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A general parametric analysis approach and its implication to sensitivity analysis in interior point methods [1]

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Abstract

Adler and Monteiro (1992) developed a parametric analysis approach that is naturally related to the geometry of the linear program. This approach is based on the availability of primal and dual optimal solutions satisfying strong complementarity. In this paper, we develop an alternative geometric approach for parametric analysis which does not require the strong complementarity condition. This parametric analysis approach is used to develop range and marginal analysis techniques which are suitable for interior point methods. Two approaches are developed, namely the LU factorization approach and the affine scaling approach.

Keywords: Linear programming; Parametric analysis; Range analysis; Marginal analysis; Post-optimality; Sensitivity analysis; Interior point method

1. Introduction

The analysis of how changes in the input data affect the optimal solution of the problem is often essential for the practical usefulness of optimization models. The ability

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to perform post-optimality analysis in a natural way in the context of the simplex method has contributed significantly to its success as a practical tool for management decision making. Our objective in this paper is to show that post-optimality analysis techniques can also be developed naturally in the context of interior point methods. First, a general framework for performing parametric analysis of linear programs is develop. Next, this framework is used to develop techniques for performing range and marginal analysis on optimal solutions obtained by interior point methods.

One possible approach to perform post-optimality analysis in connection with interior point methods is to first process the optimal solution available from these methods to generate an optimal basic solution, and then perform the analysis as in the context of the simplex method. A computational scheme for finding a pair of primal and dual optimal basic solutions from a pair of primal and dual optimal solutions is described by Megiddo [8]. One of the objectives of this paper is to explore alternative approaches for performing post-optimality analysis that do not require the availability of optimal basic solutions.

To simplify our presentation, we only provide a rigorous treatment for perturbations with respect to the right hand side (Sections 2 and 3). For the sake of completeness, the analogous results for perturbations with respect to the cost vector are given in Section 4 without any proofs. Consider the following right-hand side (RHS) parametric linear program RLP(t):

$$\begin{aligned} \phi(t) \equiv \min \quad & c^{\mathrm{T}} x \\ \text{s.t.} \quad & Ax = b + tw,\ x \geqslant 0, \end{aligned} \tag{1}$$

and its dual RLD(t):

$$\begin{aligned} \max \quad & (b + tw)^{\mathrm{T}} y \\ \text{s.t.} \quad & A^{\mathrm{T}} y + s = c,\ s \geqslant 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, c, x, $s \in \mathbb{R}^n$, b, w, $y \in \mathbb{R}^m$ and $t \in \mathbb{R}$. The purpose of the RHS *parametric analysis* is to find optimal solutions of RLP(t) for all values of t in a prespecified interval.

The following terminology is used in this paper. By RHS *range analysis* we mean the estimation of the region of parameters t for which an optimal solution of RLD(0) remains optimal for RLD(t) (see for example [6, pp. 91–92]). By RHS *marginal analysis* we mean the estimation of the left and right derivatives of the optimal value function $\phi(t)$ at $t = 0$. The term *sensitivity analysis* is used here to simultaneously refer to range analysis and marginal analysis. We use the term *post-optimality analysis* to refer to both parametric and sensitivity analysis. A similar terminology is used with respect to perturbations on the cost vector.

In [1], a parametric analysis approach is developed that is naturally related to the geometry of the linear program. In Section 2 we develop an alternative approach for parametric analysis which has several features of the approach in [1]. Similar to the approach developed in [1], our parametric analysis approach consists of solving a sequence of linear programs. The major difference of the two approaches is that we show that a sequence of linear programs can be constructed which does not depend on the concept of "optimal partition" of the variables. More specifically, it is shown that the next linear program of the sequence is determined by any optimal solution of the current linear program. At the time of revising this paper, we learned of a paper by Jansen et al. [7] that also considers a related approach for parametric and sensitivity analysis of linear programs.

The relaxation of the assumption of having points in the relative interior of the optimal face as assumed in [1] is important for both theoretical and practical reasons. On the theoretical side, it provides a natural way for unifying the optimal basis approach with the approach in [1]. From the practical side, it allows us to combine interior point methods with the simplex method while performing parametric analysis. In addition, it gives insights to develop approaches for performing sensitivity analysis on optimal solutions lying anywhere in the optimal face.

In Section 3, we use the parametric analysis framework of Section 2 to develop sensitivity analysis techniques which do not require the availability of optimal basic solutions. Two approaches are presented, namely (i) LU factorization approach and (ii) affine scaling approach. The first approach shares the same machinery as used in the approach that employs the optimal basis, and hence, it can be viewed as a natural extension of the optimal basis approach. The second approach uses machinery similar to the one employed by interior point methods.

In Section 4, we present equivalent results (without proofs) for the parametric and sensitivity analysis with respect to perturbations of the cost vector. In Section 5, we summarize our computational experience with the sensitivity analysis approach described in Section 3 and 4.

1.1. Notation and terminology

The following notation is used throughout this paper. The set $\{1, 2, \ldots, n\}$ is denoted by $\mathscr{N}$. For a given vector $x \in \mathbb{R}^n$, we define $\sigma(x) = \{i \in \mathscr{N} \mid x_i > 0\}$. For $\beta \subseteq \mathscr{N}$ we denote the complement of β with respect to $\mathscr{N}$ by $\bar{\beta}$. If $\beta \subseteq \mathscr{N}$, then A_β denotes the submatrix of the $m \times n$-matrix A whose columns correspond to indices in β. Similarly, x_β denotes the subvector of the vector $x \in \mathbb{R}^n$ whose components correspond to indices in β.

It is well known (e.g., see [4]) that any linear program has a pair of optimal primal and dual solutions that satisfies the strong complementarity property. Mathematically, this property ensures that if t is such that RLP(t) has an optimal solution, then there exists a pair of optimal primal and dual solutions $x^*(t)$ and $(y^*(t), s^*(t))$ for which $\sigma(x^*(t)) = \bar{\sigma}(s^*(t))$. The optimal solutions $x^*(t)$ and $(y^*(t), s^*(t))$ are called *strongly optimal solutions*. The partition $\mathscr{N} = \sigma(x^*(t)) \cup \bar{\sigma}(x^*(t))$ is called the *optimal partition* for problem RLP(t).

2. Right-hand side parametric analysis

Similar to the approach developed in [1], an algorithm for parametric analysis is developed that consists in the solution of a sequence of linear programs. A major difference between our approach and the one developed in [1] is that here a sequence of linear programs can be constructed without requiring the knowledge of optimal partitions. More specifically, it is shown that the next linear program of the sequence is determined by any optimal solution (not necessarily in the relative interior of the optimal face) of the current linear program.

