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Trust region affine scaling algorithms for linearly constrained convex and concave programs [1]

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Abstract

We study a trust region affine scaling algorithm for solving the linearly constrained convex or concave programming problem. Under primal nondegeneracy assumption, we prove that every accumulation point of the sequence generated by the algorithm satisfies the first order necessary condition for optimality of the problem. For a special class of convex or concave functions satisfying a certain invariance condition on their Hessians, it is shown that the sequences of iterates and objective function values generated by the algorithm converge R-linearly and Q-linearly, respectively. Moreover, under primal nondegeneracy and for this class of objective functions, it is shown that the limit point of the sequence of iterates satisfies the first and second order necessary conditions for optimality of the problem. © 1998 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

Keywords: Linearly constrained problem; Affine scaling algorithm; Trust region method; Interior point method

1. Introduction

The affine scaling (AS) algorithm for linear programming was first introduced by Dikin [6] in 1967 but remained unknown to the western community until the late 80's. The method was later rediscovered independently by Barnes [3] and Vanderbei et al. [44]. Since then, there have appeared a number of papers which study its global and local convergence [7,8,12,21,37–39,41–43], the behavior of its associated continuous trajectories [2,4,22,24,45] and its computational efficiency [1,23].

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In 1980, Dikin [9] proposed the second order affine scaling algorithm for convex quadratic programming (QP) problem, where the next iterate minimizes the objective function over the intersection of the feasible region with the ellipsoid centered at the current point and whose radius is a fixed fraction $\beta \in (0,1)$ of the radius of the largest "scaled" ellipsoid inscribed in the nonnegative orthant. This method was later rediscovered by Ye [46] and Ye and Tse [48] after the introduction of Karmarkar's algorithm [17]. The above mentioned papers all assume that the QP problem is primal nondegenerate, an assumption that considerably simplifies the analysis of convergence. In Refs. [46,48] an extra dual nondegeneracy assumption is also imposed. A global convergence proof of the second order AS algorithm which drops the primal nondegeneracy assumption but still keeps some sort of dual nondegeneracy assumption is given in Tsuchiya [40] for strictly convex QP with $\beta \in (0,1/8)$. Sun [35] gives a global convergence proof for the second order AS algorithm without imposing any nondegeneracy assumptions; however, his analysis is still restrictive since the algorithm only allows very small β, i.e. $\beta = 2^{-\mathcal{O}(L)}$, where L is the input size of the problem. Recently, Monteiro and Tsuchiya [25] proved the global convergence of the second order AS algorithm for convex QP for any $\beta \in (0,1)$ without imposing any nondegeneracy assumptions. For nonconvex QP problems, Ye [47] and Bonnans and Bouhtou [5] establish global convergence of the second order AS algorithm assuming primal nondegeneracy and some sort of dual nondegeneracy, which is satisfied by any strictly concave QP. As a special case of our results in this paper, we establish global convergence of this algorithm for any (not necessarily strictly) concave QP problem under primal nondegeneracy. Global convergence of the second order AS algorithm for indefinite QP problems under the assumption of primal nondegeneracy only is still an open question.

Computational results of the affine scaling for solving general quadratic problems are reported in Bonnans and Bouhtou [5] and Han, Pardalos and Ye [15]. Other related interior point algorithms for solving general quadratic programming are given in Kamath et al. [16] and Kamarkar et al. [18]. Interior point methods for solving general quadratic programs have been recently surveyed by Pardalos and Resende [32].

AS algorithms for solving a linearly constrained convex program have been studied by Gonzaga and Carlos [14] and Sun [36]. Ref. [14] analyzes a first-order AS algorithm, where at each iteration a line search is performed along the scaled steepest descent direction computed using the first order Taylor expansion of the objective function. Under primal nondegeneracy assumption, Gonzaga and Carlos [14] prove that every accumulation point generated by this algorithm is an optimal solution. Sun [36] studies a version of the second order AS algorithm for a certain class of convex functions whose Hessians satisfy a certain invariance property and establishes its global convergence without imposing any nondegeneracy assumption. At each iteration of his algorithm, an optimal displacement d^k that minimizes the second order Taylor expansion of the objective function over the ellipsoid with a fixed fraction $\beta > 0$ is computed, and the next iterate x^{k+1} is determined by $x^{k+1} = x^k + d^k/\kappa$, where $\kappa \geqslant 1$ is a constant which depends on the curvature of the objective function. As in his paper [35], the convergence result of Ref. [36] is restrictive in the sense that the step-length has to be small, namely $\mathcal{O}(\varepsilon)$, to insure that the algorithm finds an ε-optimal solution. Gonzaga [13] studies a trust region method which explores the shape of the trust regions to generate ellipsoidal regions adapted to the shape of the feasible region. Possible convergence results (including the case of convex objective function) of his algorithm under primal nondegeneracy assumption are given without any proofs.

In this paper, we study a version of the second order AS algorithm for solving a linearly constrained optimization problem in which the fraction β_k for the ellipsoid used at the kth iteration is selected according to a trust region strategy. Trust region methods have been an important and well studied class of iterative methods for nonlinear optimization problems. They possess strong convergence properties and are reliable and efficient in the numerical solution of optimization problems (see for example Refs. [10,11,19,20,28–31,33,34]). Moré [30] provides a comprehensive survey of trust region methods applied to unconstrained minimization problems, where ellipsoids of different shapes and sizes are used as trust regions. In the trust region methods studied by Moré, the shape of the trust region (i.e., the scaling matrix used to define the region) is only explored for the purpose of attaining good scaling and preconditioning of the variables. Convergence results for trust region methods applied to unconstrained optimization problems are obtained by assuming that the condition numbers of the scaling matrices are uniformly bounded. On the other hand, AS algorithms for linearly constrained problems explore the shape of the trust region to adapt it to the shape of the feasible region so that feasibility is achieved automatically as a by-product. In contrast to the unconstrained case, the sequence of scaling matrices used by AS algorithms has unbounded condition number. It turns out that the general theory presented in Moré [30] is also useful for the analysis of trust region methods with unbounded scaling matrices, as will be seen in the analysis of the algorithm presented in this paper.

The kth iteration of the algorithm studied in this paper can be briefly described as follows. A quadratic approximation function which agrees with the objective function in value and gradient at the current iterate is minimized over an affine scaling ellipsoid centered at the current iterate and with fraction $\beta_k > 0$. The fraction β_{k+1} is then determined from β_k according to a standard trust region strategy: the fraction is increased or decreased depending on whether the minimizer of the quadratic approximation provides a good or bad prediction of the objective function. Assuming primal nondegeneracy and that the objective function is either convex or concave, we prove that every accumulation point of the sequence of iterates generated by the algorithm satisfies the first order necessary condition for optimality of the problem; in particular, if the objective function is convex we obtain the result that any accumulation point of the sequence of iterates is an optimal solution of the problem. Assuming that the Hessians of the quadratic approximation and the objective function agree at each iteration and that the (convex or concave) objective function satisfies a certain invariance condition on its Hessian, it is shown that the sequence of iterates and objective function values converge R-linearly and Q-linearly, respectively. Moreover, if primal nondegeneracy is also assumed then it is shown that the limit point satisfies the first and second order necessary conditions for optimality of the problem.

The paper is organized as follows. In Section 2, we state the assumptions used in this paper and describe the trust region AS algorithm. We also review some basic results about this method. In Section 3, we establish the global convergence of the algorithm for solving a linearly constrained convex or concave problem under the primal nondegeneracy assumption. This section is divided into three subsections. In Subsection 3.1, we review some results that are used in the convergence analysis of trust region methods for unconstrained problems. This discussion closely follows the presentation of Moré [30]. In Subsection 3.2, by introducing a suitable change of variable, we are able to recast our algorithm into a special case of the algorithm considered in Ref. [30] and therefore use the analysis of this paper to obtain the conclusion that the complementarity product between the kth iterate and its associated dual estimate converges to 0. Finally, in Subsection 3.3, we complete the convergence analysis of the trust region AS algorithm. In Section 4, we analyze the same algorithm under the assumption that the objective function satisfies a certain invariance property and, at each iteration, the second order Taylor expansion is used for the quadratic approximation of the objective function. In Appendix A, we prove a technical result due to Gonzaga and Carlos [14] which is used in the analysis of Subsection 3.3. In Appendix B, we study some properties of the class of functions considered in Section 4.

In this paper, we make no attempt to verify the computational efficiency of the trust region AS algorithm for solving linearly constrained nonlinear problems. Our concern is mainly with the theoretical aspects of the method. However, based on the success of the affine scaling algorithm for solving linear and quadratic programs (e.g. Refs. [1,23,15]) and the effectiveness of trust region methods for solving nonlinear problems, we believe that the trust region AS algorithm will be quite successful in solving linearly constrained nonlinear problems.

The following notation is used throughout the paper. Z_+ denotes the set of all the nonnegative integers. $\mathbb{R}^p$, $\mathbb{R}^p_+$ and $\mathbb{R}^p_{++}$ denote the p-dimensional Euclidean space, the nonnegative orthant of $\mathbb{R}^p$ and the positive orthant of $\mathbb{R}^p$, respectively. The set of all $p \times q$ matrices with real entries is denoted by $\mathbb{R}^{p\times q}$. For $Q \in \mathbb{R}^{p\times p}$, $Q \geqslant 0$ ($Q \leqslant 0$) means Q is positive (negative) semi-definite and $Q > 0$ ($Q < 0$) means Q is positive (negative) definite. The diagonal matrix corresponding to a vector u is denoted by diag (u). The ith component of a vector $u \in \mathbb{R}^p$ is denoted by u_i and, for an index set $\alpha \subseteq \{1,\dots,p\}$, the subvector $[u_i]_{i\in\alpha}$ is denoted by u_α. If $\alpha \subseteq \{1,\dots,p\}$, $\beta \subseteq \{1,\dots,q\}$ and $Q \in \mathbb{R}^{p\times q}$, we let $Q_{\alpha\beta}$ denote the submatrix $[Q_{ij}]_{i\in\alpha, j\in\beta}$; if $\alpha = \{1,\dots,p\}$, $Q_{\alpha\beta}$ is simply denoted by Q_β. Given u and v in $\mathbb{R}^p$, $u \leqslant v$ means $u_i \leqslant v_i$ for every $i = 1,\dots,p$. For a vector u, the Euclidean norm, the 1-norm and the ∞-norm are denoted by $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. Given a matrix $Q \in \mathbb{R}^{p\times q}$, we let Range $(Q) \equiv \{Qv \mid v \in \mathbb{R}^q\}$ and Null $(Q) \equiv \{v \in \mathbb{R}^q \mid Qv = 0\}$. We say that (B, N) is a partition of $\{1,\dots,p\}$ if $B \cup N = \{1,\dots,p\}$ and $B \cap N = \emptyset$. The superscript $^{\mathrm{T}}$ denotes transpose. If $\{\xi_k\}$ and $\{\eta_k\}$ are two sequences of real numbers, then the notation $\xi_k = \mathcal{O}(\eta_k)$ means that there exists a scalar $r \geqslant 0$ such that $|\xi_k| \leqslant r\eta_k$ for all k sufficiently large.

2. Description of the algorithm and preliminary results

In this section, we introduce the linearly constrained problem which will be the subject of our study and state the main assumptions that will be needed in our analysis. We then describe the trust region affine scaling algorithm for solving the linearly constrained (convex or concave) problem and give some basic preliminary results that will be useful in the subsequent sections.

We consider the following linearly constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \\ & x \geqslant 0, \end{array} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We denote the feasible region of problem (1) by $\mathcal{P} \equiv \{x \in \mathbb{R}^n \mid Ax = b,\ x \geqslant 0\}$ and define the set $\mathcal{P}^0 \equiv \{x \in \mathcal{P} \mid x > 0\}$, which is the relative interior of $\mathcal{P}$ whenever it is nonempty.

