continuous trajectories for linear programs was the work of Megiddo [13] which studies the weighted barrier trajectories in the framework of the primal-dual complementarity relationship for the special case of linear and convex quadratic programming problems. A polynomial algorithm, namely, the primal-dual affine scaling algorithm, which has the weighted barrier trajectories as its continuous trajectories, was presented and analyzed by Monteiro, Adler, and Resende [15].

Using a rather different approach, this paper extends the results of [14] and [17] about the projective scaling trajectories in essentially two ways. First, no nondegeneracy assumptions are imposed on the linear program under consideration. Second, besides the analysis of the boundary behavior of the projective scaling trajectories, we also provide a detailed analysis of the convergence behavior of these trajectories. We should point out that various researchers believe the fact that any projective scaling trajectory converges to an optimal solution although no proof of this fact is available even under nondegeneracy assumptions.

The development in this paper is closely related to the one in [1] in the following sense. Let (HP) denote the homogeneous linear program that arises from problem (P) by removing the constraint $e^Tw = 1$. As showed in Bayer and Lagarias [4], every projective scaling trajectory for (P) can be obtained from an affine scaling trajectory for problem (HP) by projecting it radially into the simplex $\{x \in \mathbf{R}^n | e^Tx = 1,\ x \geqslant 0\}$. Using this fact, we will be able to translate the behavior of the affine scaling trajectories for problem (HP) to the projective scaling trajectories for problem (P). We do so under two cases: when the optimal value of (P) is zero, and when it is positive. In the first and more interesting case, it turns out that the set of optimal solutions of problem (HP) is unbounded. Since [1] analyzes the behavior of the affine scaling trajectories for a linear program that has bounded optimal face, it is necessary to extend the results of [1] in order to handle this more general case.

This paper is organized as follows. In §2, we briefly recall the definition of the projective scaling vector field and its associated trajectories for problem (P). We then discuss the relation between these trajectories and the affine scaling trajectories for problem (HP) by recalling the result of Bayer and Lagarias. In §3, we show how to extend the analysis of [1] for the affine scaling trajectories associated with linear programs that have unbounded optimal face. In §4, we analyze the convergence and boundary behavior of the projective scaling trajectories for the case in which the optimal value of problem (P) is 0. We show that the projective scaling trajectories converge to optimal solutions which depend on the starting points. We also analyze the convergence behavior of the dual estimate that naturally arises in the context of the projective scaling method as described in [18]. In §5, we consider the case when the optimal value of (P) is positive and show that every projective scaling trajectory converges to the unique point that minimizes the so-called Karmarkar potential function associated with problem (P).

2. Related results. In this section, we recall the definition of the projective scaling vector field and its associated trajectories for a linear programming problem in Karmarkar's standard form. We also motivate our approach towards analyzing the convergence and boundary behavior of the projective scaling trajectories.

Karmarkar's standard form for a linear programming problem is

$$\text{(P)} \qquad \min\{c^T w \mid Aw = 0,\ e^T w = 1,\ w \geqslant 0\},$$

where A is an $(m \times n)$-matrix $(m \leqslant n-1)$, $c, w \in \mathbf{R}^n$, and e denotes the n-vector of all ones. (In fact, Karmarkar's standard form also requires that the optimal value of (P) be equal to zero. However, we do not impose this condition beforehand.) Associated with problem (P), we also consider the homogeneous standard form linear programming problem as follows:

$$\text{(HP)} \qquad \min\{c^T x \mid Ax = 0,\ x \geqslant 0\}.$$

Problems (P) and (HP) are strongly related in the sense that information in one of them gives information in the other problem, as will be pointed out below. First, we introduce some notation to be used throughout this paper. Let

$$\begin{aligned}
&\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x > 0\},\\
&W = \{w \in \mathbf{R}^n \mid Aw = 0,\ e^T w = 1,\ w \geqslant 0\},\\
&W_I = W \cap \mathbf{R}^n_+,\\
&S = \{x \in \mathbf{R}^n \mid Ax = 0,\ x \geqslant 0\},\\
&S_I = S \cap \mathbf{R}^n_+.
\end{aligned}$$

The sets W and S are the feasible sets for problems (P) and (HP) respectively. Also, W_I and S_I are respectively the relative interiors of W and S whenever $W_I \neq \varnothing$. The lower case letter e will denote the vector of all ones whose dimension is dictated by the appropriate context. If x is a lower case letter that denotes a vector $x = (x_1, \ldots, x_n)^T$, then a capital letter will denote the diagonal matrix with the components of the vector on the diagonal, i.e., $X = \operatorname{diag}(x_1, \ldots, x_n)$. If x and z are two column n-vectors, we define their product xz to be the vector $XZe =$

$(x_1z_1, \ldots, x_nz_n)^T$. The inverse of x under this operation is denoted by x^{-1} and is given by $x^{-1} \equiv X^{-1}e$. In this way, expressions like x^{-1} and $x^{-2}z$ are defined if all the components of x are nonzero. No confusion should arise between the expressions xz and x^Tz, where the latter just denotes the inner product of x and z. Given an $(m \times n)$-matrix A and a subset B of the index set $\{1, \ldots, n\}$, we denote by A_B the submatrix of A associated with the index set B. A function x: $I \subset \mathbf{R} \to \mathbf{R}^n$, where I is an open interval, will be called a *path* or *curve*. We will say that a path x: $I \to \mathbf{R}^n$ passes through a point $x^0 \in \mathbf{R}^n$ if for some $\mu^0 \in I$, we have $x(\mu^0) = x^0$. The set of points traced by a path will be called a *trajectory*.

Let r^* denote the optimal value of problem (P). We impose the following assumptions on problem (P).

ASSUMPTION 2.1. (a) $r^* \geqslant 0$.
(b) $W_I \neq \emptyset$.
(c) *The matrix A has rank m.*

In [11], it is assumed that the optimal value of problem (P) is 0 in order to derive a polynomially convergent algorithm for (P), namely, the so-called projective scaling algorithm. However, the analysis of the projective scaling trajectories for the case $r^* > 0$ can also be derived using the same approach as for the case $r^* = 0$. Therefore, we will consider both cases simultaneously in our analysis. It can be easily shown that $r^* \geqslant 0$ if and only if problem (HP) has an optimal solution, and in this case $0 \in \mathbf{R}^n$ is an optimal solution of (HP). Hence, under our assumptions, 0 is an optimal solution of (HP). The following results relating problems (P) and (HP) can be easily shown.