The outline for this section is as follows. First, the notion of an optimality interval of an optimal dual solution is defined. Next, the equivalence of these optimality intervals with the intervals of linearity (to be defined below) of the function $\phi(t)$ is presented. Furthermore, it is shown that any optimality interval can be computed by solving a pair of linear programs. Using these results, an algorithm for parametric analysis is presented at the end of the section.

Let (y^0, s^0) be an optimal solution for RLD(t^0) for some $t^0 \in \mathbb{R}$. The *optimality interval* of (y^0, s^0) is defined as

$$\mathscr{O}(y^0, s^0) \equiv \{t \in \mathbb{R} \mid (y^0, s^0) \text{ is an optimal solution of RLD } (t)\}. \tag{2}$$

The following notation is introduced to give a computational characterization of the optimality interval (2). For $\beta \subseteq \mathscr{N}$, let

$$\Gamma(\beta) \equiv \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \mid Ax - tw = b,\ x \geqslant 0,\ x_{\bar{\beta}} = 0\},$$

$$\mathscr{T}(\beta) \equiv \{t \mid \exists x \in \mathbb{R}^n \text{ such that } (x, t) \in \Gamma(\beta)\}, \tag{3}$$

and, for any $t \in \mathscr{T}(\beta)$, define

$$\Gamma_t(\beta) \equiv \{x \in \mathbb{R}^n \mid (x, t) \in \Gamma(\beta)\}.$$

Observe that $\mathscr{T}(\beta)$ is an interval. Moreover, if $\mathscr{T}(\beta)$ is nonempty, then its left endpoint t_-^β and right endpoint t_+^β can be computed by solving the following two linear programs:

$$t_-^\beta \equiv \inf\{t \mid (x, t) \in \Gamma(\beta)\} \tag{4}$$

and

$$t_+^\beta \equiv \sup\{t \mid (x, t) \in \Gamma(\beta)\}. \tag{5}$$

Proposition 1. *For some $t^0 \in \mathbb{R}$, let (y^0, s^0) be an optimal solution of* RLD (t^0). *Let $\beta = \bar{\sigma}(s^0)$. Then,*
(i) $\mathscr{O}(y^0, s^0) = \mathscr{T}(\beta)$;
(ii) *For every $t \in \mathscr{T}(\beta)$, $\Gamma_t(\beta)$ is equal to the set of optimal solutions of* RLP (t).

Proof. The proof of (i) follows from the following equivalences: $t \in \mathscr{O}(y^0, s^0) \Leftrightarrow$ (y^0, s^0) is an optimal solution of RLD(t) $\Leftrightarrow$ there exists a feasible solution x for

RLP(t) such that x and (y^0, s^0) satisfy complementary slackness condition $\Leftrightarrow Ax = b + tw$, $x \geqslant 0$, $x_{\bar{\beta}} = 0 \Leftrightarrow t \in \mathscr{T}(\beta)$. The proof of (ii) follows from the following equivalences: $x \in \Gamma_t(\beta) \Leftrightarrow (x, t) \in \Gamma(\beta) \Leftrightarrow Ax - tw = b$, $x \geqslant 0$, $x_{\bar{\beta}} = 0 \Leftrightarrow x$ is feasible for RLP(t) and satisfies the complementary slackness condition together with $(y^0, s^0) \Leftrightarrow x$ is an optimal solution for RLP(t). □

It follows from Proposition 1(i) that the endpoints of the optimality interval $\mathscr{O}(y^0, s^0)$ can be computed by solving the two linear programs (4) and (5) in which $\beta = \bar{\sigma}(s^0)$.

It is well known that $\phi(t)$ defined in (1) is a convex piecewise linear function with a finite number of breakpoints (e.g., see [10]). We call the interval between two consecutive breakpoints an *interval of linearity* of $\phi(t)$. The following proposition expresses the left and right slopes of the function $\phi(t)$ as optimal values of certain linear programs.

Proposition 2 (Gauvin [3]). *Let $t^0 \in \mathbb{R}$ be such that* RLP (t^0) *has an optimal solution. Let $S(t^0)$ denote the set of all optimal solutions (y, s) of* RLD (t^0). *Then, the left and the right derivatives of $\phi(t)$ at $t = t^0$ are given by*

$$\phi'_-(t^0) = \inf\{w^{\mathrm{T}}y \mid (y, s) \in S(t^0)\} \tag{6}$$

and

$$\phi'_+(t^0) = \sup\{w^{\mathrm{T}}y \mid (y, s) \in S(t^0)\}, \tag{7}$$

respectively.

Note that in Eq. (6) and (7) the left and right derivatives may be equal to $-\infty$ and ∞, respectively. If the left derivative is $-\infty$ then RLP(t) is infeasible for every $t < t^0$. Similarly, if the right derivative is ∞ then RLP(t) is infeasible for every $t > t^0$.

The proof of Proposition 2 follows from well-known results from convex analysis (see for example [11]). Only an outline of the proof is provided here; for a complete proof the reader is referred to [12]. The function $F(b) \equiv \inf\{c^{\mathrm{T}}x \mid Ax = b, x \geqslant 0\}$ is a polyhedral convex function and its subgradient $\partial F(b)$ is equal to the set of optimal solutions to the dual problem $\sup\{b^{\mathrm{T}}y \mid A^{\mathrm{T}}y \leqslant c\}$. From convex analysis, we know that the directional derivative $F'(\bar{b}; w)$ of F at $\bar{b}$ along w is given by

$$F'(\bar{b}; w) = \sup\{w^{\mathrm{T}}y \mid y \in \partial F(\bar{b})\}. \tag{8}$$

Relations (6) and (7) follows by using relation (8) with the $\bar{b} = b + t^0 w$ and the formulas $\phi'_+(t^0) = F'(b + t^0 w; w)$ and $\phi'_-(t^0) = -F'(b + t^0 w; -w)$.

The relationship between the optimality intervals and the geometric shape of the function $\phi(t)$ is described next.