We make the following assumptions throughout the paper:

Assumption 1. $\operatorname{rank}(A) = m$;

Assumption 2. *there exists $x^0 \in \mathcal{P}^0$ such that $\mathcal{L}(x^0) \equiv \{x \in \mathcal{P} \mid f(x) \leqslant f(x^0)\}$ is bounded;*

Assumption 3. *the function f is either convex or concave;*

Assumption 4. *$\mathcal{P}$ is nondegenerate, i.e. AX^2A^{T} with $X \equiv \operatorname{diag}(x)$ is invertible for every $x \in \mathcal{P}$.*

Assumptions 1 and 2 will be implicitly assumed in the statement of every result of this paper. On the other hand, explicit reference will be made to the other two assumptions whenever they are needed.

Assume that $x^k \in \mathcal{P}^0$ denotes the current (the kth) iterate generated by the algorithm. A rough description of how the next iterate is computed is given as follows. The function $d \mapsto f(x^k + d) - f(x^k)$ is approximated by the quadratic function $q_k(d)$ defined as

$$q_k(d) = \nabla f(x^k)^{\mathrm{T}} d + \tfrac{1}{2} d^{\mathrm{T}} Q_k d \approx f(x^k + d) - f(x^k),$$

where $Q_k \approx \nabla^2 f(x^k)$ is an approximate Hessian (e.g., obtained by means of some quasi-Newton updating scheme), and an optimal solution d^k of the problem

$$\begin{array}{ll} \text{minimize} & q_k(d) = \nabla f(x^k)^{\mathrm{T}} d + \tfrac{1}{2} d^{\mathrm{T}} Q_k d \\ \text{subject to} & Ad = 0 \\ & \|X_k^{-1} d\| \leqslant \beta_k \end{array} \tag{2}$$

is computed, where β_k is some scalar in $(0,1)$. Clearly, $x^k + d^k$ is then an optimal solution of the function $x \mapsto q_k(x - x^k)$ over the Dikin's ellipsoid with center x^k and radius β_k:

$$\mathcal{E}(x^k, \beta_k) \equiv \{x \mid Ax = b,\ \|X_k^{-1}(x - x^k)\| \leqslant \beta_k\}.$$

Since $\mathcal{E}(x^k, \beta_k) \subseteq \mathcal{P}^0$ for any $x^k \in \mathcal{P}^0$ and any $\beta_k \in (0,1)$, we have the desired property that $x^k + d^k \in \mathcal{P}^0$. In practice, it is desirable to compute only an approximate solution of (2) since an exact solution may be hard to compute (e.g., see Refs. [5,30,31]). In our presentation, we require that an approximate solution of (2) be computed according to the following criteria:

$$d^k \in \arg\min\{q_k(d) \mid Ad = 0,\ \|X_k^{-1} d\| \leqslant \tilde{\beta}_k\},$$
$$\text{with } \tilde{\beta}_k \in [(1-\sigma)\beta_k,\ \min(\bar{\beta}, (1+\sigma)\beta_k)], \tag{3}$$

where $\sigma \in (0,1)$ and $\bar{\beta} \in (0,1)$ are given constants. In other words, $x^k + d^k$ is the exact solution of $q_k(x - x^k)$ over an ellipsoidal trust region with the same scaling matrix and with radius $\tilde{\beta}_k$ which is close to the specified β_k. Next the ratio between the *actual reduction* in f and the *predicted reduction* in q_k is computed as

$$r_k \equiv \frac{f(x^k) - f(x^k + d^k)}{-q_k(d^k)}. \tag{4}$$

For some constant $\theta_1 \in (0,1)$ (e.g., $\theta_1 = 0.25$), if $r_k > \theta_1$ then we set $x^{k+1} = x^k + d^k$; otherwise, we set $x^{k+1} = x^k$. In both cases, β_k is updated. In the second case, β_k must be reduced so that the ratio at the next iteration is improved, that is, it becomes closer to 1.

The details of the trust region affine scaling algorithm for solving (1) are given next.

Algorithm 1. *Let σ, $\bar{\beta} \in (0,1)$, $Q_0 \in \mathbb{R}^{n\times n}$ and $x^0 \in \mathcal{P}^0$ as in Assumption 2 be given. Set $\beta_0 = \bar{\beta}$.*
For $k = 0, 1, 2, \ldots$
- (a) *Determine an approximate solution d^k of (2) in the sense of (3) and compute r_k;*
- (b) *If $r_k \leqslant 0.25$, let $x^{k+1} = x^k$, $\beta_{k+1} \in (\frac{1}{8}\beta_k, \frac{1}{2}\beta_k]$;*
  If $r_k \in (0.25, 0.75)$, let $x^{k+1} = x^k + d^k$, $\beta_{k+1} \in [\frac{1}{2}\beta_k, \beta_k]$;
  If $r_k \geqslant 0.75$, let $x^{k+1} = x^k + d^k$, $\beta_{k+1} \in [\beta_k, \min(2\beta_k, \bar{\beta})]$;
- (c) *Update the matrix Q_k.*

We observe that any constants $0 < \theta_1 < \theta_2 < 1$ could be used in place of the numerical constants 0.25 and 0.75. In the three cases of Algorithm 1(b), the range of possible values for β_{k+1} is an interval; this has the advantage of providing more flexibility on the choice of β_{k+1} at each iteration. The above algorithm is in fact similar to the one described in Moré [30] but with a specific choice of the scaling matrix (see Algorithm 2 of Section 3); this observation is further explored in Subsection 3.2. Note that in Algorithm 1(b), if $r_k \leqslant 0.25$ then the displacement vector d^k is not accepted:

in this case the iteration is called *unsuccessful*; if $r_k > 0.25$ then d^k is accepted: in this case the iteration is called *successful*. We let $\mathcal{S}$ denote the set of all successful iterations:

$$\mathcal{S} \equiv \{k \in Z_+ \mid r_k > 0.25\}.$$

The main goal of Section 3 is to show that if $\bar{x} \in \mathcal{P}$ is an accumulation point of the sequence generated by Algorithm 1 then $\bar{x}$ satisfies the first order necessary condition for optimality of (1), that is, for some $(\bar{y}, \bar{s}) \in \mathbb{R}^m \times \mathbb{R}^n$,

$$A^{\mathrm{T}}\bar{y} + \bar{s} = \nabla f(\bar{x}), \quad \bar{s} \geqslant 0 \quad \text{and} \quad \bar{X}\bar{s} = 0, \tag{5}$$

where $\bar{X}$ = diag $(\bar{x})$. Moreover, Section 4 shows that, for a certain class of convex or concave functions, $\bar{x}$ satisfies the following second order necessary condition for optimality of (1):

$$Ad = 0,\ d_N = 0 \ \Rightarrow \ d^{\mathrm{T}}\nabla^2 f(\bar{x})d \geqslant 0, \tag{6}$$

where $N \equiv \{j \mid \bar{x}_j = 0\}$.

For the purpose of future reference, we note that an approximate solution d^k of (2) satisfies the first order necessary conditions for optimality of subproblem (3), namely, for some $y_+^k \in \mathbb{R}^m$ and $\mu_k \in \mathbb{R}$,

$$s_+^k + \mu_k X_k^{-2} d^k = 0, \ \text{where} \ \ s_+^k \equiv \nabla q_k(d^k) - A^{\mathrm{T}}y_+^k, \tag{7}$$

$$Ad^k = 0, \tag{8}$$

$$\mu_k \left(\|X_k^{-1}d^k\| - \tilde{\beta}_k\right) = 0, \tag{9}$$

$$\|X_k^{-1}d^k\| \leqslant \tilde{\beta}_k, \tag{10}$$

$$\mu_k \geqslant 0, \tag{11}$$

and the second order necessary condition

$$d^{\mathrm{T}}(Q_k + \mu_k X_k^{-2})d \geqslant 0, \ \forall d \in \mathrm{Null}(A). \tag{12}$$

Observe that $q_k(d^k) \leqslant 0$ due to (3) and the fact that $q_k(0) = 0$. The following result shows that, without loss of generality, we may assume that $q_k(d^k) < 0$, and hence, $d^k \neq 0$, for all $k \geqslant 0$.

Proposition 2.1. *If $q_k(d^k) = 0$ then x^k satisfies the first order necessary condition for optimality of problem (1). In particular, if $f(\cdot)$ is convex and $q_k(d^k) = 0$ then x^k is an optimal solution of problem (1). Moreover, if $Q_k = \nabla^2 f(x^k)$ then x^k also satisfies the second order necessary condition for optimality of problem (1).*

Proof. If $q_k(d^k) = 0$ then $d = 0$ is an optimal solution of (3) since $q_k(0) = 0$. Hence, we may assume that $d^k = 0$. Using this observation, relation (7) and the fact that $\nabla q_k(d) = \nabla f(x^k) + Q_k d$, we conclude that $\nabla f(x^k) - A^{\mathrm{T}}y_+^k = 0$, which shows that $\bar{x} \equiv x^k$ together with $(\bar{y}, \bar{s}) \equiv (y_+^k, 0)$ satisfy the first order condition (5). We also conclude from (9) that $\mu_k = 0$. If $Q_k = \nabla^2 f(x^k)$ then, by (12), we have that $d^{\mathrm{T}}\nabla^2 f(x^k) d \geqslant 0$ for all $d \in \mathrm{Null}(A)$, showing in particular that (6) holds with $\bar{x} \equiv x^k$. □

By Proposition 2.1, if $q_k(d^k) = 0$ for some k, then Algorithm 1 finds a point satisfying the first (and, if $Q_k = \nabla f^2(x^k)$, second) order necessary conditions for optimality of (1) in a finite number of iterations. Hence, from now on, we assume that $q_k(d^k) < 0$ and $d^k \neq 0$ for all $k \geqslant 0$. This implies in particular that expression (4) is always well-defined.

The following relations can be easily derived using relations (7) and (8) together with Assumption 1.

$$y_+^k = (AX_k^2A^{\mathrm{T}})^{-1}AX_k^2\nabla q_k(d^k), \tag{13}$$

$$s_+^k = \left(I - A^{\mathrm{T}}(AX_k^2A^{\mathrm{T}})^{-1}AX_k^2\right)\nabla q_k(d^k), \tag{14}$$

$$\mu^k \equiv \frac{\|X_k s_+^k\|}{\|X_k^{-1}d^k\|}. \tag{15}$$

Proposition 2.2. *The following statements hold:*
(a) $-q_k(d^k) = \frac{1}{2}(d^k)^{\mathrm{T}}Q_k d^k + \mu_k\|X_k^{-1}d^k\|^2$ *for all* $k \geqslant 0$;
(b) $0 < (1-\bar{\beta})x^k \leqslant x^{k+1} \leqslant (1+\bar{\beta})x^k$ *for all* $k \geqslant 0$;
(c) $f(x^k) - f(x^{k+1}) > -0.25q_k(d^k) > 0$ *for all* $k \in \mathcal{S}$;
(d) *the sequence* $\{f(x^k)\}$ *is non-increasing;*
(e) $\{x^k\} \subseteq \mathcal{L}(x^0)$ *and* $\{x^k\}$ *is bounded;*
(f) $\|X_k^{-1}d^k\| = \tilde{\beta}_k$ *for every* $k \geqslant 0$ *such that* $Q_k \leqslant 0$.