PROPOSITION 2.1. *$r^* > 0$ if and only if 0 is the unique optimal solution of* (HP).

PROPOSITION 2.2. *Assume that $r^* = 0$ and let $w \in W$ be given. Then w is an optimal solution of* (P) *if and only if for some (every) $\lambda > 0$, λw is an optimal solution of* (HP).

Observe that when $r^* = 0$, Proposition 2.2 implies that the set of optimal solutions of problem (HP) is unbounded.

We next briefly describe the projective scaling method. We refer the reader to [14] for a more detailed discussion of the projective scaling method as described below. Given a point $x \in W_I$, the projective scaling algorithm assigns to x a search direction as follows:

$$\xi(x) = [X - xx^T]\left[I - XA^T(AX^2A^T)^{-1}AX\right]Xc. \tag{2.1}$$

Note that $\xi(x)$ is a feasible direction for problem (P), that is, $\xi(x) \in \{h \in \mathbf{R}^n | Ah = 0, e^Th = 0\}$. The projective scaling algorithm determines the next iterate $\hat{x}$ according to $\hat{x} = x - \alpha(x)\xi(x)$, where $\alpha(x) > 0$ is an appropriate step size which guarantees that $\hat{x} > 0$ as well as a sufficient decrease in a potential function associated with problem (P) [11]. The transformation $\xi(x)$ then defines a vector field in W_I. In the following, we will be interested in studying the convergence and boundary behavior of the trajectories generated by $\xi(x)$, that is, the set of points traced by the solution curves of the following differential equation:

$$\dot{w}(t) = \xi(w(t)).$$

Note that, since ξ is a vector field in W_I, $w(t) \in W_I$ for every t for which $w(t)$ is defined. In our analysis, we will make use of another vector field related to problem

(HP), namely, the affine scaling vector field associated with (HP) defined as follows:

$$\Phi(x) = X\left[I - XA^T(AX^2A^T)^{-1}AX\right]Xc \tag{2.2}$$

for $x \in S_I$. The trajectories generated by $\Phi(x)$ are the sets of points traced by the solution curves of the following differential equation:

$$\dot{x}(t) = \Phi(x(t)). \tag{2.3}$$

Note that $x(t) \in S_I$ for every t for which $x(t)$ is defined. These trajectories have been studied in [1] by considering the following reparametrized differential equation:

$$\dot{x}(\mu) = \frac{1}{\mu^2}\Phi(x(\mu)), \qquad \mu > 0, \tag{2.4}$$

which yields the same family of trajectories as the one associated with (2.3) but with different parametrizations. Specifically, let $x^0 \in S_I$ and $\mu^0 > 0$ be given. If $v(t)$ is a solution curve of (2.3) satisfying $v(0) = x^0$, then $x(\mu) \equiv v((\mu^0)^{-1} - \mu^{-1})$ is a solution curve of (2.4) satisfying $x(\mu^0) = x^0$. In the next section, we will show that a solution curve of (2.4) is always defined over an open interval of the form $(0, a)$, where $a > 0$ ($a = \infty$ is also allowable).

The next result is due to Bayer and Lagarias [4] and will play a crucial role in our analysis of the projective scaling trajectories.

PROPOSITION 2.3 (Bayer & Lagarias). *If x: $I = (0, a) \to S_I$ is a solution curve of (2.4) then the curve w: $I \to W_I$ defined by $w(\mu) \equiv [1/e^Tx(\mu)]x(\mu)$ satisfies the relation $\dot{w}(\mu) = [e^Tx(\mu)/\mu^2]\xi(w(\mu))$ for all $\mu \in I$. As a consequence w: $I \to W_I$ is a parametrization of a projective scaling trajectory.*

PROOF. First note that for all $x \in S_I$ and $\lambda > 0$, we have

$$\Phi(\lambda x) = \lambda^2\Phi(x). \tag{2.5}$$

Using this fact, we obtain

$$\begin{aligned}\dot{w}(\mu) &= \frac{d}{d\mu}\left[\frac{1}{e^Tx(\mu)}x(\mu)\right] = \frac{1}{e^Tx(\mu)}\dot{x}(\mu) - \frac{e^T\dot{x}(\mu)}{\left[e^Tx(\mu)\right]^2}x(\mu)\\ &= \frac{1}{\mu^2e^Tx(\mu)}\Phi(x(\mu)) - \frac{e^T\Phi(x(\mu))}{\mu^2\left[e^Tx(\mu)\right]^2}x(\mu)\\ &= \frac{e^Tx(\mu)}{\mu^2}\left[\Phi(w(\mu)) - e^T\Phi(w(\mu))w(\mu)\right] = \frac{e^Tx(\mu)}{\mu^2}\xi(w(\mu)),\end{aligned}$$

where the fourth equality follows from (2.5) and the last equality follows from (2.1) and (2.2). □

The paths w: $I \to W_I$ obtained according to Proposition 2.3 will be called *projective scaling paths*. Obviously, given a point $w^0 \in W_I$, a projective scaling path w: $I \to W_I$ passing through w^0 can be obtained according to Proposition 2.3 from a solution curve x: $I \to S_I$ of (2.4) passing through w^0 (or any point λw^0 with $\lambda > 0$).

The projective scaling trajectories will be analyzed in the next sections by considering these parametrizations, namely, the projective scaling paths.

3. Affine scaling trajectories. In this section, we will be concerned with analyzing the convergence and boundary behavior of the affine scaling trajectories of a linear programming problem in standard form under weaker assumptions than the ones imposed in [1]. Specifically, [1] assumes that the set of optimal solutions of the linear program under consideration is nonempty and bounded. However, from the discussion in the previous section, we saw that the homogeneous standard form problem (HP) underlying the Karmarkar standard form problem (P) has unbounded optimal face when the optimal value of (P) is 0. In order to use the result of Proposition 2.3 to analyze the projective scaling trajectories, it is necessary to extend the results of the affine scaling trajectories presented in [1] to ones that do not assume that the set of optimal solutions is bounded.