Theorem 1. *Let $t^0 \in \mathbb{R}$ be such that* RLP (t^0) *has an optimal solution. Let (y^0, s^0) be an optimal solution of* RLD (t^0). *Then,*

(i) *if t^0 is not a breakpoint, then $\mathscr{O}(y^0, s^0)$ is the interval of linearity of $\phi(t)$ containing t^0;*

(ii) *if t^0 is a breakpoint, then there are three possibilities*:

(a) *if $\phi'_-(t^0) < w^{\mathrm{T}} y^0 < \phi'_+(t^0)$, then $\mathscr{O}(y^0, s^0) = \{t^0\}$*;

(b) *if $\phi'_-(t^0) = w^{\mathrm{T}} y^0$, then $\mathscr{O}(y^0, s^0)$ is the interval of linearity lying to the left of t^0*;

(c) *if $\phi'_+(t^0) = w^{\mathrm{T}} y^0$, then $\mathscr{O}(y^0, s^0)$ is the interval of linearity lying to the right of t^0.*

Proof. We first give the proof of (i). Since t^0 is not a breakpoint, $\phi'_-(t^0) = \phi'_+(t^0) = \phi'(t^0)$. Clearly, the interval of linearity $\mathscr{I}(t^0)$ of $\phi(t)$ containing t^0 can be expressed as

$$\mathscr{I}(t^0) = \{t \mid \phi(t) = \phi(t^0) + \phi'(t^0)(t - t^0)\}. \tag{9}$$

On the other hand,

$$\mathscr{O}(y^0, s^0) = \{t \mid \phi(t) = (b + tw)^{\mathrm{T}} y^0\} = \{t \mid \phi(t) = \phi(t^0) + (t - t^0) w^{\mathrm{T}} y^0\}, \tag{10}$$

where the last equality follows from the fact that $\phi(t^0) = (b + t^0 w)^{\mathrm{T}} y^0$. By proposition 2, we know that $w^{\mathrm{T}} y^0 = \phi'(t^0)$. Hence, it follows from expressions (9) and (10) that $\mathscr{I}(t^0) = \mathscr{O}(y^0, s^0)$. We next show (ii). Assume that t^0 is a breakpoint, that is, $\phi'_-(t^0) < \phi'_+(t^0)$. The interval of linearity $\mathscr{I}_+(t^0)$ to the right of t^0 and the interval of linearity $\mathscr{I}_-(t^0)$ to the left of t^0 can be expressed as

$$\mathscr{I}_+(t^0) = \{t \mid \phi(t) = \phi(t^0) + \phi'_+(t^0)(t - t^0)\} \tag{11}$$

and

$$\mathscr{I}_-(t^0) = \{t \mid \phi(t) = \phi(t^0) + \phi'_-(t^0)(t - t^0)\}, \tag{12}$$

respectively. Clearly, expressions (10) and (11) imply that $\mathscr{O}(y^0, s^0) = \mathscr{I}_+(t^0)$ *if* $\phi'_+(t^0) = w^{\mathrm{T}} y^0$. Similarly, expressions (10) and (12) imply that $\mathscr{O}(y^0, s^0) = \mathscr{I}_-(t^0)$ if $\phi'_-(t^0) = w^{\mathrm{T}} y^0$. Lastly, if $\phi'_-(t^0) < w^{\mathrm{T}} y^0 < \phi'_+(t^0)$ then $\phi(t^0) + (t - t^0) w^{\mathrm{T}} y^0 < \phi(t)$ for every $t \neq t^0$. Hence, $\mathscr{O}(y^0, s^0) = \{t^0\}$. □

The previous results lead to the following method for parametric analysis. Assuming that an optimal solution is available for $t = 0$, the algorithm is described only for increasing values of t.

Algorithm 2.1 (*An algorithm for the RHS parametric analysis*).

Input. Solutions x^0 and (y^0, s^0) which are optimal for RLP(0) and RLD(0), respectively. Set $k = 0$, and $t^0 = 0$.

Step 1. Let $\beta = \bar{\sigma}(s^k)$. Solve the linear program

$$t^{k+1} = \sup\{t \mid Ax - tw = b,\ x \geqslant 0,\ x_{\bar{\beta}} = 0\}. \tag{13}$$

If t^{k+1} is ∞, **stop**. Otherwise, let (x^{k+1}, t^{k+1}) be an optimal solution of this problem. Then, x^{k+1} is an optimal solution of RLP(t^{k+1}) and $\lambda x^k + (1 - \lambda) x^{k+1}$ is an optimal solution of RLP($\lambda t^k + (1 - \lambda) t^{k+1}$) for every $\lambda \in [0, 1]$.

Step 2. Let $\alpha \equiv \sigma(x^{k+1})$. Solve the linear program

$$\sup\{w^T y \mid A^T y + s = c,\ s_\alpha = 0,\ s \geqslant 0\}. \tag{14}$$

If this problem is unbounded, RLP(t) is infeasible for all $t > t^{k+1}$; **stop**. Otherwise, let (y^{k+1}, s^{k+1}) be an optimal solution of (14).

Step 3. Set $k = k + 1$, return to Step 1.

The correctness of Algorithm 2.1 is now discussed. In order to simplify the arguments, assume that 0 is not a breakpoint. By Theorem 1(i), the optimal value t^1 (possibly $t^1 = \infty$) computed at Step 1 is equal to the first breakpoint larger than 0. Using Proposition 1(ii), it is easily seen that the point x^1 computed at Step 1 is an optimal solution of RLP(t^1) and that $\lambda x^0 + (1 - \lambda)x^1$ is an optimal solution of RLP($\lambda t^0 + (1 - \lambda)t^1$) for every $\lambda \in [0, 1]$. Since $\alpha = \sigma(x^1)$ and x^1 is an optimal solution for RLP(t^1), it follows by the complementarity slackness condition that the feasible region of problem (14) is exactly the set of optimal solutions of problem RLD(t^1). In particular, the solution (y^1, s^1) is an optimal solution of RLD(t^1). By Propositon 2, the optimal value of the linear program (14) gives the slope of $\phi(t)$ over the interval of linearity to the right of t^1. Hence, $\phi'_+(t^1) = w^T y^1$. Since t^1 is a breakpoint, by using Theorem 1(ii-c), it follows that t^2 is the first breakpoint larger than t^1 (possibly $t^2 = \infty$). The above arguments justify one step of Algorithm 2.1. The additional steps are similarly justified.

Note that there is no algorithmic restriction on how (13) and (14) are solved. In particular, either the simplex method or an interior point method can be used for this purpose.

3. RHS sensitivity analysis in interior point methods

The theory studied in Section 2 gives the foundation for the development of a sensitivity analysis approach in the absence of an optimal basis. The main goal of this section is to describe how this can be done. Two approaches are presented, namely the LU factorization approach and the affine scaling approach. The first approach shares the same computational machinery as used in the optimal basis approach, and hence, it can be viewed as a natural extension of the optimal basis approach. The second approach uses machinery similar to the one employed by interior point methods.

The outline for this section is as follows. In Subsection 3.1, approaches for performing range analysis, i.e. for estimating the optimality interval(s) containing 0 are given. In Subsection 3.2, approaches for performing marginal analysis, i.e. for estimating the left and right slopes of $\phi(t)$ at $t = 0$ are given.