Proof. We first prove (a). Using relations (7) and (8), we have for all $k \geqslant 0$,

$$\begin{aligned} q_k(d^k) &= \nabla f(x^k)^{\mathrm{T}}d^k + \tfrac{1}{2}(d^k)^{\mathrm{T}}Q_k d^k = (\nabla f(x^k) + Q_k d^k)^{\mathrm{T}}d^k - \tfrac{1}{2}(d^k)^{\mathrm{T}}Q_k d^k \\ &= (d^k)^{\mathrm{T}}\nabla q_k(d^k) - \tfrac{1}{2}(d^k)^{\mathrm{T}}Q_k d^k = -\mu_k\|X_k^{-1}d^k\|^2 - \tfrac{1}{2}(d^k)^{\mathrm{T}}Q_k d^k. \end{aligned}$$

We now prove (b). Since (b) clearly holds if the kth iteration is unsuccessful, we may assume that the kth iteration is successful. Then, $x^{k+1} = x^k + d^k$ and, by relations (3) and (10), we have $\|X_k^{-1}(x^{k+1} - x^k)\| \leqslant \tilde{\beta}_k \leqslant \bar{\beta}$, which clearly implies that $(1-\bar{\beta})x^k \leqslant x^{k+1} \leqslant (1+\bar{\beta})x^k$. Since $\bar{\beta} < 1$, this implies that if $x^k > 0$ then $x^{k+1} \geqslant (1-\bar{\beta})x^k > 0$. Using the fact that $x^0 > 0$ and a simple induction argument, we conclude that $x^k > 0$ for all $k \geqslant 0$, and hence (b) holds. Statement (c) follows from (4) and the fact that $r_k > 0.25$ and $x^{k+1} = x^k + d^k$ for every $k \in \mathcal{S}$. Statement (d) follows from (c) and the fact that $x^{k+1} = x^k$ for all $k \notin \mathcal{S}$. Statement (e) is an immediate consequence of (b), (d) and Assumption 2. We now prove statement (f). Assume for contradiction that there exists a $k \geqslant 0$ such that $Q_k \leqslant 0$ and $\|X_k^{-1}d^k\| < \tilde{\beta}_k$. Then, by (9), we have $\mu_k = 0$. It then follows from (12) that $d^{\mathrm{T}}Q_k d \geqslant 0$ for all $d \in$ Null (A). But since $Q_k \leqslant 0$, we must have $d^{\mathrm{T}}Q_k d = 0$ for all $d \in$ Null (A). Hence, by (2) and (3), it follows that

$$d^k \in \arg\min\{\nabla f(x^k)^{\mathrm{T}}d \mid Ad = 0,\ \|X_k^{-1}d\| \leqslant \tilde{\beta}_k\}. \tag{16}$$

Using the assumption that $q_k(d^k) < 0$ for all $k \geqslant 0$ (see paragraph after Proposition 2.1) and the fact that the objective function of (16) is linear, it is easy to see that $\|X_k^{-1}d^k\| = \tilde{\beta}_k$, a contradiction. □

3. Global convergence

The purpose of this section is to establish the global convergence of Algorithm 1 for solving problem (1). This section is divided into three subsections. In the first subsection, we review some results that are used in the convergence analysis of trust region methods for unconstrained problems. This discussion closely follows the presentation of Moré [30], which analyzes a general trust region method in which ellipsoids of different shapes and sizes are used as trust regions. This general theory turns out to be useful for the convergence analysis of Algorithm 1. In the second subsection, by introducing a suitable change of variables that eliminates the constraints $Ax = b$ from problem (1), we are able to translate Algorithm 1 into a special case of the algorithm considered in Ref. [30] and therefore use the analysis of this paper to obtain the conclusion that $\lim_{k\to\infty} \|X_k s(x^k)\| = 0$, where $s(x)$ is defined for all $x \in \mathcal{P}$ by

$$s(x) \equiv \left(I - A^{\mathrm{T}}[AX^2A^{\mathrm{T}}]^{-1}AX^2\right)\nabla f(x). \tag{17}$$

(Note that Assumptions 1 and 4 are needed here to guarantee that $s(\cdot)$ is well-defined on the relative boundary of $\mathcal{P}$.) Finally, in the third subsection, we complete the convergence analysis of Algorithm 1. Specifically, under the assumptions stated in Section 2, we show that every accumulation point $\bar{x}$ of the sequence $\{x^k\}$ generated by Algorithm 1 satisfies the first order necessary condition for optimality of problem (1); in particular, if f is convex then $\bar{x}$ is an optimal solution of (1).

3.1. Trust region algorithms for unconstrained problems

In this subsection we review some results discussed in Moré [30] that play an important role in the analysis of Algorithm 1.

Moré [30] studies a general trust region method for minimizing a continuously differentiable function $h : \mathbb{R}^p \to \mathbb{R}$ over $\mathbb{R}^p$, in which the trust region subproblems are of the form

$$\begin{array}{ll} \text{minimize} & \phi_k(w) = \nabla h(z^k)^{\mathrm{T}} w + \frac{1}{2} w^{\mathrm{T}} B_k w \\ \text{subject to} & \|D_k w\| \leqslant \beta_k, \end{array} \tag{18}$$

where the minimization is with respect to the displacement vector $w \in \mathbb{R}^p$, z^k denotes the kth iterate, $B_k \in \mathbb{R}^{p\times p}$, the scaling matrix $D_k \in \mathbb{R}^{p\times p}$ is invertible (not necessarily diagonal or symmetric) and $\beta_k > 0$.

Since an exact optimal solution of problem (18) may be hard to find, an approximate solution is computed instead. Let $\gamma_1 > 0$, $\gamma_2 > 0$ be two constants. According to Ref. [30], w^k is called an approximate solution of (18) if

$$\phi_k(w^k) \leqslant \gamma_1 \min\{\phi_k(w) \mid \|D_k w\| \leqslant \beta_k\}, \quad \|D_k w^k\| \leqslant \gamma_2 \beta_k. \tag{19}$$

The following ratio between the real reduction in h and its predicted reduction

$$\rho_k = \frac{h(z^k) - h(z^k + w^k)}{-\phi_k(w^k)} \tag{20}$$

is used to determine whether w^k is accepted as the displacement vector: if $\rho_k > 0.25$, w^k is accepted and we set $z^{k+1} = z^k + w^k$; otherwise, we set $z^{k+1} = z^k$.

We are now ready to state the complete algorithm studied in Ref. [30].

Algorithm 2. *Trust Region Method.*
Let $\gamma_1 > 0$, $\gamma_2 > 0$, $z^0 \in \mathbb{R}^p$, $\beta_0 > 0$, $B_0 \in \mathbb{R}^{p\times p}$ and a nonsingular matrix $D_0 \in \mathbb{R}^{p\times p}$ be given.
For $k = 0, 1, 2, \ldots$

(a) *Determine an approximate solution w^k of (18) according to (19), and compute ρ_k;*

(b) *If $\rho_k \leqslant 0.25$ then $z^{k+1} = z^k$, $\beta_{k+1} \in (0, \frac{1}{2}\beta_k]$;*
If $\rho_k \in (0.25, 0.75)$ then $z^{k+1} = z^k + w^k$, $\beta_{k+1} \in [\frac{1}{2}\beta_k, \beta_k]$;
If $\rho_k \geqslant 0.75$ then $z^{k+1} = z^k + w^k$, $\beta_{k+1} \in [\beta_k, 2\beta_k]$;

(c) *Update the matrix B_k and the scaling matrix D_{k+1}.*

It is useful at this point to make a few observations about Algorithm 2. As in Algorithm 1, any constants $0 < \theta_1 < \theta_2 < 1$ could be used in place of the numerical constants 0.25 and 0.75. If, for some k, $\phi_k(w^k) = 0$ then it is easy to see that $\nabla h(z^k) = 0$; in this case, having computed a critical point z^k of $h(\cdot)$, Algorithm 2 terminates at the kth iteration. From now on, we assume that $\phi_k(w^k) < 0$, and hence $w^k \neq 0$, for all $k \geqslant 0$. At this point no restriction is imposed on the way the sequence of matrices $\{B_k\}$ and $\{D_k\}$ are updated; however, to obtain meaningful results about the behavior of Algorithm 2, some conditions on $\{B_k\}$ and $\{D_k\}$ will be needed (see (21) and (23)). As in Algorithm 1, we say that the kth iteration is *successful* if $\rho_k > 0.25$, and *unsuccessful*, otherwise. Finally, we note that $\{h(z^k)\}$ is a non-increasing sequence.

A complete convergence analysis of Algorithm 2 in the context of unconstrained minimization problems can be found in Moré [30]. Here we are only interested in some of the technical results stated in Ref. [30]. For the purpose of future reference, we next state these results here. We start with Lemma 4.8 of Ref. [30].

Lemma 3.1. *If w^k satisfies* (19), *then*

$$-\phi_k(w^k) \geqslant \tfrac{1}{2}\gamma_1 \|D_k^{-T}\nabla h(z^k)\| \min\left\{\beta_k, \frac{\|D_k^{-T}\nabla h(z^k)\|}{\|D_k^{-T}B_k D_k^{-1}\|}\right\}.$$

The following condition is used in the statement of the next two results: there exist constants $\sigma_1 > 0$ and $\sigma_2 > 0$ such that

$$\|D_k^{-T}B_k D_k^{-1}\| \leqslant \sigma_1, \quad \|D_k^{-1}\| \leqslant \sigma_2, \quad \forall k \geqslant 0. \tag{21}$$

The next result is Theorem 4.10 of Ref. [30].

Theorem 3.2. *Let $\{z^k\}$ be the sequence generated by Algorithm 2. Assume that condition* (21) *holds and that $\{h(z^k)\}$ is bounded below. Then,*

$$\liminf_{k\to\infty} \|D_k^{-T}\nabla h(z^k)\| = 0. \tag{22}$$

The application of the above result to the convergence analysis of Algorithm 1 would lead to the conclusion that $\liminf_{k\to\infty} \|X_k s(x^k)\| = 0$. This result alone is not sufficient to prove convergence of Algorithm 1. What is really needed is the stronger result that $\lim_{k\to\infty} \|X_k s(x^k)\| = 0$, which will be obtained by means of the result stated below. It is a modification of Theorem 4.14 of Ref. [30] in the sense that instead of assuming the condition: there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|z^k - z^m\| \leqslant \delta_1 \text{ for } m \leqslant k \leqslant l \implies \|D_m - D_l\| \leqslant \delta_2,$$

the following condition is assumed: as m and l tend to ∞, we have

$$\|z^m - z^l\| \to 0 \implies \|D_m^{-1}D_m^{-T} - D_l^{-1}D_l^{-T}\| \to 0. \tag{23}$$

Although the proof of the next result is a slight modification of the one given for Theorem 4.14 of Ref. [30], we include it here for the sake of completeness.

Theorem 3.3. *Let $\{z^k\}$ be the sequence generated by Algorithm 2. Assume that* (21) *and* (23) *hold and that $\{z^k\}$ is bounded. Then*

$$\lim_{k\to\infty} \|D_k^{-T}\nabla h(z^k)\| = 0. \tag{24}$$

Proof. Assume for contradiction that (24) does not hold. Then there exists a constant $\varepsilon_1 > 0$ such that the index set $\mathcal{K} \equiv \{k \mid \|D_k^{-T}\nabla h(z^k)\| \geqslant \varepsilon_1\}$ is infinite. In view of Theorem 3.2, for any given $\varepsilon_2 \in (0, \varepsilon_1)$, it is easy to construct two index sequences $\{m_i\} \subseteq \mathcal{K}$ and $\{l_i\}$ such that for all i, $m_i < l_i < m_{i+1}$,

$$\|D_{l_i}^{-T}\nabla h(z^{l_i})\| < \varepsilon_2, \text{ and } \|D_k^{-T}\nabla h(z^k)\| \geqslant \varepsilon_2, \text{ for all } k = m_i, \ldots, l_i - 1. \tag{25}$$

Using the fact that z^{k+1} is either equal to z^k or $z^k + w^k$ and relations (19) and (21), we obtain

$$\|z^{k+1} - z^k\| \leqslant \|D_k^{-1}\|\|D_k(z^{k+1} - z^k)\| \leqslant \sigma_2\gamma_2\beta_k, \qquad \forall k \geqslant 0. \tag{26}$$

Using Lemma 3.1, relations (20), (21), (25), (26) and the fact that $\rho_k > 0.25$ if the kth iteration is successful, we conclude that if $m_i \leqslant k < l_i$ and the k-th iteration is successful then

$$\begin{aligned} h(z^k) - h(z^{k+1}) &\geqslant 0.25(-\phi_k(w^k)) \geqslant \tfrac{1}{8}\gamma_1\varepsilon_2 \min\left\{\beta_k, \frac{\varepsilon_2}{\sigma_1}\right\} \\ &\geqslant \tfrac{1}{8}\gamma_1\varepsilon_2 \min\left\{\frac{\|z^{k+1} - z^k\|}{\sigma_2\gamma_2}, \frac{\varepsilon_2}{\sigma_1}\right\}. \end{aligned} \tag{27}$$