The discussion in this section is intended to be brief in order to avoid repetition of arguments already presented in [1]. Only the proofs of those results that require different arguments are presented. Specifically, these results are stated in Proposition 3.2 and Theorem 3.2. The proof of Theorem 3.2 presented here, under our more general assumptions that allow the optimal face of the linear program to be unbounded, is essentially the major change in the analysis presented in [1]. However, we strongly advise the reader to skip the proof of Theorem 3.2 on a first reading since its content is only used in the proof of Theorem 3.3 which is not given here. The main results of this section are stated in Theorems 3.3 and 3.4. Theorem 3.3 completely describes the convergence and boundary behavior of the affine scaling trajectories and their corresponding asymptotic directions when they approach the optimal face. The conclusions of this theorem are exactly the same as its corresponding counterpart presented in [1]. Theorem 3.4 is new and shows that every optimal solution which lies in the relative interior of the optimal face of a linear program is a limit point of an affine scaling trajectory. Theorem 3.4 is essentially an important corollary of Theorem 3.3. Further remarks regarding the differences between the exposition in [1] and the one presented in this section will be given in a more appropriate place later on.

We start by introducing our terminology. Consider the linear programming problem in standard form

$$(\tilde{\mathrm{P}}) \qquad \min\{c^T x \mid Ax = b,\ x \geqslant 0\}$$

and its dual problem

$$(\tilde{\mathrm{D}}) \qquad \max\{b^T y \mid A^T y + z = c,\ z \geqslant 0\},$$

where A is an $(m \times n)$-matrix and b, c are vectors of length m and n respectively. The following notation will be used throughout this section:

$$S_A = \{x \in \mathbf{R}^n;\ Ax = b\},$$

$$S_F = \{x;\ x \in S_A,\ x \geqslant 0\},$$

$$S_I = \{x;\ x \in S_A,\ x > 0\},$$

$$T_A = \{(y, z) \in \mathbf{R}^m \times \mathbf{R}^n;\ A^T y + z = c\},$$

$$T_F = \{(y, z);\ (y, z) \in T_A,\ z \geqslant 0\},$$

$$T_I = \{(y, z);\ (y, z) \in T_A,\ z > 0\}.$$

The sets S_F and T_F are the feasible sets of problems ($\tilde{\mathrm{P}}$) and ($\tilde{\mathrm{D}}$) respectively, the sets S_A and T_A are the affine hulls of S_F and T_F respectively, and the sets S_I and T_I, if they are nonempty sets, are the relative interior of S_F and T_F respectively.

Regarding problems ($\tilde{\mathrm{P}}$) and ($\tilde{\mathrm{D}}$), we have the following useful result.

PROPOSITION 3.1. *Assume that the set S_I is nonempty. Then the following conditions are equivalent:*

(a) *For all (or some) $\mu > 0$, the problem* $\min\{c^T x - \mu \sum_{j=1}^n \ln x_j;\ x \in S_I\}$ *has a (unique) global solution.*

(b) *The set T_I is nonempty.*

(c) *The set of optimal solutions of problem* (P) *is nonempty and bounded.*

(d) *For all (or some) $\bar{x} \in S_F$, the set $\{x \in S_F;\ c^T x \leqslant c^T \bar{x}\}$ is bounded.*

In fact, (b), (c), and (d) are also equivalent under the weaker assumption that $S_F \neq \varnothing$. We refer the reader to [7] and [13] for arguments that lead to the proof of Proposition 3.1.

We impose the following assumptions on problem ($\tilde{\mathrm{P}}$).

ASSUMPTION 3.1. (a) *The set S_I is nonempty.*

(b) *Problem* ($\tilde{\mathrm{P}}$) *has an optimal solution.*

(c) $\operatorname{rank}(A) = m$.

The affine scaling vector field is given by

$$\Phi(x) = X\left[I - XA^T(AX^2A^T)^{-1}AX\right]Xc$$

for $x \in S_I$. In the following, the trajectories of this vector field will be analyzed by studying the solution curves of the differential equation

$$\dot{x}(\mu) = \frac{1}{\mu^2}\Phi(x(\mu)), \tag{3.1}$$

where $\mu > 0$. In particular, we are interested in the behavior of a solution curve of (3.1) as μ decreases, that is, as $c^T x(\mu)$ monotonically decreases.

We now turn our efforts towards characterizing the solution curves of (3.1) as a path of solutions of a logarithmic barrier family of problems. Consider the following family of problems parametrized by the penalty parameter $\mu > 0$:

$$(\mathrm{P}_\mu) \qquad \begin{aligned} \min \quad & c^T x - \mu\left[p^T x + \sum_{j=1}^n \ln x_j\right] \\ \text{s.t.} \quad & Ax = b \\ & x > 0, \end{aligned}$$

where $p \in \mathbf{R}^n$. Since the objective function of problem (P_μ) is a strictly convex function, it follows that (P_μ) has at most one global solution. The global solution $x(\mu)$ of problem (P_μ), if it exists, satisfies the following Karush-Kuhn-Tucker optimality condition:

$$z(\mu) - \mu x(\mu)^{-1} = \mu p, \tag{3.2.a}$$

$$Ax(\mu) = b, \qquad x(\mu) > 0, \tag{3.2.b}$$

$$A^T y(\mu) + z(\mu) = c \tag{3.2.c}$$

for some $y(\mu) \in \mathbf{R}^m$ and $z(\mu) \in \mathbf{R}^n$ which are uniquely determined. Let $I(p)$ denote the set of parameters $\mu > 0$ such that problem (P_μ) (and hence system (3.2)) has a solution. In view of the equivalence of (a) and (b) of Proposition 3.1, it follows that problem (P_μ) has a global solution if and only if the set $Y(p, \mu) \equiv \{y;\ A^T y < c - \mu p\} \neq \varnothing$, and, as a consequence, we have $I(p) = \{\mu > 0;\ Y(p, \mu) \neq \varnothing\}$. If the set $I(p)$ is nonempty, it has to be an open interval having 0 as one of its extreme points. This fact is stated in the following result.