Sensitivity analysis is usually performed along many specified directions (i.e., along all unit vectors). Therefore, sensitivity analysis estimation along a direction should be performed as cheap as possible. Preferably, only one square system of linear equations with an already factored coefficient matrix should be solved for each direction. The corresponding computational task involves one forward and one back substitution for each direction, or simply, one *solve* for each direction. The optimal basis approach performs one solve for each direction. However, by performing little computational work for each direction, we can only expect to get an approximation of the optimality interval containing 0, or estimates of the slopes of $\phi(t)$ at $t=0$.

We believe that the LU factorization approach is computationally less expensive than the affine scaling approach. Indeed, the LU factorization approach has the advantage that it requires the factorization of one matrix for estimating the ranges and the slopes corresponding to all directions. On the other hand, the affine scaling approach requires the factorization of two matrices; one for the range analysis and the other for the marginal analysis. A detailed computational study is required to find the approach which gives better approximations of the optimality intervals and better estimates for the slopes of $\phi(t)$.

In this section, it is convenient to simplify the notation used in Section 2. It is assumed that a pair of strongly complementary optimal solutions x^* and (y^*, s^*) for problems RLP(0) and RLD(0) are available. Let $\beta \equiv \sigma(x^*)$ throughout this section and assume that $A=[P:Z]$ where the columns of P correspond to variables with indices in β.

It has been shown that some interior point algorithms, when properly terminated, are able to compute a strongly optimal solution pair (e.g., see [5,9]). Therefore, the assumption that a strongly optimal pair is available is not restrictive in the context of interior point methods and simplifies our development considerably.

The discussion in this section also applies to the case when a strongly complementary solution pair is not available. If x^0 and (y^0, s^0) are optimal solutions for RLP(0) and RLD(0), respectively, then the discussion of Subsection 3.1 holds if $\beta=\overline{\sigma}(s^0)$ is taken and, similarly, the discussion of Subsection 3.2 holds if $\beta=\sigma(x^0)$ is taken. However, if the strong complementarity property does not hold then we may fail to find a nontrivial estimate (i.e., a interval of positive length) of a nontrivial optimality interval of (y^0, s^0). The possibility of this failure also exists in the optimal basis approach. In summary, degeneracy of the linear program and the need for performing little computational work for each direction causes the possibility of trivial estimates be generated in both approaches. Finally, we note that the LU factorization approach discussed below reduces to the optimal basis approach if β is taken to be the set of basic variable indices of an optimal basic feasible solution.

The following proposition shows that the optimality interval associated with a strongly complementary optimal solution of problem RLD(0) is either equal to the singleton $\{0\}$ or is a nontrivial closed interval containing 0 in its interior.

Proposition 3. *Suppose that x^* and (y^*, s^*) are strongly complementary optimal solutions for problems* RLP(0) *and* RLD(0), *respectively. Consider Theorem* 1 *with $t^0=0$, $x^0=x^*$ and $(y^0, s^0)=(y^*, s^*)$. Then, Cases* (ii-b) *and* (ii-c) *of Theorem* 1 *cannot occur.*

Proof. By Proposition 2, $\phi'_-(0)$ and $\phi'_+(0)$ are exactly the minimum and maximum value of the linear function $w^{\mathrm{T}}y$ over the set

$$\mathscr{S}(0) = \left\{(y, s) \mid P^{\mathrm{T}}y = c_\beta,\ Z^{\mathrm{T}}y + s_{\bar\beta} = c_{\bar\beta},\ s_\beta = 0,\ s_{\bar\beta} \geqslant 0\right\},$$

that is the set of all optimal solutions of the problem RLD(0). Using the fact that the range space of P and the null space of P^{T} are orthogonal complements, one can easily verify that exactly one of the following two systems of linear equations has a solution:

$$Pd_\beta = w \tag{15}$$

or

$$P^{\mathrm{T}}d_y = 0, \qquad w^{\mathrm{T}}d_y = 1. \tag{16}$$

If (15) has a solution, say d_β^*, then $w^{\mathrm{T}}y = (d_\beta^*)^{\mathrm{T}}P^{\mathrm{T}}y = (d_\beta^*)^{\mathrm{T}}c_\beta$ for every (y, s) in the optimal face $\mathscr{S}(0)$ of RLD(0). Hence, $w^{\mathrm{T}}y$ is constant over $S(0)$, and so $\phi'_-(0) = \phi'_+(0)$. Thus, 0 is not a breakpoint of $\phi(t)$, that is Case (i) of Theorem 1 holds. Assume now that (16) has a solution, say d_y^*. Clearly, (y^*, s^*) is a feasible solution of $\mathscr{S}(0)$ for which $s_{\bar\beta}^* > 0$. Hence, one can take positive steps from (y^*, s^*) along the directions (d_y^*, d_s^*) and $-(d_y^*, d_s^*)$, where $d_s^* = -A^{\mathrm{T}}d_y^*$, to obtain points in $\mathscr{S}(0)$ with smaller and larger values (in terms of the function $w^{\mathrm{T}}y$) than (y^*, s^*). Hence, $\phi'_-(0) < w^{\mathrm{T}}y^* < \phi'_+(0)$ which shows that Case (ii-a) of Theorem 1 holds. □

Proposition 3 is not only interesting in its own right but it also motivates approaches for obtaining an estimate for the optimality interval containing 0 when 0 is not a breakpoint or estimates for the derivatives $\phi'_-(0)$ and $\phi'_+(0)$ when 0 is a breakpoint. The details of how to compute these estimates are the subject of the following subsections.

3.1. Estimating right-hand side ranges

It follows from the results of Section 2 that the optimality interval $[t_-^*, t_+^*] \equiv \mathscr{O}(y^*, s^*)$ of the optimal solution (y^*, s^*) is determined by solving the two linear programs $t_+^* = \sup\{t \mid (x, t) \in \Gamma(\beta)\}$ and $t_-^* = \inf\{t \mid (x, t) \in \Gamma(\beta)\}$, where $\Gamma(\beta)$ is defined by relation (3). Equivalently, t_+^* and t_-^* are determined by

$$t_+^* = \sup\left\{t \mid Px_\beta - tw = b,\ x_\beta \geqslant 0\right\} \tag{17}$$

and

$$t_-^* = \inf\left\{t \mid Px_\beta - tw = b,\ x_\beta \geqslant 0\right\}, \tag{18}$$

where $P \equiv A_\beta$. To obtain estimates of the interval $[t_-^*, t_+^*]$, a solution of the system of linear equations

$$Pd_\beta = w \tag{19}$$

is computed. If (19) has no solution, that is w is not in the range of P, then the optimality interval of (y^0, s^0) is {0} (see the proof of Proposition 3). If (19) has a solution, say d_β^*, then minimum ratio tests are performed as follows. If $(d_\beta^*)_i \geqslant 0$ for all i, let $t_+ := \infty$; otherwise, let

$$t_+ := \min\left\{-\left(x_\beta^*\right)_i / \left(d_\beta^*\right)_i \mid \left(d_\beta^*\right)_i < 0\right\}. \tag{20}$$