Since $\{h(z^k)\}$ is a nonincreasing sequence, h is continuous and the sequence $\{z^k\}$ is bounded, we have $\lim_{k\to\infty} h(z^k) - h(z^{k+1}) = 0$. In view of (27), this implies the existence of some $i_0 \geqslant 0$ such that

$$h(z^k) - h(z^{k+1}) \geqslant \frac{\gamma_1 \varepsilon_2}{8\sigma_2\gamma_2}\|z^{k+1} - z^k\|, \quad \forall k \text{ successful}, \ m_i \leqslant k < l_i \text{ and } i \geqslant i_0.$$

Since the above relation holds trivially for all unsuccessful iterations, it follows by the triangle inequality that

$$h(z^{m_i}) - h(z^{l_i}) \geqslant \frac{\gamma_1 \varepsilon_2}{8\sigma_2\gamma_2}\|z^{m_i} - z^{l_i}\|, \qquad \forall i \geqslant i_0.$$

This clearly implies that $\lim_{i\to\infty} \|z^{m_i} - z^{l_i}\| = 0$. Using this relation, the boundedness of $\{z^k\}$, the continuity of ∇h and (23), we conclude that, for some $i_1 \geqslant i_0$,

$$\|\nabla h(z^{m_i}) - \nabla h(z^{l_i})\| \leqslant \varepsilon_2^2 \ \text{ and } \ \|D_{m_i}^{-1}D_{m_i}^{-T} - D_{l_i}^{-1}D_{l_i}^{-T}\| \leqslant \varepsilon_2^2, \quad \forall i \geqslant i_1. \quad (28)$$

The assumption that $\{z^k\}$ is bounded implies the existence of a constant $\sigma_3 > 0$ such that $\|\nabla h(z^k)\| \leqslant \sigma_3$ for all k. This together with (21), (25) and (28) imply

$$\begin{aligned}\|D_{m_i}^{-T}\nabla h(z^{m_i})\|^2 &= \nabla h(z^{m_i})^{\mathrm{T}}\left(D_{m_i}^{-1}D_{m_i}^{-T} - D_{l_i}^{-1}D_{l_i}^{-T}\right)\nabla h(z^{m_i}) \\ &\quad + \nabla h(z^{l_i})^{\mathrm{T}} D_{l_i}^{-1}D_{l_i}^{-T}\nabla h(z^{l_i}) \\ &\quad + \left(\nabla h(z^{m_i}) + \nabla h(z^{l_i})\right)^{\mathrm{T}} D_{l_i}^{-1}D_{l_i}^{-T}\left(\nabla h(z^{m_i}) - \nabla h(z^{l_i})\right) \\ &\leqslant (\sigma_3^2 + 1 + 2\sigma_3\sigma_2^2)\varepsilon_2^2.\end{aligned}$$

Noting that $\|D_{m_i}^{-T}\nabla h(z^{m_i})\|^2 \geqslant \varepsilon_1^2$ due to the fact that $\{m_i\} \subseteq \mathcal{K}$, we conclude that

$$\varepsilon_1^2 \leqslant (\sigma_3^2 + 1 + 2\sigma_3\sigma_2^2)\varepsilon_2^2$$

for any $\varepsilon_2 \in (0, \varepsilon_1)$, a contradiction. □

3.2. Convergence of the complementarity product

In this section we show that Algorithm 1 can be recast as a special case of Algorithm 2 by means of a suitable change of coordinates. Using the analysis of the previous subsection, we then obtain the main result of this section, namely $\lim_{k\to\infty} \|X_k s(x^k)\| = 0$, where $X_k \equiv \operatorname{diag}(x^k)$ and $s(\cdot)$ is defined in (17). We note that Assumptions 3 and 4 are not used to derive this result.

Throughout this subsection and the next one, we make the following assumptions on the sequence of matrices $\{Q_k\}$. The first one is all we need for deriving the main result of this subsection; the other assumption will be used in the next subsection, where we provide the complete convergence analysis of Algorithm 1.

Assumption 5. *There exists $M > 0$ such that $|d^{\mathrm{T}}Q_kd| \leqslant M\|d\|^2$ for all $d \in Null(A)$ and $k \geqslant 0$.*

Assumption 6. *Either one of the following conditions holds:*
(a) $d^TQ_kd \geqslant 0$ *for all* $d \in Null(A)$ *and* $k \geqslant 0$, *or;*
(b) $d^TQ_kd \leqslant 0$ *for all* $d \in Null(A)$ *and* $k \geqslant 0$.

Observe that if $Q_k = \nabla^2 f(x^k)$ for all $k \geqslant 0$ then both Assumptions 5 and 6 are automatically implied by Assumptions 2 and 3 of Section 2.

Let $H \in \mathbb{R}^{n\times(n-m)}$ be a matrix whose columns form a basis for the null space of A, and define the mapping $\Pi : \mathbb{R}^{n-m} \to \mathrm{Null}(A)$ by $\Pi(z) = Hz$. Obviously, Π is an isomorphism. Let $x^0 \in \mathcal{P}^0$ be the point as in Assumption 2. Consider the function $h : \mathbb{R}^{n-m} \to \mathbb{R}$ given by

$$h(z) \equiv f(x^0 + Hz), \quad \forall z \in \mathbb{R}^{n-m}, \tag{29}$$

and define the sequences of $(n-m) \times (n-m)$-matrices $\{B_k\}$ and $\{D_k\}$ for all $k \geqslant 0$ as

$$B_k \equiv H^TQ_kH, \qquad D_k \equiv (H^TX_k^{-2}H)^{1/2}. \tag{30}$$

We have the following straightforward result whose proof is left to the reader.

Proposition 3.4. *Assumption* 5 *holds if and only if the sequence* $\{B^k\} = \{H^TQ_kH\}$ *is bounded. Moreover, Assumption* 6(a) (*respectively* 6(b)) *holds if and only if* $B_k \geqslant 0$ (*respectively* $B_k \leqslant 0$) *for all* $k \geqslant 0$.

We next show that Algorithm 1 can be recast as a special case of Algorithm 2. We start by pointing out the relationship between the approximate solutions of the trust region subproblems (2) and (18).

Proposition 3.5. *Let the point* $x^k \in \mathcal{P}^0$ *be given and define* $z^k = \Pi^{-1}(x^k - x^0)$. *If* d^k *is an approximate solution of* (2) *according to* (3), *then* $w^k = \Pi^{-1}(d^k)$ *is an approximate solution of* (18) (*with* B_k *and* D_k *given by* (30)) *according to* (19) *with* $\gamma_1 = (1-\sigma)^2$ *and* $\gamma_2 = 1 + \sigma$.

Proof. Using the fact that $w^k = \Pi^{-1}(d^k)$ and d^k is an approximate solution of (2) according to (3), it is easy to verify that

$$w^k \in \arg\min\{\phi_k(w) \mid \|D_kw\| \leqslant \tilde{\beta}_k\}, \text{ where } \tilde{\beta}_k \in [(1-\sigma)\beta_k, (1+\sigma)\beta_k]. \tag{31}$$

The result now follows from Lemma 3.13 in Moré and Sorensen [31] which states that (31) implies that w^k satisfies (19) with $\gamma_1 = (1-\sigma)^2$ and $\gamma_2 = 1 + \sigma$. □

The following result follows as an immediate consequence of Proposition 3.5.

Proposition 3.6. *Let* $\{x^k\}$ *denote the sequence generated by Algorithm 1 and consider the sequence* $\{z^k\}$ *defined by* $z^k = \Pi^{-1}(x^k - x^0)$ *for all* $k \geqslant 0$. *Then Algorithm 2, applied to the function h defined in* (29), *with initial point* $z^0 = 0$, *and with the* *sequences* $\{D_k\}$ *and* $\{B_k\}$ *given by* (30), *generates the sequence* $\{z^k\}$ *whenever the sequence of approximate solutions* $\{w^k\}$ *of subproblem* (18) *are chosen according to* $w^k = \Pi^{-1}(d^k)$ *for all* $k \geqslant 0$, *where* $\{d^k\}$ *is the sequence of approximate solutions of subproblem* (2) *generated by Algorithm 1.*

From now on, we let $\{z^k\}$ denote the sequence defined as

$$z^k = \Pi^{-1}(x^k - x^0) = (H^{\mathrm{T}}H)^{-1}H^{\mathrm{T}}(x^k - x^0), \qquad \forall k \geqslant 0. \tag{32}$$

Our next goal is to show that the sequences $\{B_k\}$, $\{D_k\}$ and $\{z^k\}$ satisfy conditions (21) and (23), and hence, the hypothesis of Theorem 3.3. Condition (21) turns out to hold under a mild condition on the sequence of matrices $\{Q_k\}$, namely, that the sequence $\{H^{\mathrm{T}}Q_kH\}$ be bounded.

Lemma 3.7. *Suppose Assumption 5 holds. Then, there exist constants* $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *such that*

$$\|D_k^{-T}B_kD_k^{-1}\| \leqslant \sigma_1, \text{ and } \|D_k^{-1}\| \leqslant \sigma_2, \quad \forall k \geqslant 0.$$

Proof. In view of Proposition 3.4, Assumption 5 implies that $\{B_k\}$ is bounded. Hence, it suffices to show that $\{D_k^{-1}\}$ is bounded. Letting $P(x)$ denote the projection matrix onto Null(AX) and noting that Null(AX_k) = Range$(X_k^{-1}H)$, we have

$$P(x^k) = X_k^{-1}H(H^{\mathrm{T}}X_k^{-2}H)^{-1}H^{\mathrm{T}}X_k^{-1}. \tag{33}$$

From (30), (33) and some simple matrix manipulation, we obtain

$$D_k^{-2} = (H^{\mathrm{T}}X_k^{-2}H)^{-1} = (H^{\mathrm{T}}H)^{-1}H^{\mathrm{T}}X_kP(x^k)X_kH(H^{\mathrm{T}}H)^{-1}. \tag{34}$$

The result now follows from the fact that sequences $\{P(x^k)\}$ and $\{x^k\}$ are bounded. □

We observe that the condition that the sequence $\{H^{\mathrm{T}}Q_kH\}$ be bounded is independent of the choice of H, and hence, it is a property of the sequence $\{Q_k\}$ alone.

In the next four lemmas we show that the sequences $\{D_k\}$ and $\{z^k\}$ defined in (30) and (32), respectively, satisfy condition (23). We note that if Assumption 4 holds, then this fact is easily proved using the fact that $P(x)$ is a continuous function over the nonnegative orthant $\mathbb{R}^n_+$. (Recall that $P(x)$ denotes the projection matrix onto the null space of AX.) For the sake of generality, we prove this result without using Assumption 4.

For the purpose of stating the first lemma, we introduce the following notation. For a partition (B, N) of $\{1, \dots, n\}$, we let $\tilde{P}_B(x_B)$ denote the projection matrix onto the null space of A_BX_B and $\tilde{P}_N(x_N)$ denote the projection matrix onto the subspace $\{p_N \mid A_NX_Np_N \in \text{Range}(A_B)\}$, where $X_B = \text{diag}(x_B)$ and $X_N = \text{diag}(x_N)$. The following result is due to Tsuchiya [39] and a simplified proof can be found in Monteiro and Tsuchiya [25].

Lemma 3.8. *Let (B,N) be a partition of $\{1,\dots,n\}$. Then, for every $x \in \mathbb{R}^n_{++}$, there holds*

$$P(x) = \begin{pmatrix} \tilde{P}_N(x_N) & 0 \\ 0 & \tilde{P}_B(x_B) \end{pmatrix} + \triangle P,$$

where $\|\triangle P\| = \mathcal{O}(\|x_N\|\|x_B^{-1}\|)$.