PROPOSITION 3.2. *If the set $I(p)$ is nonempty, then $I(p) = (0, d_p)$ for some $d_p > 0$.*

PROOF. Assume that $I(p) \neq \varnothing$ and let $\mu \in I(p)$ be given. By the definition of $I(p)$, it follows that there exists $y \in \mathbf{R}^m$ such that $A^T y < c - \mu p$. On the other hand, (b) of Assumption 3.1 implies that the dual problem $(\tilde{\mathrm{D}})$ is feasible, that is, there exists $y^* \in \mathbf{R}^m$ such that $A^T y^* \leqslant c$. Let $\lambda \in (0,1)$ be given and consider the vector $y_\lambda \equiv (1 - \lambda) y^* + \lambda y$. Since $A^T y_\lambda < c - \lambda \mu p$, it follows that $\lambda \mu \in I(p)$. Since this holds for any $\lambda \in (0,1)$ and $\mu \in I(p)$, the result follows. □

It has been shown in [1] that whenever the set of optimal solutions of problem $(\tilde{\mathrm{P}})$ is bounded, the set $I(p)$ is nonempty for all $p \in \mathbf{R}^n$. Moreover, if the set S_F is bounded, then $I(p) = (0, \infty)$ for all $p \in \mathbf{R}^n$. However, when the set of optimal solutions of problem $(\tilde{\mathrm{P}})$ is unbounded, it may happen that $I(p) = \varnothing$. Indeed, when the set of optimal solutions of $(\tilde{\mathrm{P}})$ is unbounded, $I(p) = \varnothing$ for any $p \geqslant 0$. This last observation is a consequence of the equivalence of (b) and (c) of Proposition 3.1.

For a point $x \in S_I$, we define the affine dual estimates $y^E(x)$ and $z^E(x)$ at x by

$$y^E(x) = (AX^2A^T)^{-1} AX^2 c, \tag{3.3}$$

$$z^E(x) = c - A^T (AX^2A^T)^{-1} AX^2 c. \tag{3.4}$$

In the discrete affine scaling algorithm (see, for example, [19]), if $x \in S_I$ is a near optimal point, then $(y^E(x), z^E(x))$ can be used in an attempt to obtain a near optimal (possibly infeasible) solution for the dual problem $(\tilde{\mathrm{D}})$ (see statement (c) of Theorem 3.3 below).

The following result establishes the important relationship between the path of solutions of system (3.2) with solutions of the differential equation (3.1) and the dual estimates (3.3) and (3.4).

THEOREM 3.1. *Assume that $p \in \mathbf{R}^n$ is given such that $I(p) \neq \varnothing$. Let $(x(\mu), y(\mu), z(\mu))$, $\mu \in I(p)$, denote the path of solutions of the parametrized system of equations* (3.2) *corresponding to the given p. Then $x(\mu)$, $\mu \in I(p)$, is a solution of the differential equation* (3.1) *and $y(\mu)$ and $z(\mu)$ satisfy the relations*

$$y(\mu) - \mu \dot{y}(\mu) = y^E(x(\mu)), \qquad z(\mu) - \mu \dot{z}(\mu) = z^E(x(\mu))$$

for all $\mu \in I(p)$.

The proof of Theorem 3.1 is analogous to the proof of Theorem 2.1 in [1]. We have already observed that, when the optimal face of $(\tilde{\mathrm{P}})$ is unbounded, it may happen that $I(p) = \varnothing$ for a given $p \in \mathbf{R}^n$. The next result essentially characterizes those vectors $p \in \mathbf{R}^n$ for which $I(p) \neq \varnothing$.

COROLLARY 3.1. *Let $x^0 \in S_I$, $(y^0, z^0) \in T_A$, and $\mu^0 > 0$ be given. Let $p = (z^0/\mu^0 - (x^0)^{-1})$. Then $(0, \mu^0] \subset I(p)$ and the path of solutions $x(\mu)$ of problem (P_μ) is a solution curve of* (3.1) *satisfying $x(\mu^0) = x^0$.*

PROOF. By the choice of p, it follows that (x^0, y^0, z^0) is a solution of system (3.2) with $\mu = \mu^0$. Hence, $\mu^0 \in I(p)$ and by Proposition 3.2 it follows that $(0, \mu^0] \subset I(p)$. Also, since the solution of system (3.2) for a given $\mu > 0$ is unique, it follows that $(x(\mu^0), y(\mu^0), z(\mu^0)) = (x^0, y^0, z^0)$. □

Note that, in Corollary 3.1, any choice of a point $(y^0, z^0) \in T_A$ can be used to determine a solution curve $x(\mu)$ of (3.1) such that $x(\mu^0) = x^0$. In the following, let θ^* denote the common optimal value of problems $(\tilde{\text{P}})$ and $(\tilde{\text{D}})$. The next result shows that the trajectory $x(\mu)$ and its associated "dual trajectory" $(y(\mu), z(\mu))$ converge in objective value to the optimal value θ^*.

THEOREM 3.2. *Let $p \in \mathbf{R}^n$ be given such that $I(p) \neq \varnothing$ and consider the solution $(x(\mu), y(\mu), z(\mu))$, $\mu \in I(p)$, of system* (3.2). *Then $\lim_{\mu \to 0} c^T x(\mu) = \lim_{\mu \to 0} b^T y(\mu) = \theta^*$. Moreover, both $x(\mu)$ and $(y(\mu), z(\mu))$ lie in a bounded set for μ sufficiently small.*

PROOF. Since $I(p) \neq \varnothing$, select an arbitrary $\mu^0 \in I(p)$ and let $(x^0, y^0, z^0) \equiv (x(\mu^0), y(\mu^0), z(\mu^0))$. Multiplying (3.2.a) by $x(\mu)^T$, we obtain

$$x(\mu)^T z(\mu) = n\mu + \mu p^T x(\mu).$$

Using the fact that $x^T z = c^T x - b^T y$ for any $x \in S_A$ and $(y, z) \in T_A$, one can easily verify that the last expression is equivalent to the following relation:

$$\text{(3.5)} \quad \left(1 - \frac{\mu}{\mu^0}\right)\left[c^T x(\mu) - b^T y(\mu)\right] = \frac{\mu}{\mu^0}\left[\left(c^T x^0 - b^T y^0\right) - \left(z^0 - \mu^0 p\right)^T x(\mu) - \left(x^0\right)^T z(\mu)\right] + n\mu.$$