Similarly, if $(d_\beta^*)_i \leqslant 0$ for all i, let $t_- := -\infty$; otherwise, let

$$t_- := \max\left\{-\left(x_\beta^*\right)_i / \left(d_\beta^*\right)_i \mid \left(d_\beta^*\right)_i > 0\right\}. \tag{21}$$

The interval $[t_-, t_+]$ is then a nontrivial estimate of the optimality interval $[t_-^*, t_+^*]$, since $x_\beta^* > 0$. Observe that if P is an optimal basis, then the above procedure reduces to the optimal basis approach.

Two approaches for computing d_β^* are given below. The first approach is based on factoring a submatrix of the constraint matrix. The second approach is based on computing the affine scaling direction.

3.1.1. LU factorization approach

If the optimal partition is generated by the approach discussed in [8] then a suitable factorization of P is already available. In this factorization, P is written as

$$P = \begin{bmatrix} B & N \\ Q & R \end{bmatrix}, \tag{22}$$

where the rows of $[Q : R]$ are linearly dependent on the rows of $[B : N]$, and the columns of N are linearly dependent on the columns of B. In addition, LU factors of B are also assumed to be available.

If the components of the vector d_β corresponding to the columns of the matrix N are set to zero in system (19), the LU factors of B can be used to solve for the remaining components of d_β. Note that the computational requirement in this approach is comparable to the optimal basis approach.

If w is in the range of P, then (19) can have either a unique solution or multiple solutions. System (19) will have a unique solutions if and only if P has full column rank. Clearly, if P is an optimal basis generated by the simplex method then (19) has a unique solution, namely $d_\beta^* = P^{-1} w$, and the ranges determined by the LU factorization and the optimal basis approaches coincide.

3.1.2. Affine scaling approach

An alternative approach based on the computation of affine scaling directions is described in this section. Since the solution obtained by an interior point algorithm is usually in the relative interior of the optimal face (and this is assumed to be the case here), it is a natural idea to use the affine scaling direction to obtain estimates of the optimal values of the linear programs (17) and (18).

First note that the point $(x_\beta, t) \equiv (x_\beta^*, 0)$ is feasible for the linear programs (17) and

(18). The affine scaling direction for (17) at $(x_\beta^*, 0)$ is given by the solution of the following problem

$$\begin{aligned} \max_{(d_\beta,\lambda)} \quad & \lambda \\ \text{s.t.} \quad & Pd_\beta - \lambda w = 0 \\ & \| D^{-1} d_\beta \|^2 \leqslant 1, \end{aligned} \tag{23}$$

where D is the diagonal matrix whose elements are given by the components of x_β^*. After writing the optimality conditions for this problem and doing some elementary algebraic manipulation, one can show that the optimal solution $(\hat{d}_\beta, \hat{\lambda})$ of problem (23) is determined by the relations $\| D^{-1}\hat{d}_\beta \| = 1$ and $\hat{d}_\beta = \hat{\lambda} D^2 P^{\mathrm{T}} v$, where v is a solution of $PD^2P^{\mathrm{T}}v = w$. Note that $\hat{\lambda} > 0$ since $Pd_\beta = w$ has a solution. The scaled direction $d_\beta^* = \hat{d}_\beta / \hat{\lambda} = D^2 P^{\mathrm{T}} v$ is a solution of (19). The vector d_β^* can also be computed by solving the system of linear equations

$$\begin{bmatrix} -D^{-2} & P^{\mathrm{T}} \\ P & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ w \end{pmatrix},$$

and then taking $d_\beta^* = D^2 P^{\mathrm{T}} v$. Here u and v are vectors of appropriate dimensions. The above augmented system may be useful if P has some dense columns (e.g., see [2]).

3.2. *Marginal analysis*

By the proof of Proposition 3, we know that $\phi'_-(0) < w^{\mathrm{T}} y^* < \phi'_+(0)$ whenever 0 is a breakpoint. In this case, approaches for estimating $\phi'_-(0)$ and $\phi'_+(0)$ are given in this subsection. Clearly, $w^{\mathrm{T}} y^*$ is a straightforward (or trivial) estimate for both $\phi'_-(0)$ and $\phi'_+(0)$. In fact, this is the only estimate that is usually provided by the optimal basis approach.

The relations (6) and (7) imply that $\phi'_+(0)$ and $\phi'_-(0)$ can be obtained by solving

$$\phi'_+(0) = \sup\left\{ w^{\mathrm{T}} y \mid A^{\mathrm{T}} y + s = c,\ s_\beta = 0,\ s \geqslant 0 \right\}, \tag{24}$$

and

$$\phi'_-(0) = \inf\left\{ w^{\mathrm{T}} y \mid A^{\mathrm{T}} y + s = c,\ s_\beta = 0,\ s \geqslant 0 \right\}. \tag{25}$$

Clearly, y^* is a feasible solution for problems (24) and (25). Nontrivial estimates for $\phi'_+(0)$ and $\phi'_-(0)$ can be obtained as follows. We first compute a direction d_y^* which is a solution of the system

$$P^{\mathrm{T}} d_y = 0, \qquad w^{\mathrm{T}} d_y = 1, \tag{26}$$

and then compute $d_s^* = -A^{\mathrm{T}} d_y^*$. Next, minimum ratio tests are performed as follows. If $(d_s^*)_i \geqslant 0$ for all $i \in \bar{\beta}$ then $\phi'_+(0) = \infty$; otherwise, let

$$\nu^+ = \min\left\{ -s_i^* / (d_s^*)_i \mid i \in \bar{\beta},\ (d_s^*)_i < 0 \right\}, \tag{27}$$

and take $\phi'_+(0) \approx w^T(y^* + \nu^+ d_y^*)$ as an estimate for $\phi'_+(0)$. Similarly, if $(d_s^*)_i \leqslant 0$ for all $i \in \bar{\beta}$, then $\phi'_-(0) = -\infty$; otherwise let

$$\nu^- = \min\left\{ s_i^* / (d_s^*)_i \mid i \in \bar{\beta},\ (d_s^*)_i > 0 \right\}, \tag{28}$$

and take $\phi'_-(0) \approx w^{\mathrm{T}}(y^* - \nu^- d_y^*)$ as an estimate for $\phi'_-(0)$.