Proposition 3.9. *Let $C \subseteq \mathbb{R}^l$ be a nonempty set and let $m : \mathrm{cl}\, C \to \mathbb{R}^p$ be a function such that m restricted to C is continuous. Assume also that for every $c \in \mathrm{cl}\, C \setminus C$ and every sequence $\{c^k\} \subseteq C$ converging to c there holds $\lim_{k\to\infty} m(c^k) = m(c)$. Then, m is continuous.*

Proof. To prove continuity of m, let $\{c^k\} \subseteq \mathrm{cl}\ C$ be a sequence converging to c. We will show that $\lim_{k\to\infty} m(c^k) = m(c)$. We may assume that either $\{c^k\} \subseteq C$ or $\{c^k\} \subseteq \mathrm{cl}\ C \setminus C$. If $\{c^k\} \subseteq C$ then $\lim_{k\to\infty} m(c^k) = m(c)$ follows easily from the assumption that m restricted to C is continuous or from the other limiting assumption. So, assume now that $\{c^k\} \subseteq \mathrm{cl}\ C \setminus C$. It is easy to show the existence of a sequence $\{b^k\} \subseteq C$ such that $\|b^k - c^k\| \leqslant 1/k$ and $\|m(c^k) - m(b^k)\| \leqslant 1/k$ for all $k \geqslant 0$. Clearly, this implies that $\lim_{k\to\infty} b^k = c$. Hence, by the first case, we conclude that $\lim_{k\to\infty} m(b^k) = m(c)$. This together with the fact that $\|m(c^k) - m(b^k)\| \leqslant 1/k$ for all k implies that $\lim_{k\to\infty} m(c^k) = m(c)$. We have thus shown that m is continuous. □

Lemma 3.10. *The sequences $\{D_k\}$ and $\{z^k\}$ defined in* (30) *and* (32) *satisfy condition* (23).

Proof. We first show that the matrix-valued function $x \mapsto P(x)X$ is continuous on the nonnegative orthant $\mathbb{R}^n_+$, where $X \equiv \mathrm{diag}\,(x)$. It is sufficient to show that the assumptions of Proposition 3.9 is satisfied for the set $C = \mathbb{R}^n_{++}$ and the function $m(x) = P(x)X$ for all $x \in \mathbb{R}^n_+$. It is obvious that m restricted to $\mathbb{R}^n_{++}$ is continuous since the function $x \mapsto P(x)$ is obviously continuous on $\mathbb{R}^n_{++}$. We now verify the other limiting assumption of Proposition 3.9. Let $\bar{x} \in \mathbb{R}^n_+ \setminus \mathbb{R}^n_{++}$ be given and let $\{x^k\} \subseteq \mathbb{R}^n_{++}$ be an arbitrary sequence converging to $\bar{x}$. Let $B = \{i \mid \bar{x}_i > 0\}$ and $N = \{i \mid \bar{x}_i = 0\}$. It is easy to see that

$$m(\bar{x}) = \begin{pmatrix} \tilde{P}_B(\bar{x}_B)\bar{X}_B & 0 \\ 0 & 0 \end{pmatrix}.$$

Using Lemma 3.8 and the fact that $\|\tilde{P}_N(x_N)\| \leqslant 1$ for all $x_N > 0$ and that $x_B \mapsto \tilde{P}_B(x_B)$ is a continuous function for $x_B > 0$, it is now easy to see that $\lim_{k\to\infty} m(x^k) = m(\bar{x})$.

In view of (34) and the above observation, we see that D_k^{-2}, viewed as a function of x^k, varies continuously with $x^k \in \mathbb{R}^n_+$. Hence, since $\{x^k\}$ is bounded, it follows that as m and l tend to ∞, we have

$$\|x^m - x^l\| \to 0 \implies \|D_m^{-2} - D_l^{-2}\| \to 0. \tag{35}$$

Condition (23) now follows from the above implication by noting that, as m and l tend to ∞, $\|x^m - x^l\| = \|H(z^m - z^l)\| \to 0$ if and only if $\|z^m - z^l\| \to 0$. □

We are now ready to use Theorem 3.3 to obtain the conclusion that $\lim_{k\to\infty} \|X_k s(x^k)\| = 0$. We observe that the result below does not require Assumptions 3 and 4 to hold.

Theorem 3.11. *Suppose Assumption 5 holds and let $\{x^k\}$ be the sequence generated by Algorithm 1. Then,*

$$\lim_{k\to\infty} \|X_k s(x^k)\| = 0.$$

Proof. Consider the sequences $\{B_k\}$, $\{D_k\}$ and $\{z^k\}$ defined in (30) and (32). In view of Proposition 3.6, Lemma 3.7, Lemma 3.10 and Theorem 3.3, we have $\lim_{k\to\infty} \|D_k^{-1}\nabla h(z^k)\|^2 = 0$. Using the fact that, due to (29), $\nabla h(w) = H^{\mathrm{T}}\nabla f(x)$ whenever $x = Hw + x^0$ and relation (30), we conclude

$$\lim_{k\to\infty} \nabla f(x^k)^{\mathrm{T}} H(H^{\mathrm{T}} X_k^{-2} H)^{-1} H^{\mathrm{T}} \nabla f(x^k) = 0,$$

which, in view of (17) and (33), is equivalent to

$$\begin{aligned}\lim_{k\to\infty} \|X_k s(x^k)\|^2 &= \lim_{k\to\infty} \|P(x^k) X_k \nabla f(x^k)\|^2 \\ &= \lim_{k\to\infty} \nabla f(x^k)^{\mathrm{T}} X_k P(x^k) X_k \nabla f(x^k) = 0. \qquad \square\end{aligned}$$

3.3. Convergence analysis of algorithm 1

The purpose of this subsection is to complete the convergence analysis of Algorithm 1. The main result of this subsection (namely, Theorem 3.14) states that, under Assumptions 3, 4, 5 and 6, every accumulation point of the sequence $\{x^k\}$ generated by Algorithm 1 satisfies the first order necessary condition for optimality of problem (1); in particular, if f is convex, we obtain the result that every accumulation point of $\{x^k\}$ is an optimal solution of (1).

The following notation is used throughout this subsection. Let $\bar{x}$ be an accumulation point of $\{x^k\}$ and let $\bar{s} \equiv s(\bar{x})$. (Here, we are assuming that Assumption 4 holds.) We define the index sets $N \equiv \{i \mid \bar{s}_i \neq 0\}$ and $B \equiv \{1, \ldots, n\} \setminus N$.

The arguments used in the proof of the following result are similar to the ones used in Gonzaga and Carlos [14]. For the sake of completeness, its proof is given in the Appendix A.

Lemma 3.12. *Suppose Assumptions 3, 4 and 5 hold. Then* $\lim_{k\to\infty} s(x^k) = \bar{s}$.

We point out that Lemma 3.12 does not require Assumption 6; this assumption is used for the first time in the next result. Recall that $\mathcal{S}$ denotes the index set of all successful iterations of Algorithm 1.

Lemma 3.13. *Suppose Assumptions 3, 4, 5 and 6 hold. Then* $\lim_{k\in\mathcal{S}} s_+^k = \bar{s}$.

Proof. By Lemma 3.12, it suffices to show that $\lim_{k\in\mathcal{S}}[s_+^k - s(x^k)] = 0$. By (14) and (17), we have

$$s_+^k - s(x^k) = \left(I - A^{\mathrm{T}}(AX_k^2A^{\mathrm{T}})^{-1}AX_k^2\right)Q_kd^k, \qquad \forall k \geqslant 0.$$

Since the matrix $I - A^{\mathrm{T}}(AX_k^2A^{\mathrm{T}})^{-1}AX_k^2$ maps every vector in Range(A^{T}) into the zero vector, the limit above will follow if we show that the projection of the vector Q_kd^k onto Null(A) converges to 0 as $k \in \mathcal{S}$ tends to ∞, or equivalently,

$$\lim_{k\in\mathcal{S}} H^{\mathrm{T}}Q_kd^k = 0, \tag{36}$$

where H is any matrix whose columns form a basis for Null(A). Indeed, (a) and (c) of Proposition 2.2 imply

$$f(x^k) - f(x^{k+1}) > -0.25q_k(d^k) = \tfrac{1}{8}(d^k)^{\mathrm{T}}(Q_k + \mu_kX_k^{-2})d^k + \tfrac{1}{8}\mu_k(d^k)^{\mathrm{T}}X_k^{-2}d^k,$$
$$\forall k \in \mathcal{S}.$$

Since, by (d) and (e) of Proposition 2.2, $\lim_{k\in\mathcal{S}} f(x^k) - f(x^{k+1}) = 0$ and both terms on the right hand side of the above inequality are nonnegative due to (12), we conclude that

$$\lim_{k\in\mathcal{S}}(d^k)^{\mathrm{T}}(Q_k + \mu_kX_k^{-2})d^k = 0, \quad \lim_{k\in\mathcal{S}}\mu_k(d^k)^{\mathrm{T}}X_k^{-2}d^k = 0, \tag{37}$$

which immediately yields $\lim_{k\in\mathcal{S}}(d^k)^{\mathrm{T}}Q_kd^k = 0$. Let $\{w^k\}$ be a sequence such that $Hw^k = d^k$ for all $k \geqslant 0$. Then, we have $\lim_{k\in\mathcal{S}}(w^k)^{\mathrm{T}}(H^{\mathrm{T}}Q_kH)w^k = 0$. Assume first that (a) of Assumption 6 holds. Proposition 3.4 and Assumption 5 imply that $\{H^{\mathrm{T}}Q_kH\}$ is a bounded sequence of positive semi-definite matrices. Using these two facts together with $\lim_{k\in\mathcal{S}}(w^k)^{\mathrm{T}}(H^{\mathrm{T}}Q_kH)w^k = 0$, it is easy to see that $\lim_{k\in\mathcal{S}}(H^{\mathrm{T}}Q_kH)w^k = 0$, or equivalently, (36) holds. If (b) of Assumption 6 holds then a similar argument as above, with $H^{\mathrm{T}}Q_kH$ replaced by $-H^{\mathrm{T}}Q_kH$, shows that (36) also holds. □

Theorem 3.14. *Suppose Assumptions 3, 4, 5 and 6 hold. Then every accumulation point of* $\{x^k\}$ *satisfies the first order necessary condition for optimality of* (1). *In particular, if* $f(\cdot)$ *is convex then any accumulation point of* $\{x^k\}$ *is an optimal solution of* (1).

Proof. Since the accumulation point $\bar{x}$ of the sequence $\{x^k\}$ considered at the beginning of this subsection is arbitrary, it suffices to prove that $\bar{x}$ satisfies the first order necessary condition for optimality of (1). Let $\bar{s} \equiv s(\bar{x})$ and let $\bar{y} \equiv (A\bar{X}^2A^{\mathrm{T}})^{-1}A\bar{X}^2\nabla f(\bar{x})$; we will show that $(\bar{y}, \bar{s})$ satisfies (5). By relation (17), we have $\nabla f(\bar{x}) = s(\bar{x}) + A^{\mathrm{T}}\bar{y} = \bar{s} + A^{\mathrm{T}}\bar{y}$. Moreover, continuity of $s(\cdot)$ over $\mathcal{P}$ and Theorem 3.11 imply that $\bar{X}\bar{s} = 0$. We next verify that $\bar{s} \geqslant 0$. Indeed, assume for contradiction that there exists an index $l \in N$ such that $\bar{s}_l < 0$. Then, $\bar{x}_l = 0$ since $\bar{x}_l\bar{s}_l = 0$. By Lemma 3.13, there exists an integer $\bar{k} > 0$ such that $(s_+^k)_l < 0$ for all $k \geqslant \bar{k}$ and $k \in \mathcal{S}$. Hence, in view of (7) and

(11), we have $d_l^k > 0$, and hence, $x_l^{k+1} = x_l^k + d_l^k > x_l^k$ for all $k \geqslant \bar{k}$ and $k \in \mathcal{S}$. Since $x_l^{k+1} = x_l^k$ for all $k \notin \mathcal{S}$, we conclude that $x_l^{k+1} \geqslant x_l^k > 0$ for all $k \geqslant \bar{k}$, a contradiction with the fact that $\bar{x}_l = 0$ is an accumulation point of $\{x_l^k\}$. □

We conclude the section by noting that all the results in Section 3 hold if $f(\cdot)$ is assumed to be continuously differentiable only. The assumption that $f(\cdot)$ is twice continuously differentiable will be fully used in the next section.