Note that $x(\mu) > 0$ and $z(\mu) - \mu p > 0$ for all $\mu \in I(p)$ and, in particular, we have that $x^0 > 0$ and $z^0 - \mu^0 p > 0$. This observation and relation (3.5) imply

$$\left(1 - \frac{\mu}{\mu^0}\right)\left[c^T x(\mu) - b^T y(\mu)\right] \leqslant \frac{\mu}{\mu^0}\left[\left(c^T x^0 - b^T y^0\right) - \mu\left(x^0\right)^T p\right] + n\mu.$$

Hence, as μ tends to 0, we obtain

$$\text{(3.6)} \qquad \limsup_{\mu \to 0} c^T x(\mu) - b^T y(\mu) \leqslant 0.$$

Obviously, since $x(\mu) \in S_I$, we have

$$\text{(3.7)} \qquad c^T x(\mu) \geqslant \theta^*$$

for all $\mu \in I(p)$, and therefore

$$\text{(3.8)} \qquad \liminf_{\mu \to 0} c^T x(\mu) \geqslant \theta^*.$$

Let x^* be an optimal solution of $(\tilde{\text{P}})$ (cf. (b) of Assumption 3.1). Since $z(\mu) - \mu p > 0$, we have $(z(\mu) - \mu p)^T x^* \geqslant 0$, which implies

$$c^T x^* - b^T y(\mu) - \mu p^T x^* \geqslant 0$$

or equivalently,

$$\theta^* - \mu p^T x^* \geqslant b^T y(\mu). \tag{3.9}$$

Hence, we have

$$\limsup_{\mu \to 0} b^T y(\mu) \leqslant \theta^*. \tag{3.10}$$

Now from relations (3.6), (3.8), and (3.10), it follows that $\lim_{\mu \to 0} c^T x(\mu) = \theta^*$ and $\lim_{\mu \to 0} b^T y(\mu) = \theta^*$. To show that $x(\mu)$ and $(y(\mu), z(\mu))$ lie in a bounded set for all μ sufficiently small, first note that the components of $x(\mu)$ and $z(\mu)$ are bounded from below since $x(\mu) > 0$ and $z(\mu) > \mu p$. On the other hand, from relations (3.5), (3.7) and (3.9), it follows that

$$\mu\left(1 - \frac{\mu}{\mu^0}\right)p^T x^* \leqslant \frac{\mu}{\mu^0}\left[\left(c^T x^0 - b^T y^0\right) - \left(z^0 - \mu^0 p\right)^T x(\mu) - \left(x^0\right)^T z(\mu)\right] + n\mu.$$

Dividing the last expression by μ/μ^0 and rearranging, yields

$$\left(z^0 - \mu^0 p\right)^T x(\mu) + \left(x^0\right)^T z(\mu) \leqslant n\mu^0 + \left(c^T x^0 - b^T y^0\right) - \mu^0\left(1 - \frac{\mu}{\mu^0}\right)p^T x^*,$$

which shows that the components of $x(\mu)$ and $z(\mu)$ are bounded from above and therefore bounded for all μ sufficiently small. Since A has full rank, it also follows that $y(\mu)$ is bounded for all μ sufficiently small. □

Theorem 3.2 has a much simpler proof (see Theorem 3.1 in [1]) when it is assumed that the optimal face of $(\tilde{\mathrm{P}})$ is bounded. For example, the fact that $x(\mu)$ is bounded for all $\mu \in I(p)$ sufficiently small follows by using the equivalence of (c) and (d) of Proposition 3.1 and the fact that $c^T x(\mu)$ decreases as μ decreases. At this point, we should point out that the necessary adaptations for the development in [1] to hold under our present assumptions (cf. Assumption 3.1) have already been presented in Proposition 3.2, Corollary 3.1, and Theorem 3.2.

Before stating the next result, we need to introduce some terminology. The optimal face of $(\tilde{\mathrm{P}})$ (resp. $(\tilde{\mathrm{D}})$) is the set of points $S_O \equiv \{x;\ x \in S_F,\ c^T x = \theta^*\}$ (resp. $T_O \equiv \{y, z);\ (y, z) \in T_F,\ b^T y = \theta^*\}$). The set S_O is a face of the polyhedron S_F and therefore can be expressed as the set of points $\{x \in S_F;\ x_j = 0,\ j \in N\}$ for some index set $N \subseteq \{1, \dots, n\}$. We may assume that N is the maximal set (with respect to inclusion) satisfying this property, that is, $j \in N$ if and only if $x_j = 0$ for every $x \in S_O$. Let B denote the set of indices $j \in \{1, \dots, n\}$ such that $j \notin N$. It is well known that T_O is the face of the polyhedron T_F given by $\{(y, z) \in T_F;\ z_j = 0,\ j \in B\}$ and that B is the maximal set with this property (see, for example, Schrijver [16]). We are now ready to state the main result of this section.

THEOREM 3.3. *Let $p \in \mathbf{R}^n$ be given such $I(p) \neq \varnothing$ and consider the path of solutions $(x(\mu), y(\mu), z(\mu))$, $\mu \in I(p)$, of system (3.2) with respect to the given p. Then, we have:*

(a) *The limit of $x(\mu)$ as μ tends to 0 exists and is equal to the optimal solution x^* of problem $(\tilde{\mathrm{P}})$ (and hence $x_N^* = 0$) such that x_B^* is the (unique) optimal solution of the problem*

$$\begin{aligned} \max \quad & -\left[\left(x_B^0\right)^{-1}\right]^T x_B + \sum_{j \in B} \ln x_j \\ s.t. \quad & A_B x_B = b, \quad x_B > 0, \end{aligned}$$

where x^0 is any point in the path $x(\mu)$, that is, $x^0 = x(\mu^0)$ for some $\mu^0 \in I(p)$.