As in Subsection 3.1, we next describe two approaches for computing d_y^* satisfying (26), namely the LU factorization approach and the affine scaling approach.

3.2.1. LU factorization approach

As in Subsection 3.1.1, assume that the factorization of P given by (22) is available. After deleting the linearly dependent columns from P, (26) reduces to

$$\begin{bmatrix} B^{\mathrm{T}} & Q^{\mathrm{T}} \\ w_B^{\mathrm{T}} & w_Q^{\mathrm{T}} \end{bmatrix} d_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{29}$$

Here w is partitioned according to the row partition of P. Since B is a nonsingular matrix, in order to solve (29) we only need one additional column from $[Q\ w_Q]^{\mathrm{T}}$ which is linearly independent from the columns of $[B\ w_B]^{\mathrm{T}}$. The components of the solution of (29) corresponding to the other columns in $[Q\ w_Q]^{\mathrm{T}}$ are then set to zero.

Let q_i be the ith column of Q^{T} and the corresponding element in w be $(w_Q)_i$. The inversion formula for bordered matrices gives,

$$\begin{bmatrix} B^{\mathrm{T}} & q_i \\ w_B^{\mathrm{T}} & (w_Q)_i \end{bmatrix}^{-1} = \begin{bmatrix} C & u \\ v^{\mathrm{T}} & 1/\delta \end{bmatrix},$$

where

$$C = B^{-\mathrm{T}} + (1/\delta) B^{-\mathrm{T}} q_i w_B^{\mathrm{T}} B^{-\mathrm{T}}, \quad \delta = (w_Q)_i - w_B^{\mathrm{T}} B^{-\mathrm{T}} q_i,$$
$$u = -(1/\delta) B^{-\mathrm{T}} q_i, \qquad v^{\mathrm{T}} = -(1/\delta) w_B^{\mathrm{T}} B^{-\mathrm{T}}.$$

Clearly, the inverse above exists if $\delta \neq 0$. Therefore, to identify a linearly independent column it is sufficient to find an i such that

$$(w_Q)_i - w_B^{\mathrm{T}} B^{-\mathrm{T}} q_i \neq 0.$$

This computation can be done efficiently by first computing $B^{-1} w_B$. In fact, $B^{-1} w_B$ is computed as part of the computations for estimating the right hand side ranges. After a linearly independent column is identified, a solution of (29) can be computed by using the formulae given above. Note that this requires the computation of $B^{-\mathrm{T}} q_i$, which can be obtained by using the LU factors of B.

3.2.2. Affine scaling approach

It is also possible to compute d_y^* as the search direction in the affine scaling method. Let Z_i denote the ith column of Z. The affine scaling direction is given by the solution of the following problem:

$$\begin{aligned} \max \quad & w^{\mathrm{T}} d_y \\ \text{s.t.} \quad & P^{\mathrm{T}} d_y = 0 \\ & \sum_{i \in \bar{\beta}} \frac{1}{(s_{\bar{\beta}}^*)_i^2} (Z_i^{\mathrm{T}} d_y)^2 \leqslant 1 \end{aligned}$$

Define $D_{ii} \equiv 1/s_i^*$ for every $i \in \bar{\beta}$ and let D denote the diagonal matrix with the elements D_{ii}, $i \in \bar{\beta}$, on its diagonal. The affine scaling direction can be computed by solving the system of equations

$$\begin{bmatrix} 0 & P^{\mathrm{T}} \\ P & ZD^2Z^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} d^1 \\ \hat{d}_y \end{pmatrix} = \begin{pmatrix} 0 \\ w \end{pmatrix}, \tag{30}$$

and then taking a positive scalar multiple of $\hat{d}_y$. The vector d^1 is of appropriate dimension. Alternatively, the augmented system

$$\begin{bmatrix} 0 & 0 & P^{\mathrm{T}} \\ 0 & -D^{-2} & Z^{\mathrm{T}} \\ P & Z & 0 \end{bmatrix} \begin{pmatrix} d^1 \\ d^2 \\ \hat{d}_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \tag{31}$$

can be solved. Here d^1 and d^2 are vectors of appropriate dimensions. The solutions of (30) and (31) require factorization of a symmetric indefinite matrix.

4. Parametric and sensitivity analyses for the cost vector

In this section we describe approaches for the parametric and sensitivity analyses with respect to the cost vector. The results given here are analogous to the ones developed in Sections 2 and 3 for RHS parametric and sensitivity analyses and are stated without proofs.

4.1. Cost parametric analysis

Consider the following cost parametric linear program CLP(t):

$$\begin{aligned} \psi(t) \equiv \min \quad & (c+th)^{\mathrm{T}} x \\ \text{s.t.} \quad & Ax = b, \ x \geqslant 0, \end{aligned} \tag{32}$$

and its dual CLD(t):

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ \text{s.t.} \quad & A^{\mathrm{T}} y + s = c + th, \ s \geqslant 0, \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $c, h, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ and $t \in \mathbb{R}$. The purpose of the cost *parametric analysis* is to find optimal solutions of CLP(t) for all values of t in a prespecified interval.

The following notation is used to develop the approach for the cost parametric analysis. If x^0 is an optimal solution of CLP(t^0) for some $t^0 \in \mathbb{R}$, then the optimality interval of x^0 is defined as

$$\mathscr{O}^c(x^0) \equiv \{t \in \mathbb{R} \mid x^0 \text{ is an optimal solution of CLP}(t)\}. \tag{33}$$

For $\beta \subset \mathcal{N}$, let

$$\Gamma^c(\beta) \equiv \{(y, s, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \mid A^T y + s = c + th,\ s \geqslant 0,\ s_\beta = 0\}, \tag{34}$$

$$\mathcal{T}^c(\beta) \equiv \{t \mid \exists (y, s) \in \mathbb{R}^m \times \mathbb{R}^n \text{ such that } (y, s, t) \in \Gamma^c(\beta)\},$$

and, for any $t \in \mathcal{T}^c(\beta)$, define

$$\Gamma_t^c(\beta) \equiv \{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n \mid (y, s, t) \in \Gamma^c(\beta)\}.$$

Clearly, $\mathcal{T}^c(\beta)$ is an interval (possibly empty). If $\mathcal{T}^c(\beta)$ is nonempty, then its endpoints are determined by the optimal values of the two linear programs $\inf\{t \mid (y, s, t) \in \Gamma^c(\beta)\}$ and $\sup\{t \mid (y, s, t) \in \Gamma^c(\beta)\}$.