4. Additional results for a class of objective functions

In this section, we consider the behavior of Algorithm 1 when $Q_k = \nabla^2 f(x^k)$ for all $k \geqslant 0$ and the objective function satisfies the invariance property that the null space of $\nabla^2 f(x)$ is constant for every $x \in \mathcal{P}$. Under these conditions, we show that the sequence $\{x^k\}$ converges R-linearly to a point satisfying first and second order necessary conditions for optimality of (1) and that the sequence $\{f(x^k)\}$ converges (monotonically and) Q-linearly.

Throughout this section we make the following assumptions.

Assumption 7. *$Q_k \equiv \nabla^2 f(x^k)$ for all $k \geqslant 0$;*

Assumption 8. *$Null(\nabla^2 f(x)) = Null(\nabla^2 f(z))$ for any $x, z \in \mathcal{P}$.*

We observe that Assumptions 2, 3 and 7 together automatically imply Assumptions 5 and 6. Hence, all the results obtained in Section 3 hold under Assumptions 1–4 and Assumption 7.

From now on we denote the constant subspace $\mathrm{Null}(\nabla^2 f(x))$, $x \in \mathcal{P}$, by $\mathcal{N}$ and its orthogonal subspace by $\mathcal{N}^\perp$. Given any vector $d \in \mathbb{R}^n$, we let d_0 and $d_\perp$ denote the orthogonal projections of d onto $\mathcal{N}$ and $\mathcal{N}^\perp$, respectively.

Lemma 4.1. *Suppose Assumptions 3 and 8 hold. Then given any compact set $S \subseteq \mathcal{P}$, there exist constants $\lambda_1 = \lambda_1(S) > 0$ and $\lambda_2 = \lambda_2(S) > 0$ such that $\lambda_1 \|d_\perp\|^2 \leqslant |d^{\mathrm{T}} \nabla^2 f(x) d| \leqslant \lambda_2 \|d_\perp\|^2$ for all $x \in S$ and $d \in \mathbb{R}^n$.*

Proof. By considering $-f$ if f is concave, we may assume that f is convex. Since $d^{\mathrm{T}} \nabla^2 f(x) d = d_\perp^{\mathrm{T}} \nabla^2 f(x) d_\perp$ for all $d \in \mathbb{R}^n$, it is sufficient to show that, for some $\lambda_2 \geqslant \lambda_1 > 0$, we have $\lambda_1 \|u\|^2 \leqslant u^{\mathrm{T}} \nabla^2 f(x) u \leqslant \lambda_2 \|u\|^2$ for all $(x, u) \in S \times \mathcal{N}^\perp$, or equivalently, $\lambda_1 \leqslant u^{\mathrm{T}} \nabla^2 f(x) u \leqslant \lambda_2$ for all $(x, u) \in C \equiv S \times \{u \in \mathcal{N}^\perp \mid \|u\| = 1\}$. But this trivially follows by noting that the set C is compact, the values assumed by the function $(x, u) \in C \mapsto u^{\mathrm{T}} \nabla^2 f(x) u \in \mathbb{R}$ are strictly positive and by using the fact that a continuous function defined on a compact set achieves a minimum and a maximum value. □

The following lemma gives an alternative characterization for a function satisfying Assumptions 3 and 8.

Lemma 4.2. *Let f be a function satisfying Assumption 3. Then, f satisfies Assumption 8 if and only if, for every compact set $S \subseteq \mathcal{P}$, there exists a constant $\kappa = \kappa(S) \geqslant 1$ such that $|d^T \nabla^2 f(x) d| \leqslant \kappa |d^T \nabla^2 f(z) d|$ for all $x, z \in S$ and all $d \in \mathbb{R}^n$.*

Proof. To prove the "if" part, let $x, z \in \mathcal{P}$ be given. Letting $S = \{x, z\}$, we conclude that there exists a constant $\kappa \geqslant 1$ such that $|d^T \nabla^2 f(x) d| \leqslant \kappa |d^T \nabla^2 f(z) d|$ for all $d \in \mathbb{R}^n$. Hence, if $d \in$ Null $(\nabla^2 f(z))$ then $|d^T \nabla^2 f(x) d| \leqslant \kappa |d^T \nabla^2 f(z) d| = 0$, from which we conclude that $d \in$ Null $(\nabla^2 f(x))$, due to Assumption 3. Hence, Null $(\nabla^2 f(z)) \subseteq$ Null $(\nabla^2 f(x))$. Since this inclusion holds for every $x, z \in \mathcal{P}$, we have in fact that Null $(\nabla^2 f(z))$ = Null $(\nabla^2 f(x))$ for every $x, z \in \mathcal{P}$, i.e. Assumption 8 holds. The "only if" part is an immediate consequence of Lemma 4.1: if $\lambda_1 = \lambda_1(S)$ and $\lambda_2 = \lambda_2(S)$ are as in Lemma 4.1 then $\kappa \equiv \lambda_2/\lambda_1 \geqslant 1$ is easily seen to satisfy the condition of this lemma. □

The class of functions satisfying the alternative condition of Lemma 4.2 has already been considered in Sun [36]. It is observed in his paper that any convex quadratic function or any convex function having positive definite Hessian everywhere satisfies this condition. More generally, it is easily seen that any function of the form $f(x) = u(Ex) + c^T x$, where $E \in \mathbb{R}^{l \times n}$, $c \in \mathbb{R}^n$ and $u : \mathbb{R}^l \to \mathbb{R}$ is a twice continuously differentiable function such that $\nabla^2 u(y) > 0$ (or, $\nabla^2 u(y) < 0$) for any $y \in \mathbb{R}^l$, also satisfies Assumption 8, and hence the conclusion of Lemma 4.2. Conversely, in Appendix B, we give a partial characterization for the reverse implication (see Lemma B.3). As a special case, we show that if Null$(\nabla^2 f(x))$ is constant for every $x \in \mathbb{R}^n$ then f has the form mentioned above.

Lemma 4.3. *Suppose Assumption 7 hold. Then d^k is an optimal solution to the problem*

$$\begin{array}{ll} \textit{minimize} & \psi_k(d) \equiv \nabla f(x^k)^T d + \frac{1}{2} d^T \nabla^2 f(x^k) d + \frac{1}{2}\mu_k \|X_k^{-1} d\|^2 \\ \textit{subject to} & Ad = 0, \end{array} \tag{38}$$

and $\psi_k(d)$ is convex in Null(A).

Proof. Follows immediately from relations (7), (8) and (12) and the fact that, by Assumption 7, $Q_k = \nabla^2 f(x^k)$, and hence $\nabla q_k(d) = \nabla f(x^k) + \nabla^2 f(x^k) d$, for all $d \in \mathbb{R}^n$. □

Lemma 4.4. *Suppose Assumptions 3, 7 and 8 hold. Then there exists a constant $\lambda_1 > 0$ such that for all $k \geqslant 0$,*

$$\lambda_1 \|d_\perp^k\|^2 \leqslant \tfrac{1}{2} \left|(d^k)^T \nabla^2 f(x^k) d^k\right| \leqslant -q_k(d^k). \tag{39}$$

Proof. Since by Assumption 2, $\mathcal{L}(x^0)$ is compact, Lemma 4.1 implies the existence of a constant $\lambda_1 > 0$ such that

$$\tfrac{1}{2} \left|d^T \nabla^2 f(x) d\right| \geqslant \lambda_1 \|d_\perp\|^2, \quad \forall d \in \mathbb{R}^n, \forall x \in \mathcal{L}(x^0). \tag{40}$$

By Assumption 3, $f(\cdot)$ is either convex or concave. If $f(\cdot)$ is convex then (a) and (e) of Proposition 2.2, Assumption 7 and (40) imply

$$\begin{aligned}-q_k(d^k) &= \tfrac{1}{2}(d^k)^{\mathrm{T}}\nabla^2 f(x^k)d^k + \mu_k(d^k)^{\mathrm{T}}X_k^{-2}d^k \geqslant \tfrac{1}{2}\left|(d^k)^{\mathrm{T}}\nabla^2 f(x^k)d^k\right| \\ &\geqslant \lambda_1\|d_\perp^k\|.\end{aligned}$$

If $f(\cdot)$ is concave then (a) and (e) of Proposition 2.2, Assumption 7, (12) and (40) imply

$$\begin{aligned}-q_k(d^k) &= -\tfrac{1}{2}(d^k)^{\mathrm{T}}\nabla^2 f(x^k)d^k + (d^k)^{\mathrm{T}}\left(\nabla^2 f(x^k) + \mu_k X_k^{-2}\right)d^k \\ &\geqslant \tfrac{1}{2}\left|(d^k)^{\mathrm{T}}\nabla^2 f(x^k)d^k\right| \geqslant \lambda_1\|d_\perp^k\|^2. \qquad \square\end{aligned}$$

Lemma 4.5. *Suppose Assumption 8 hold and let $\bar{x} \in \mathcal{P}$ be fixed. Then for any $x \in \mathcal{P}$ and any $d \in \mathcal{N}$, there holds*

$$\nabla f(x)^{\mathrm{T}}d = \nabla f(\bar{x})^{\mathrm{T}}d.$$

Proof. Let $x \in \mathcal{P}$ and $d \in \mathcal{N}$ be given and define the function $F(\theta) \equiv \nabla f(\bar{x} + \theta(x - \bar{x}))^{\mathrm{T}}d$. Clearly, $F(1) = \nabla f(x)^{\mathrm{T}}d$ and $F(0) = \nabla f(\bar{x})^{\mathrm{T}}d$. Applying the mean value theorem to $F(\theta)$ and using Assumption 8 and the fact that $d \in \mathcal{N}$, we obtain

$$\nabla f(x)^{\mathrm{T}}d - \nabla f(\bar{x})^{\mathrm{T}}d = (x - \bar{x})^{\mathrm{T}}\nabla^2 f(z)d = 0,$$

where z is a point lying in the line segment between x and $\bar{x}$. $\square$

The proof of the following lemma can be found in Monteiro and Wright [27]. It unifies Theorem 2.5 and Lemma A.1 of Monteiro, Tsuchiya and Wang [26], which in turn are based on Theorem 2 of Tseng and Luo [37].

Lemma 4.6. *Let $c \in \mathbb{R}^q$ and $G \in \mathbb{R}^{p\times q}$ be given. Then there exists a nonnegative constant $L = L(c, G)$ with the property that for any diagonal matrix $D > 0$ and any vector $g \in Range(G)$, the (unique) optimal solution $\bar{w} = \bar{w}(c, G)$ of*

$$\min_w c^{\mathrm{T}}w + \tfrac{1}{2}\|Dw\|^2, \quad \text{subject to } Gw = g,$$

satisfies

$$\|\bar{w}\| \leqslant L\left(|c^{\mathrm{T}}\bar{w}| + \|g\|_\infty\right).$$

The next result is due to Sun (see Theorem 2 of Ref. [35]). The proof given below, based on Lemma 4.6, is however simpler than its original proof.