(b) *The limit of* $(y(\mu), z(\mu))$, *as* μ *tends to* 0, *exists and is equal to the optimal solution* (y^*, z^*) *of problem* $(\tilde{D})$ *(and hence* $z_B^* = 0$*) such that* (y^*, z_N^*) *is the (unique) optimal solution of the problem*

$$\max\left\{ \sum_{j \in N} \ln z_j \middle| A_N^T y + z_N = c_N,\ A_B^T y = c_B,\ z_N > 0 \right\}.$$

(c) *The limit of the affine dual estimates along the trajectory* $x(\mu)$, $\mu \in I(p)$, *as* μ *tends to* 0, *exists and is given by*

$$\lim_{\mu \to 0} \left(y^E(x(\mu)), z^E(x(\mu)) \right) = (y^*, z^*),$$

where (y^*, z^*) *is the optimal solution of problem* $(\tilde{P})$ *as in statement* (b) *above.*

(d) *The limit of* $\dot{x}(\mu)$, *as* μ *tends to* 0, *exists and its value is given by*

$$\lim_{\mu \to 0} \dot{x}_N(\mu) = (z_N^*)^{-1},$$

and $\lim_{\mu \to 0} \dot{x}_B(\mu)$ *is equal to the (unique) optimal solution of the following problem where the minimization is with respect to* v_B:

$$\begin{aligned} &\min \tfrac{1}{2} \| (x_B^*)^{-1} v_B \|^2 \\ &s.t.\ A_B v_B = -A_N (z_N^*)^{-1}. \end{aligned}$$

(e) *The limit of* $(\dot{y}(\mu), \dot{z}(\mu))$, *as* μ *tends to* 0, *exists and its value is given by*

$$\lim_{\mu \to 0} \dot{z}_B(\mu) = p_B + (x_B^*)^{-1},$$

and $\lim_{\mu \to 0}(\dot{y}(\mu), \dot{z}_N(\mu))$ *is equal to the (unique) optimal solution of the following problem where the minimization is with respect to* (r, s_N):

$$\begin{aligned} &\min \tfrac{1}{2} \| (z_N^*)^{-1} (s_N - p_N) \|^2 \\ &s.t.\ A_B^T r = -p_B - (x_B^*)^{-1}, \\ &\qquad A_N^T r + s_N = 0. \end{aligned}$$

The proof of Theorem 3.3 follows along the same line as the results proved in [1] and therefore we do not provide the details here.

Statement (a) of Theorem 3.3 shows that every affine scaling trajectory converges to an optimal solution lying in the relative interior of the optimal face and which depends on the trajectory itself. Note that statement (d) guarantees that all affine scaling trajectories converging to the same optimal point do so along the same asymptotic direction of approach. However, the asymptotic directions of approach for trajectories converging to distinct optimal points are in general different. The situation is quite different for the dual trajectories $(y(\mu), z(\mu))$. They always converge to the same optimal point (usually called the *center* of the dual optimal face), but along different asymptotic directions of approach.

A question that naturally arises from the discussion above is whether any point in the relative interior of the optimal face is a limit of an affine scaling trajectory. Toward answering this question, we introduce some more notation. Let S_O^+ denote the relative interior of S_O, that is, $S_O^+ = \{(x_B, 0) \in S_O | x_B > 0\}$. Given $x_N^0 \in \mathbf{R}_+^N$, let $S_I(x_N^0) = \{x \in S_I | x_N = x_N^0\}$. We observe that any two points in $S_I(x_N^0)$ have the same objective function value, that is, if $x, x' \in S_I(x_N^0)$ then $c^T x = c^T x'$. Also, let Φ: $S_I \to S_O^+$ denote the function defined as follows: given $x^0 \in S_I$, let $\Phi(x^0)$ denote the point in S_O^+ to which the affine scaling trajectory through x^0 converges, or equivalently, by (a) of Theorem 3.3, $\Phi(x^0) = (x_B^*, 0)$ where $x_B^* \in \mathbf{R}_+^B$ is the (unique) point which maximizes $-[(x_B^0)^{-1}]^T x_B + \sum_{j \in B} \ln x_j$ subject to $A_B x_B = b$. It is a consequence of the next result that the function Φ maps S_I onto S_O^+.

THEOREM 3.4. *Let $x_N^0 \in \mathbf{R}_+^N$ satisfy $S_I(x_N^0) \neq \emptyset$. Then, the function Φ maps the set $S_I(x_N^0)$ one-to-one and onto the set S_O^+.*

PROOF. Let $(x_B^*, 0) \in S_O^+$ be given. By (a) of Theorem 3.3, it is enough to show that there exists a unique point $x_B^0 \in \mathbf{R}_+^B$ satisfying the following properties:
(a) $x^0 \equiv (x_B^0, x_N^0) \in S_I$.
(b) x_B^* is the (unique) optimal solution of the problem

$$(3.11) \qquad \max_{x_B} \left\{ -\left[\left(x_B^0\right)^{-1} \right]^T x_B + \sum_{j \in B} \ln x_j | A_B x_B = b,\ x_B > 0 \right\}.$$

Indeed, let x_B^0 be the (unique) optimal solution of the following problem:

$$(3.12) \qquad \max_{x_B} \left\{ -\left[(x_B^*)^{-1} \right]^T x_B + \sum_{j \in B} \ln x_j | A_B x_B = b - A_N x_N^0,\ x_B > 0 \right\}.$$

First note that problem (3.12) is feasible since, by assumption, $S_I(x_N^0) \neq \emptyset$. That problem (3.12) has an optimal solution follows from the fact that the set $\{y | A_B^T y < (x_B^*)^{-1}\}$ is nonempty (since $y = 0$ clearly belongs to it) and the equivalence of (a) and (c) of Proposition 3.1. The point x_B^0 then satisfies the optimality conditions corresponding to problem (3.12) and, in particular, we have $(x_B^*)^{-1} - (x_B^0)^{-1} \in \text{range}(A_B^T)$.
This last relation together with the relation $A_B x_B^* = b$ shows that x_B^* satisfies the optimality conditions for (3.11) and hence that x_B^* is the optimal solution of (3.11). This shows the existence statement. If x_B^1 is any point satisfying properties (a) and (b) above with x_B^0 replaced by x_B^1, then one can easily see, by using arguments similar to the ones above, that x_B^1 is optimal for (3.12) and hence $x_B^1 = x_B^0$. This shows the uniqueness statement. □

4. The behavior of the projective scaling trajectories ($r^* = 0$). In this section, we study the convergence and boundary behavior of the projective scaling trajectories for the linear programming problem in Karmarkar's standard form (P) stated in §2 under the assumption that the optimal value r^* of problem (P) is 0. The case $r^* > 0$ will be briefly analyzed in the next section. Since by Proposition 2.3 every projective scaling trajectory $w(\mu)$ for problem (P) can be obtained from an affine scaling trajectory by radial projection onto $\{x | e^T x = 1\}$, it is enough to use the results of the previous section in order to analyze the behavior of the projective scaling trajectories.