The following proposition is similar to Proposition 1.

Proposition 4. *For some $t^0 \in \mathbb{R}$, let x^0 be an optimal solution of* CLP(t^0). *Let $\beta = \sigma(x^0)$. Then,*

(i) $\mathcal{O}^c(x^0) = \mathcal{T}^c(\beta)$;

(ii) *for every $t \in \mathcal{T}^c(\beta)$, $\Gamma_t^c(\beta)$ is equal to the set of optimal solutions of* CLD(t).

It is well known that $\psi(t)$ defined in (32) is a concave piecewise linear function with a finite number of breakpoints (e.g., see [10]). The next proposition expresses the left and right slopes of $\psi(t)$ as optimal values of certain linear programs (see Shapiro [12] for a proof of this result).

Proposition 5. *Let $t^0 \in \mathbb{R}$ be such that* CLP(t^0) *has an optimal solution. Let $S(t^0)$ denote the set of all optimal solutions x of* CLP(t^0). *Then, the left and the right derivatives of $\psi(t)$ at $t = t^0$ are given by*

$$\psi'_-(t^0) = \sup\{h^T x \mid x \in S(t^0)\} \tag{35}$$

and

$$\psi'_+(t^0) = \inf\{h^T x \mid x \in S(t^0)\}, \tag{36}$$

respectively.

The derivatives $\psi'_-(t^0)$ or $\psi'_+(t^0)$ may take the values ∞ and $-\infty$, respectively. If the left derivative is ∞ then CLP(t) is unbounded for every $t < t^0$. Similarly, if the right derivative is $-\infty$, then CLP(t) is unbounded for every $t > t^0$.

We call the interval between two consecutive breakpoints of ψ(t) an *interval of linearity* of $\psi(t)$. The next theorem is analogous to Theorem 1 and is stated without proof.

Theorem 2. *Let $t^0 \in \mathbb{R}$ be such that* CLP(t^0) *has an optimal solution. Let x^0 be an optimal solution of* CLP(t^0). *Then,*

(i) *if t^0 is not a breakpoint, then $\mathcal{O}^c(x^0)$ is the interval of linearity of $\psi(t)$ containing t^0;*

(ii) *if t^0 is a breakpoint, then there are three possibilities:*

(a) *if $\psi'_-(t^0) > h^T x^0 > \psi'_+(t^0)$, then $\mathcal{O}^c(x^0) = \{t^0\}$;*

(b) *if* $\psi'_-(t^0) = h^{\mathrm{T}}x^0$, *then* $\mathscr{O}^c(x^0)$ *is the interval of linearity lying to the left of* t^0;

(c) *if* $\psi'_+(t^0) = h^{\mathrm{T}}x^0$, *then* $\mathscr{O}^c(x^0)$ *is the interval of linearity lying to the right of* t^0;

These results are now translated into the following cost parametric analysis algorithm. As in Algorithm 2.1, we describe the method only for increasing values of t. The algorithm can be validated in the same way as Algorithm 2.1.

An algorithm for the cost parametric analysis

Input. Solutions x^0 and (y^0, s^0) which are optimal for CLP(0) and CLD(0), respectively. Set $k = 0$, and $t^0 = 0$.

Step 1. Let $\beta = \sigma(x^k)$. Solve the linear program

$$t^{k+1} = \sup\{t \mid A^{\mathrm{T}}y + s - th = c,\ s \geqslant 0,\ s_\beta = 0\}.$$

If t^{k+1} is ∞, **stop**. Otherwise, let $(y^{k+1}, s^{k+1}, t^{k+1})$ be an optimal solution of this problem. Then, (y^{k+1}, s^{k+1}) is an optimal solution of CLD(t^{k+1}) and $(\lambda y^k + (1-\lambda)y^{k+1}, \lambda s^k + (1-\lambda)s^{k+1})$ is an optimal solution of CLD($\lambda t^k + (1-\lambda)t^{k+1}$) for every $\lambda \in [0, 1]$.

Step 2. Let $\alpha \equiv \sigma(s^{k+1})$. Solve the linear program

$$\inf\{h^{\mathrm{T}}x \mid Ax = b,\ x_\alpha = 0,\ x \geqslant 0\}. \tag{37}$$

If this problem is unbounded, CLP(t) is unbounded for all $t > t^{k+1}$; **stop**. Otherwise, let x^{k+1} be an optimal solution of this problem.

Step 3. Set $k = k + 1$, return to *Step 1*.

4.2. Cost sensitivity analysis in interior point methods

In this subsection, we assume that x^* and (y^*, s^*) are given strongly complementary optimal solutions for problems CLP(0) and CLD(0). We also let $\beta = \sigma(x^*)$ and $A = [P : Z]$ where the columns of P correspond to variables with indices in β.

4.2.1. Estimating the cost ranges

By Proposition 4, we know that the optimality interval $[t_-^*, t_+^*] \equiv \mathscr{O}^c(x^*)$ of the optimal solution x^* is determined by solving the linear programs:

$$t_+^* = \sup\{t \mid A^{\mathrm{T}}y + s - th = c,\ s \geqslant 0,\ s_\beta = 0\}$$

and

$$t_-^* = \inf\{t \mid A^{\mathrm{T}}y + s - th = c,\ s \geqslant 0,\ s_\beta = 0\}.$$

To obtain estimates of the interval $[t_-^*, t_+^*]$, we solve the following system of linear equations:

$$P^{\mathrm{T}}d_y = h_\beta. \tag{38}$$

If (38) has no solution then $\mathscr{O}^c(x^*) = \{0\}$. Otherwise, if (38) has a solution, say d_y^*, then we perform the minimum ratio tests as follows. Let $d_s^* \equiv h - A^{\mathrm{T}} d_y^*$. If $(d_s^*)_i \geqslant 0$ for all i, let $t_+ := \infty$; otherwise, let

$$t_+ := \min\left\{-s_i^*/(d_s^*)_i \mid (d_s^*)_i < 0,\ i \in \bar{\beta}\right\}.$$

Similarly, if $(d_s^*)_i \leqslant 0$ for all i, let $t_- := -\infty$; otherwise, let

$$t_- := \max\left\{-s_i^*/(d_s^*)_i \mid (d_s^*)_i > 0,\ i \in \bar{\beta}\right\}.$$

The interval $[t_-, t_+]$ is then a nontrivial estimate of the optimality interval $[t_-^*, t_+^*]$, since $s_{\bar{\beta}}^* > 0$.