Lemma 4.7. *Suppose that Assumptions 3, 7 and 8 hold. Then $\|d^k\|^2 = \mathcal{O}\left(-q_k(d^k)\right)$.*

Proof. Let E be a matrix such that $\mathrm{Null}(E) = \mathcal{N}$. In view of Lemma 4.3 and the fact that $Ed^k = Ed_0^k + Ed_\perp^k = Ed_\perp^k$, we have

$$d^k = \arg\min\{\nabla f(x^k)^{\mathrm{T}} d + \tfrac{1}{2} d^{\mathrm{T}} \nabla^2 f(x^k) d + \tfrac{1}{2}\mu_k \|X_k^{-1} d\|^2 \mid Ad = 0,\ Ed = Ed_\perp^k\}. \tag{41}$$

The objective function of (41) can be simplified as follows. Assume that d is a feasible solution of (41). Then $d - d_\perp^k \in \mathrm{Null}(E) = \mathcal{N}$. Using Lemma 4.5 and the definition of $\mathcal{N}$, it is easy to see that

$$\begin{aligned} d^{\mathrm{T}} \nabla^2 f(x^k) d &= (d_\perp^k)^{\mathrm{T}} \nabla^2 f(x^k) d_\perp^k, \\ \nabla f(x^k)^{\mathrm{T}} d &= \nabla f(\bar{x})^{\mathrm{T}} d - \nabla f(\bar{x})^{\mathrm{T}} d_\perp^k + \nabla f(x^k)^{\mathrm{T}} d_\perp^k. \end{aligned} \tag{42}$$

Since the quantities $(d_\perp^k)^{\mathrm{T}} \nabla^2 f(x^k) d_\perp^k$, $\nabla f(\bar{x})^{\mathrm{T}} d_\perp^k$ and $\nabla f(x^k)^{\mathrm{T}} d_\perp^k$ do not depend on d, it follows from (41) and (42) that

$$d^k = \arg\min\{\nabla f(\bar{x})^{\mathrm{T}} d + \tfrac{1}{2}\mu_k \|X_k^{-1} d\|^2 \mid Ad = 0, Ed = Ed_\perp^k\}.$$

From Lemma 4.6, we know that there exists a constant $C_2 > 0$ which only depends on $\nabla f(\bar{x})$, A and E such that

$$\begin{aligned} \|d^k\| &\leqslant C_2(|\nabla f(\bar{x})^{\mathrm{T}} d^k| + \|Ed_\perp^k\|) \\ &\leqslant C_2(|\nabla f(\bar{x})^{\mathrm{T}} d_\perp^k| + |\nabla f(\bar{x})^{\mathrm{T}} d_0^k| + \|Ed_\perp^k\|). \end{aligned} \tag{43}$$

Using Lemma 4.4, Lemma 4.5, the definition of $q_k(\cdot)$ in (2) and the fact that $d^k = d_0^k + d_\perp^k$ and $d_0^k \in \mathcal{N}$, we obtain

$$\begin{aligned} \left|\nabla f(\bar{x})^{\mathrm{T}} d_0^k\right| = \left|\nabla f(x^k)^{\mathrm{T}} d_0^k\right| &= \left|\nabla f(x^k)^{\mathrm{T}} d^k - \nabla f(x^k)^{\mathrm{T}} d_\perp^k\right| \\ &= \left|q_k(d^k) - \tfrac{1}{2}(d^k)^{\mathrm{T}} \nabla^2 f(x^k) d^k - \nabla f(x^k)^{\mathrm{T}} d_\perp^k\right| \\ &\leqslant \left|q_k(d^k)\right| + \left|\tfrac{1}{2}(d^k)^{\mathrm{T}} \nabla^2 f(x^k) d^k\right| + \left|\nabla f(x^k)^{\mathrm{T}} d_\perp^k\right| \\ &\leqslant 2\left(-q_k(d^k)\right) + \|\nabla f(x^k)\| \|d_\perp^k\|. \end{aligned} \tag{44}$$

Since $\{x^k\} \subseteq \mathcal{L}(x^0)$, which by Assumption 2 is a compact set, we conclude that $\{\nabla f(x^k)\}$ is bounded. This observation, relations (43) and (44) and Lemma 4.4 imply

$$\begin{aligned} \|d^k\| &= \mathcal{O}\left(-q_k(d^k)\right) + \mathcal{O}\left(\|d_\perp^k\|\right) = \mathcal{O}\left(-q_k(d^k)\right) + \mathcal{O}\left(-q_k(d^k)\right)^{1/2} \\ &= \mathcal{O}\left(-q_k(d^k)\right)^{1/2}. \quad \square \end{aligned}$$

The next corollary follows immediately from Lemma 4.7.

Corollary 4.8. Suppose Assumptions 3, 7 and 8 hold. Then

$$\|x^{k+1} - x^k\| = \mathcal{O}\left(f(x^k) - f(x^{k+1})\right)^{1/2}.$$

Proof. It follows immediately from Lemma 4.7, Proposition 2.2(c) and the fact that $x^{k+1} = x^k + d^k$ if $k \in \mathcal{S}$ and $x^{k+1} = x^k$ if $k \notin \mathcal{S}$. $\square$

We next prove the geometric convergence of the subsequence $\{f(x^k) - f^\infty\}_{k\in\mathcal{S}}$, where $f^\infty \equiv \lim_{k\to\infty} f(x^k)$. We first state and prove two technical lemmas.

Lemma 4.9. *Let $\mathcal{A} \equiv \{x \in \mathcal{P} \mid f(x) = f^\infty\}$. Then there exists an integer $\bar{k} > 0$ such that*

$$\min_{x \in \mathcal{A}} \|X_k^{-1}(x - x^k)\| \leqslant \sqrt{n}, \ \forall k \geqslant \bar{k}.$$

Proof. Assume for contradiction that there exists a subsequence $\{x^k\}_{k \in K}$ such that

$$\|X_k^{-1}(x - x^k)\| > \sqrt{n}, \quad \forall x \in \mathcal{A}, \quad \forall k \in K. \tag{45}$$

By Proposition 2.2(e), $\{x^k\}_{k \in K}$ is bounded. By passing to a subsequence, we may assume that $\lim_{k \in K} x^k = x^*$. Clearly, $x^* \in \mathcal{A}$. Observe that

$$\lim_{k \in K} \|X_k^{-1}(x^* - x^k)\|^2 = \lim_{k \in K} \sum_{i=1}^n (x_i^*/x_i^k - 1)^2 = |N(x^*)|,$$

where $N(x^*) \equiv \{i \mid x_i^* = 0\}$ and $|N(x^*)|$ denotes the cardinality of $N(x^*)$. Hence, if $|N(x^*)| < n$ then $\|X_k^{-1}(x^* - x^k)\| \leqslant \sqrt{n}$ for all k sufficiently large. Also, if $|N(x^*)| = n$ then $x^* = 0$, and hence, $\|X_k^{-1}(x^* - x^k)\| = \sqrt{n}$ for all $k \geqslant 0$. Since both cases contradict (45) with $x = x^* \in \mathcal{A}$, the result follows. □

The next lemma shows that $\{\beta_k\}$ is bounded away from 0.

Lemma 4.10. *Suppose that Assumptions 3, 7 and 8 hold. Then,* $\liminf_{k \to \infty} \beta_k > 0$.

Proof. Assume for contradiction that $\liminf_{k \to \infty} \beta_k = 0$. Let $K \equiv \{k \in Z_+ \mid \beta_k > \beta_{k+1}\}$. It is easy to show that $\liminf_{k \in K} \beta_{k+1} = 0$, or equivalently, that $\lim_{k \in K_1} \beta_{k+1} = 0$, for some infinite subset $K_1 \subseteq K$. Observing that, by Algorithm 1(b), we have $\beta_k \leqslant 8\beta_{k+1}$ for all $k \geqslant 0$, we conclude that $\lim_{k \in K_1} \beta_k = 0$. Since, by (3),

$$\|d^k\| \leqslant \|X_k\| \|X_k^{-1} d^k\| \leqslant (1 + \sigma) \|X_k\| \beta_k,$$

it follows from the fact that $\lim_{k \in K_1} \beta_k = 0$ and Proposition 2.2(e) that $\lim_{k \in K_1} \beta_k = 0$ that $\lim_{k \in K_1} \|d^k\| = 0$. By Taylor Expansion Theorem, the definition of $q_k(\cdot)$ in (2) and Assumption 7, we have

$$\begin{aligned} f(x^k) - f(x^k + d^k) &= -\nabla f(x^k)^{\mathrm{T}} d^k - (d^k)^{\mathrm{T}} \nabla^2 f(x^k + \theta_k d^k) d^k \\ &= -q_k(d^k) + (d^k)^{\mathrm{T}} \left(\nabla^2 f(x^k) - \nabla^2 f(x^k + \theta_k d^k) \right) d^k, \\ &\qquad \forall k \geqslant 0, \end{aligned}$$

where $\theta_k \in (0, 1)$. This relation, relation (4) and Lemma 4.7 imply

$$\begin{aligned} |r_k - 1| &= \frac{|(d^k)^{\mathrm{T}} \left(\nabla^2 f(x^k) - \nabla^2 f(x^k + \theta_k d^k) \right) d^k|}{-q_k(d^k)} \\ &= \mathcal{O} \left(\|\nabla^2 f(x^k) - \nabla^2 f(x^k + \theta_k d^k)\| \right). \end{aligned}$$

Using the continuity of $\nabla^2 f(\cdot)$, the fact that $\lim_{k \in K_1} \|d^k\| = 0$ and Proposition 2.2(e), it follows from the last expression that $\lim_{k \in K_1} r_k = 1$. However, since $\beta_k > \beta_{k+1}$ for every $k \in K_1$, it follows from Algorithm 1(b) that $r_k < 0.75$ for every $k \in K_1$, a fact that contradicts the conclusion that $\lim_{k \in K_1} r_k = 1$. $\square$

We are now ready to establish the geometric convergence of the subsequence $\{f(x^k) - f^\infty\}_{k \in \mathcal{S}}$.

Lemma 4.11. *Suppose that Assumptions 3, 7 and 8 hold and let* $f^\infty \equiv \lim_{k \to \infty} f(x^k)$. *Then, there exist a constant* $\gamma > 0$ *and an integer* $\bar{k}$ *such that*

$$\frac{f(x^k) - f(x^{k+1})}{f(x^k) - f^\infty} \geqslant \gamma, \quad \forall k \in \mathcal{S}, \ k \geqslant \bar{k}. \tag{46}$$

In particular, the subsequence $\{f(x^k) - f^\infty\}_{k \in \mathcal{S}}$ *converges geometrically, that is, if* $\mathcal{S} = \{k_0 < k_1 < k_2 < \dots\}$ *then* $f(x^{k_{l+1}}) - f^\infty \leqslant (1 - \gamma)(f(x^{k_l}) - f^\infty)$ *for all* $l \geqslant 0$ *sufficiently large.*

Proof. Using Proposition 2.2(c), it suffices to show that $-q_k(d^k) \geqslant \tilde{\gamma}(f(x^k) - f^\infty)$ for some constant $\tilde{\gamma} > 0$ and all $k \geqslant \bar{k}$. Indeed, let $\bar{x}^k \in \mathcal{A} \equiv \{x \in \mathcal{P} \mid f(x) = f^\infty\}$ be such that

$$\bar{x}^k = \arg\min_{x \in \mathcal{A}} \|X_k^{-1}(x - x^k)\|, \tag{47}$$

and define $x_\alpha^k \equiv x^k + \alpha(\bar{x}^k - x^k)$, where $\alpha \in [0, 1]$ is arbitrary. Using the definition of $\psi_k(\cdot)$ in (38), Lemma 4.3 and the fact that $x_\alpha^k - x^k \in \mathrm{Null}(A)$, we obtain

$$\begin{aligned} q_k(d^k) + \frac{\mu_k}{2}\|X_k^{-1}d^k\|^2 &= \psi_k(d^k) \leqslant \psi_k(x_\alpha^k - x^k) \\ &= q_k(x_\alpha^k - x^k) + \frac{\mu_k}{2}\|X_k^{-1}(x_\alpha^k - x^k)\|^2. \end{aligned}$$

By second order Taylor expansion and the fact that $\bar{x}^k \in \mathcal{A}$, we have

$$\begin{aligned} f^\infty - f(x^k) &= f(\bar{x}^k) - f(x^k) \\ &= \nabla f(x^k)^{\mathrm{T}}(\bar{x}^k - x^k) + \tfrac{1}{2}(\bar{x}^k - x^k)^{\mathrm{T}}\nabla^2 f(z^k)(\bar{x}^k - x^k), \end{aligned}$$

for some z^k lying on the line segment between x^k and $\bar{x}^k$. Using the above two relations, relation (9) and the definitions of x_α^k and $q_k(\cdot)$, we obtain

$$\begin{aligned} -q_k(d^k) &\geqslant -q_k(x_\alpha^k - x^k) - \frac{\mu_k}{2}\|X_k^{-1}(x_\alpha^k - x^k)\|^2 + \frac{\mu_k}{2}\|X_k^{-1}d^k\|^2 \\ &= -\alpha\nabla f(x^k)^{\mathrm{T}}(\bar{x}^k - x^k) - \frac{\alpha^2}{2}(\bar{x}^k - x^k)^{\mathrm{T}}\nabla^2 f(x^k)(\bar{x}^k - x^k) \\ &\quad - \frac{\mu_k\alpha^2}{2}\|X_k^{-1}(\bar{x}^k - x^k)\|^2 + \frac{\mu_k}{2}\tilde{\beta}_k^2 \\ &= \alpha\left(f(x^k) - f^\infty\right) + \frac{\alpha}{2}(\bar{x}^k - x^k)^{\mathrm{T}}\left(\nabla^2 f(z^k) - \alpha\nabla^2 f(x^k)\right)(\bar{x}^k - x^k) \\ &\quad - \frac{\mu_k\alpha^2}{2}\|X_k^{-1}(\bar{x}^k - x^k)\|^2 + \frac{\mu_k}{2}\tilde{\beta}_k^2. \end{aligned} \tag{48}$$