Before going through the analysis of the behavior of the projective scaling trajectories, we point out the relationship between the optimal faces of problems (P) and (HP). Let N denote the set of indices j such that $x_j = 0$ whenever x is an optimal solution of problem (HP). It follows from Proposition 2.2 that N is also the set of indices j for which $x_j = 0$ in every optimal solution of problem (P). Moreover, the set $B \equiv \{1, \dots, n\} - N$ is nonempty.

The dual of problem (P) is the problem stated as follows:

$$\text{(D)}\qquad \begin{array}{ll} \max & \rho \\ \text{s.t.} & A^T y + e\rho + z = 0, \\ & z \geqslant 0. \end{array}$$

With respect to the dual problem (D), we have: (y, ρ, z) is an optimal solution of problem (D) if and only if $A_B^T y = c_B$, $A_N^T y + z_N = c_N$, $z_N \geqslant 0$, $z_B = 0$, and $\rho = 0$.

We start by showing that a projective scaling trajectory always converges to an optimal solution of problem (P). In general, when the dimension of the optimal face of problem (P) is greater than 0, two different projective scaling trajectories may converge to two distinct points on the optimal face of (P).

THEOREM 4.1. *Let $w(\mu)$ be a projective scaling path passing through a point $w^0 \in W_I$. Then, the limit of $w(\mu)$, as μ tends to 0, exists and has the value*

$$\lim_{\mu \to 0} w(\mu) = \frac{x^*}{e^T x^*},$$

where x^* is the optimal solution of problem (HP) (and hence $x_N^* = 0$) such that x_B^* is the (unique) optimal solution of the problem

$$\begin{array}{ll} \max & -\left[\left(w_B^0\right)^{-1}\right]^T x_B + \sum_{j \in B} \ln x_j \\ s.t. & A_B x_B = 0, \quad x_B > 0. \end{array}$$

As a consequence, the limit of $w(\mu)$, as μ tends to 0, is an optimal solution of (P). *Moreover, every optimal solution in the relative interior of the optimal face of* (P) *is the limit of a projective scaling path.*

The proof of Theorem 4.1 follows immediately from statement (a) of Theorem 3.3, Theorem 3.4, and the definition of a projective scaling path.

The next result translates statement (c) of Theorem 3.3, namely, the convergence of the affine dual estimates along an affine scaling trajectory, to the context of the projective scaling trajectories. First, we need to define dual estimates, at a point $x \in W_I$, which naturally arise from applying the projective scaling algorithm to problem (P). The dual estimates for the projective algorithm were first introduced in [18]. They are defined as follows. Given a point $x \in W_I$, define the projective dual estimates $\hat{y}^E(x)$, $\hat{\rho}^E(x)$, and $\hat{z}^E(x)$ at x as

$$\hat{y}^E(x) \equiv \left(AX^2A^T\right)^{-1} X^2 c, \tag{4.1}$$

$$\hat{\rho}^E(x) \equiv \min_j \left\{c_j - A_j^T \hat{y}^E(x)\right\}, \tag{4.2}$$

$$\hat{z}^E(x) \equiv c - A^T \hat{y}^E(x) - e\hat{\rho}^E(x), \tag{4.3}$$

where A_j denotes the jth column of the matrix A. Observe that the projective dual estimate $(\hat{y}^E(x), \hat{\rho}^E(x), \hat{z}^E(x))$, at a point $x \in W_I$, is a feasible solution to problem (D). It has been shown in [18] that when x^k, $k = 1, 2, \ldots,$ is a sequence converging to a nondegenerate optimal solution of the linear program (P), the projective dual estimate $(\hat{y}^E(x^k), \hat{\rho}^E(x^k), \hat{z}^E(x^k))$ converges to the unique optimal solution $(y^*, 0, z^*)$ of the dual problem (D). The analog of this fact with respect to the projective scaling trajectories without nondegeneracy assumptions is as follows.

THEOREM 4.2. *Let $w: I \to W_I$, where $I = (0, a)$, be a projective scaling path. Then the limit, as μ tends to 0, of the projective dual estimate along the projective scaling path $w(\mu)$ exists and is given as follows*:

$$\lim_{\mu \to 0} \hat{y}^E(x(\mu)) = y^*, \tag{4.4}$$

$$\lim_{\mu \to 0} \hat{\rho}^E(x(\mu)) = 0, \tag{4.5}$$

$$\lim_{\mu \to 0} \hat{z}^E(x(\mu)) = z^*, \tag{4.6}$$

where $(y^*, 0, z^*)$ is the optimal solution of problem (D) (*hence $z_B^* = 0$*) *such that (y^*, z_N^*) is the* (*unique*) *optimal solution of the problem*

$$\max\left\{ \sum_{j \in N} \ln z_j | A_N^T y + z_N = c_N,\ A_B^T y = c_B,\ z_N > 0 \right\}.$$

PROOF. Let $x(\mu)$ be the solution curve of (3.1) such that $w(\mu) = [1/e^T x(\mu)] x(\mu)$ for all $\mu \in I$. From (3.3) and (4.1), it follows that $y^E(x(\mu)) = \hat{y}^E(w(\mu))$ for all $\mu \in I$. By statement (c) of Theorem 3.3, we have that $\lim_{\mu \to 0} y^E(x(\mu)) = y^*$, which implies (4.4). It then follows from (4.2) that

$$\lim_{\mu \to 0} \hat{\rho}(w(\mu)) = \min_j \left\{ c_j - A_j^T y^* \right\} = \min_j z_j^* = 0$$

since $z_N^* = 0$ and $N \neq \varnothing$. This shows (4.5). Relation (4.6) now follows from (4.3), (4.4), and (4.5). □

The next result shows the behavior of the derivatives of the projective scaling paths.