Note that if P is an optimal basis then the above procedure reduces to the optimal basis approach. Observe also that if h is equal to the i-unit vector with $i \in \bar{\beta}$, then $h_\beta = 0$ and $d_y^* \equiv 0$ is a solution of (38). In this case, the above scheme yields the estimates $t_- = -s_i^*$ and $t_+ = \infty$.

An LU factorization approach similar to the one described in Subsection 3.1.1 can be used to compute a solution of (38). Another possibility is to use an affine scaling approach in which a solution for the problem

$$\begin{array}{lll} \max_{(\lambda, d_y, d_s)} & \lambda & \\ \text{s.t.} & A^{\mathrm{T}} d_y + d_s - \lambda h = 0 & \qquad (39) \\ & \| D(d_s)_{\bar{\beta}} \|^2 \leqslant 1,\ (d_s)_\beta = 0, & \end{array}$$

is computed, where D is the diagonal matrix with the elements $(s_i^*)^{-1}$, $i \in \bar{\beta}$, on its diagonal. If $(\lambda^*, d_y^*, d_s^*)$ denotes an optimal solution of (39), then $\hat{d}_y = d_y^*/\lambda^*$ together with some vector d^1 solve the linear system of equations

$$\begin{bmatrix} 0 & P^{\mathrm{T}} \\ P & ZD^2Z^{\mathrm{T}} \end{bmatrix} \begin{pmatrix} d^1 \\ \hat{d}_y \end{pmatrix} = \begin{pmatrix} h_\beta \\ ZD^2 h_{\bar{\beta}} \end{pmatrix}. \qquad (40)$$

The $\hat{d}_y$-component of a solution of (40) is then a solution of (38) determined by the affine scaling approach.

4.2.2. Cost marginal analysis

If 0 is a breakpoint then $\psi'_-(0) > h^{\mathrm{T}} x^* > \psi'_+(0)$. In this case, we can use the procedure described next to obtain estimates for the derivatives $\psi'_-(0)$ and $\psi'_+(0)$. Relations (35) and (36) show that $\psi'_+(0)$ and $\psi'_-(0)$ can be obtained by solving

$$\psi'_+(0) = \inf\left\{h^{\mathrm{T}} x \mid Ax = b,\ x_{\bar{\beta}} = 0,\ x \geqslant 0\right\} = \inf\left\{h_\beta^{\mathrm{T}} x_\beta \mid P x_\beta = b,\ x_\beta \geqslant 0\right\} \qquad (41)$$

and

$$\psi'_-(0) = \sup\left\{h^{\mathrm{T}} x \mid Ax = b,\ x_{\bar{\beta}} = 0,\ x \geqslant 0\right\} = \sup\left\{h_\beta^{\mathrm{T}} x_\beta \mid P x_\beta = b,\ x_\beta \geqslant 0\right\}, \qquad (42)$$

Clearly, x^* is a feasible solution for the first LP problems in relations (41) and (42). Therefore, $h^{\mathrm{T}} x^*$ is a trivial estimate for $\psi'_-(0)$ and $\psi'_+(0)$. Better estimates for $\psi'_-(0)$ and $\psi'_+(0)$ can be obtained as follows. A direction d_β which is a solution of the system

$$P d_\beta = 0, \qquad h_\beta^{\mathrm{T}} d_\beta = 1 \qquad (43)$$

is computed. If $(d_\beta)_i \geqslant 0$ for all $i \in \beta$ then $\phi'_-(0) = \infty$; otherwise, let

$$v^- = \min\{-(x^*_\beta)_i/(d_\beta)_i \mid i \in \beta, (d_\beta)_i < 0\},$$

and take $\psi'_-(0) \approx h^T_\beta(x^*_\beta + v^- d_\beta) = h^T_\beta x^*_\beta + v^-$ as an estimate for $\psi'_-(0)$. Similarly, if $(d_\beta)_i \leqslant 0$ for all $i \in \beta$, then $\psi'_+(0) = -\infty$, otherwise, let

$$v^+ = \min\{(x^*_\beta)_i/(d_\beta)_i \mid i \in \beta, (d_\beta)_i > 0\}$$

and take $\psi'_+(0) \approx h^T_\beta(x^*_\beta - v^+ d_\beta) = h^T_\beta x^*_\beta - v^+$ as an estimate for $\psi'_+(0)$.

An approach similar to the LU factorization approach of Section 3.2.1 can be used to compute a solution of (43). Another possibility is the affine scaling approach in which the (scaled) affine scaling direction $\hat{d}_\beta$ for the second LP problem in (41) is computed. Namely, we solve the system $(PD^2P^T)u = PD^2h_\beta$ for u and set $\hat{d}_\beta = D^2h_\beta - D^2P^Tu$. Then, $d^*_\beta \equiv \hat{d}_\beta / h^T_\beta \hat{d}_\beta$ is solution of (43).

5. Numerical experiences with sensitivity analysis

We end this paper by summarizing our numerical experience with the sensitivity analysis approach described in Sections 3 and 4.

The computational results indicate that the affine scaling approach generally provides better range and slope estimates than the LU approach.

A comparison between the range and slope estimates generated by the optimal basis approach and interior point approach provides mixed indications. The optimal basis approach provided better RHS range estimates than the interior point approach even when $t = 0$ is not a breakpoint. The RHS slope estimates obtained by the interior point approach are however more accurate when $t = 0$ is a breakpoint. The conclusions are mixed with respect to the cost range estimates.

We should remark that the above conclusions are very limited since it is based on the results obtained for just a few small problems. It would be necessary to conduct the experiments in a more extensive set of problems involving large scale linear programs to obtain better conclusions about the two approaches.

We have not performed any computational analysis for the parametric analysis approach described in this paper in which an interior point code is used to solve the subproblems. However, we think that such an approach would be a valuable alternative to the parametric simplex algorithm for analyzing the solution of highly degenerate LP problems. Indeed, for these problems the parametric simplex algorithm generally performs many (degenerate) pivots without making substantial progress while an interior point method performs a certain number of iterations that depends little on the problem size and its degree of degeneracy.

For the interior point parametric analysis approach to be successful, the interior point code used to solve the subproblems should be able to solve LP problems with any type of linear constraints without having the need to change them to an equivalent set of linear constraints in standard form and/or dual form. In this respect, the approach and the linear algebra machinery described in the paper [2] would be useful. Without giving any details, we observe that after one subproblem is solved, it is easy to show that its primal and dual optimal solutions can be used to construct a point lying in the relative interior of the feasible region of next subproblem. The next subproblem could then be solved by an interior point code (having the above requirements) starting from this point. This approach for solving the subproblem eliminates the potential complication of guessing a good warm start and of using an infeasible point as a starting point.

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