THEOREM 4.3. *Let $w: I \to W_I$, where $I = (0, a)$, be a projective scaling path. Let (y^*, z^*) be as in the statement of Theorem* 4.1. *Then the limit of $\dot{w}(\mu)$, as μ tends to* 0, *exists and the vector $\lim_{\mu \to 0} \dot{w}_N(\mu)$ is a positive multiple of the vector $(z_N^*)^{-1}$.*

PROOF. Let $x(\mu)$ be the solution curve (3.1) such that $w(\mu) = [1/e^T x(\mu)] x(\mu)$ for all $\mu \in I$. Then

$$\dot{w}(\mu) = \frac{1}{e^T x(\mu)} \dot{x}(\mu) - \frac{e^T \dot{x}(\mu)}{\left[e^T x(\mu)\right]^2} x(\mu).$$

The theorem now follows from statements (a) and (d) of Theorem 3.3. □

It is also a consequence of statement (d) of Theorem 3.3 that all projective scaling trajectories converging to the same optimal solution do so along the same asymptotic direction of approach. The next result is an immediate application of Theorem 4.3.

COROLLARY 4.1. *If the optimal face of problem* (P) *consists of a unique vertex of W, then all projective scaling trajectories approach this unique vertex with a common asymptotic direction $d \in \mathbf{R}^n$ given by*

$$d_N = (z_N^*)^{-1}, \qquad d_B = -\left(\tilde{A}_B^T \tilde{A}_B\right)^{-1} \tilde{A}_B^T \tilde{A}_N d_N, \tag{4.7}$$

where

$$\tilde{A}_B = \begin{pmatrix} A_B \\ e^T \end{pmatrix} \quad \text{and} \quad \tilde{A}_N = \begin{pmatrix} A_N \\ e^T \end{pmatrix},$$

and z_N^* *is as in the statement of Theorem* 4.1.

PROOF. Assume that a projective scaling trajectory approaches the unique optimal solution of problem (P) according to the asymptotic direction d. By Theorem 4.3, we may assume that $d_N = (z_N^*)^{-1}$. Obviously d satisfies

$$\tilde{A}_B d_B + \tilde{A}_N d_N = 0. \tag{4.8}$$

Since problem (P) has a unique optimal solution, it follows that the matrix $\tilde{A}_B$ has linearly independent columns. Hence, (4.8) implies (4.7) and the theorem follows. □

A proof of Corollary 4.1 was given in [17] for the case when problem (P) is primal nondegenerate.

5. The behavior of the projective scaling trajectories ($r^* > 0$). In this section, we consider the convergence behavior of the projective scaling trajectories for the case when the optimal value r^* of problem (P) is positive. In this case, the projective scaling trajectories do not converge to an optimal solution of problem (P). Instead, we show that they converge to a unique point $w^* \in W_I$ which minimizes the so-called Karmarkar potential function over W_I as follows:

$$P(x) = n \ln c^T x - \sum_{j=1}^{n} \ln x_j.$$

This fact was implicitly proved in §9 of [14] under the assumption that problem (P) is primal nondegenerate.

OBSERVATION 5.1. Even though the function $P(x)$ is not strictly convex over W_I, the function

$$Q(x) \equiv \exp P(x) = \frac{(c^T x)^n}{\prod_{j=1}^n x_j}$$

is strictly convex over W_I. For a proof of this fact, see Imai [10]. This implies that $Q(x)$, and hence $P(x)$, has a unique minimizer w^* over W_I. Also, the minimizer w^* is completely characterized by Karush-Kuhn-Tucker conditions for the problem $\min\{P(x) | x \in W_I\}$.

Before stating the main result of this section, we need to introduce a lemma which is an immediate consequence of the results of §2.

LEMMA 5.1. *Assume that the optimal value of problem* (P) *is positive and let* w: $I \to W_I$, *where* $I = (0, a)$, *be a projective scaling path. Then*

$$\lim_{\mu \to 0} x(\mu) = 0 \text{ and}$$

$$\lim_{\mu \to 0} \dot{x}(\mu) = (z^*)^{-1}, \tag{5.1}$$

where z^* is the z-component of the (unique) optimal solution of the problem

(5.2)
$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \ln z_j \\ \text{s.t.} \quad & A^T y + z = c, \quad z > 0. \end{aligned}$$

PROOF. Since $r^* > 0$, it follows from Proposition 2.1 that the optimal face of problem (HP) consists only of the vector $0 \in \mathbf{R}^n$. Hence, in this case, $N = \{1, \ldots, n\}$. The result now follows from statements (a) and (d) of Theorem 3.3. □

We are now ready to state the main result of this section.

THEOREM 5.1. *Assume that the optimal value of problem* (P) *is positive and let* $w: I \to W_I$, *where* $I = (0, a)$, *be a projective scaling path. Then the limit of* $w(\mu)$, *as* μ *tends to* 0, *exists and is equal to the (unique) optimal solution of the problem*

(5.3)
$$\begin{aligned} \min \quad & n \ln c^T x - \sum_{j=1}^{n} \ln x_j \\ \text{s.t.} \quad & Ax = 0, \quad e^T x = 1, \quad x > 0. \end{aligned}$$

PROOF. Let $x(\mu)$ be a solution curve of the differential equation (3.1) such that

$$w(\mu) = \frac{1}{e^T x(\mu)} x(\mu).$$

Then, by Lemma 5.1, we have that $\lim_{\mu \to 0} x(\mu) = 0$, which implies that $\lim_{\mu \to 0} e^T x(\mu) = 0$. Hence, applying L'Hopital's rule, we obtain

$$\lim_{\mu \to 0} w(\mu) = \lim_{\mu \to 0} \frac{\dot{x}(\mu)}{e^T \dot{x}(\mu)} = \frac{(z^*)^{-1}}{e^T (z^*)^{-1}},$$

where the last equality follows from (5.1). Using the fact that z^* satisfies the optimality conditions for problem (5.2), one can easily verify that $[1/e^T(z^*)^{-1}](z^*)^{-1}$ satisfies the optimality conditions for problem (5.3). The theorem now follows from Observation 5.1. □

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