Using Lemma 4.2, the compactness of $\mathcal{L}(x^0)$, we conclude the existence of a constant $\kappa \geqslant 1$ such that

$$|d^{\mathrm{T}}\nabla^2 f(x)d| \leqslant \kappa|d^{\mathrm{T}}\nabla^2 f(z)d|, \quad \forall x,z \in \mathcal{L}(x^0), \ \forall d \in \mathrm{Null}(A). \tag{49}$$

We now consider the following two cases separately: f is convex and f is concave. Assume first that f is convex. Let

$$\alpha_k \equiv \min\left\{\frac{1}{\kappa}, \ \frac{\tilde{\beta}_k}{\|X_k^{-1}(\bar{x}^k - x^k)\|}\right\} \leqslant 1. \tag{50}$$

Using (49) and the fact that $\alpha_k \leqslant 1/\kappa$ and $\nabla^2 f(x^k)$ is positive semi-definite, we obtain

$$\begin{aligned}&(\bar{x}^k - x^k)^{\mathrm{T}}(\nabla^2 f(z^k) - \alpha_k \nabla^2 f(x^k))(\bar{x}^k - x^k)\\ &\quad \geqslant (1/\kappa - \alpha_k)(\bar{x}^k - x^k)^{\mathrm{T}}\nabla^2 f(x^k)(\bar{x}^k - x^k) \geqslant 0.\end{aligned}$$

This relation, relation (48) with $\alpha = \alpha_k$ and relation (50) imply

$$-q_k(d^k) \geqslant \alpha_k\left(f(x^k) - f^\infty\right).$$

We next consider the case where f is concave. Let

$$\alpha_k \equiv \min\left\{1, \ \frac{\tilde{\beta}_k^2}{(\kappa+1)\|X_k^{-1}(\bar{x}^k - x^k)\|^2}\right\}. \tag{51}$$

Using relations (12), (48) with $\alpha = \alpha_k$, (49) and (51) and the fact that $\alpha_k^2 \leqslant \alpha_k$ and $\nabla^2 f(x^k)$ is negative semi-definite, we obtain

$$\begin{aligned}-q_k(d^k) &\geqslant \alpha_k\left(f(x^k) - f^\infty\right) + \frac{\alpha_k}{2}(\bar{x}^k - x^k)^{\mathrm{T}}\nabla^2 f(z^k)(\bar{x}^k - x^k)\\ &\quad - \frac{\mu_k\alpha_k}{2}\|X_k^{-1}(\bar{x}^k - x^k)\|^2 + \frac{\mu_k}{2}\tilde{\beta}_k^2\\ &\geqslant \alpha_k\left(f(x^k) - f^\infty\right) + \frac{\kappa\alpha_k}{2}(\bar{x}^k - x^k)^{\mathrm{T}}\nabla^2 f(x^k)(\bar{x}^k - x^k) - \frac{\mu_k\tilde{\beta}_k^2}{2(\kappa+1)}\\ &\quad + \frac{\mu_k}{2}\tilde{\beta}_k^2\\ &\geqslant \alpha_k\left(f(x^k) - f^\infty\right) - \frac{\kappa\alpha_k}{2}\mu_k\|X_k^{-1}(\bar{x}^k - x^k)\|^2 + \frac{\kappa\mu_k\tilde{\beta}_k^2}{2(\kappa+1)}\\ &\geqslant \alpha_k\left(f(x^k) - f^\infty\right).\end{aligned}$$

Hence, in both cases, we have shown that $f(x^k) - f(x^{k+1}) \geqslant \alpha_k(f(x^k) - f^\infty)$ for all $k \in \mathcal{S}$. The result now follows by observing that $\liminf_{k\to\infty}\alpha_k > 0$, due to relations (47), (50) and (51) and Lemmas 4.9 and 4.10. □

Theorem 4.12. *Suppose that Assumptions 3, 7 and 8 hold. Then the following statements hold.*

(a) $\lim_{k\to\infty} x^k = x^*$ *for some* $x^* \in \mathcal{P}$*;*

(b) *there exists a constant* $C_3 > 0$ *such that* $\|x^k - x^*\| \leqslant C_3(f(x^k) - f(x^*))^{1/2}$ *for all* $k \geqslant 0$.

Proof. Since $\lim_{k\to\infty} f(x^k) - f^\infty = 0$, (a) and (b) follows if we show the existence of a constant $C_3 > 0$ such that $\|x^k - x^m\| \leqslant C_3(f(x^k) - f^\infty)^{1/2}$ for any $k \geqslant 0$ sufficiently large and any $m \geqslant k$. Indeed, assume that $k \geqslant \bar{k}$, where $\bar{k}$ is the index mentioned in Lemma 4.11, and let l denote the smallest index in $\mathcal{S}$ such that $l \geqslant k$. By Corollary 4.8, there exists a constant $C > 0$ such that $\|x^k - x^{k+1}\| \leqslant C(f(x^k) - f(x^{k+1}))^{1/2}$ for all $k \geqslant 0$. Using this fact, Lemma 4.11 and the fact that $x^k = x^l$, we obtain for any $m \geqslant k$ that

$$
\begin{aligned}
\|x^k - x^m\| &\leqslant \sum_{t=k}^{\infty} \|x^t - x^{t+1}\| \leqslant C \sum_{t=k}^{\infty} \left(f(x^t) - f(x^{t+1})\right)^{1/2} \\
&= C \sum_{t\in\mathcal{S}, t\geqslant l} \left(f(x^t) - f(x^{t+1})\right)^{1/2} \leqslant C \sum_{t\in\mathcal{S}, t\geqslant l} \left(f(x^t) - f^\infty\right)^{1/2} \\
&\leqslant C \left(\sum_{i=0}^{\infty} (1-\gamma)^{i/2}\right) \left(f(x^l) - f^\infty\right)^{1/2} = C_3 \left(f(x^k) - f^\infty\right)^{1/2}. \qquad \square
\end{aligned}
$$

The next theorem is an immediate consequence of Theorem 3.14 and 4.12. The next result is the main result of this section.

Theorem 4.13. *Suppose that Assumptions 3, 4, 7 and 8 hold. Then the sequence* $\{x^k\}$ *converges to a point satisfying the first and second order necessary conditions for optimality of problem (1). If* $f(\cdot)$ *is convex, then the limit point is an optimal solution of problem (1).*

Proof. In view of Theorems 3.14 and 4.12, we conclude that $\lim_{k\to\infty} x^k = x^*$ and x^* satisfies the first order necessary condition for optimality of problem (1). In particular, when $f(\cdot)$ is convex, x^* is an optimal solution of problem (1). We now prove that x^* satisfies the second order necessary condition for optimality of problem (1), that is, $d^{\mathrm{T}}\nabla^2 f(x^*)d \geqslant 0$ for every $d \in \{d \mid Ad = 0, d_{N^*} = 0\}$, where $N^* \equiv \{i \mid x_i^* = 0\}$. Since this conclusion is obviously true when $f(\cdot)$ is convex, it is enough to show it when $f(\cdot)$ is concave. By Theorem 3.11, Corollary 3.12, Lemma 3.13, we have $\lim_{k\in\mathcal{S}} \|X_k s_+^k\| = 0$. Moreover, by Proposition 2.2(f), Assumption 7, the fact that $\nabla^2 f(x^k) \leqslant 0$ for all $k \geqslant 0$ and relation (3), we have $\|X_k^{-1}d^k\| = \tilde{\beta}_k \geqslant (1-\sigma)\beta_k$. These two observations together with Lemma 4.10 and relation (15) imply that $\lim_{k\in\mathcal{S}} \mu_k = 0$. By letting $k \in \mathcal{S}$ tend to infinity in relation (12), we easily see that $d^{\mathrm{T}}\nabla^2 f(x^*)d \geqslant 0$ for every $d \in \{d \mid Ad = 0, d_{N^*} = 0\}$. $\square$

Appendix A

In this appendix, we provide a proof of Lemma 3.12.

Recall that $\bar{x}$ denotes an accumulation point of $\{x^k\}$, $\bar{s} \equiv s(\bar{x})$, $N \equiv \{i \mid \bar{s}_i \neq 0\}$ and $B \equiv \{1,\ldots,n\} \setminus N$. We define

$$\Omega \equiv \{x \in \mathcal{P} \mid x_N = 0,\ f(x) = f(\bar{x})\}. \tag{A.1}$$

The next five lemmas establish the fact that every accumulation point of $\{x^k\}$ is in Ω.

Lemma A.1. *Suppose Assumptions 3, 4 and 5 hold. Then the set Ω is convex.*

Proof. Let $\bar{y} \equiv (A\bar{X}^2A^{\mathrm{T}})^{-1}A\bar{X}^2\nabla f(\bar{x})$ where $\bar{X} \equiv \operatorname{diag}(\bar{x})$. By (17), we have $\nabla f(\bar{x}) = s(\bar{x}) + A^{\mathrm{T}}\bar{y} = \bar{s} + A^{\mathrm{T}}\bar{y}$, and by the definition of B, we have $\bar{s}_B = 0$. Using these relations and Theorem 3.11, it is easy to see that $\bar{x}$ is an optimal solution of the problem

$$\text{optimize } \{f(x) \mid Ax = b, x_B \geqslant 0,\ x_N = 0\}, \tag{A.2}$$

where "optimize" should be read as "minimize" when $f(\cdot)$ is convex, and, "maximize" when $f(\cdot)$ is concave. Hence, Ω is the set of optimal solutions of (A.2). Since the set of optimal solutions of a minimization (maximization) problem with a convex (concave) objective function over a convex set must be convex, the result follows. □

The next lemma is a well-known result in convex analysis.

Lemma A.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex or concave continuously differentiable function. If f is constant on a convex set $C \subseteq \mathbb{R}^n$, then $\nabla f(\cdot)$ is constant on C.*

Lemma A.3. *Suppose Assumptions 3, 4 and 5 hold. Then $s(x) = \bar{s}$ for every $x \in \Omega$.*

Proof. Let $x \in \Omega$ be given. It then follows from Lemmas A.1 and A.2 and the fact that $f(\cdot)$ is constant on Ω that $\nabla f(x) = \nabla f(\bar{x})$. Hence, letting $\bar{y} \equiv (A\bar{X}^2A^{\mathrm{T}})^{-1}A\bar{X}^2\nabla f(\bar{x})$ and using (17) and the fact that $x_N = 0$, we obtain

$$\begin{aligned} s(x) &= \left[I - A^{\mathrm{T}}(AX^2A^{\mathrm{T}})^{-1}AX^2\right]\nabla f(\bar{x}) \\ &= \left[I - A^{\mathrm{T}}(AX^2A^{\mathrm{T}})^{-1}AX^2\right](\bar{s} + A^{\mathrm{T}}\bar{y}) \\ &= \bar{s} - A^{\mathrm{T}}(AX^2A^{\mathrm{T}})^{-1}AX^2\bar{s} = \bar{s}. \qquad \square \end{aligned}$$

Since $s(\cdot)$ is a continuous function over $\mathcal{P}$ and the level set $\mathcal{L}(x^0)$ is assumed to be bounded (and hence, compact), it follows that $s(\cdot)$ is uniformly continuous over $\mathcal{L}(x^0)$. Hence, there exists a constant $\tau > 0$ such that

$$\|s(x) - s(z)\| \leqslant \min_{i\in N} \frac{|\bar{s}_i|}{2} \quad \text{for all } x, z \in \mathcal{L}(x^0) \text{ such that } \|x - z\| \leqslant \tau. \tag{A.3}$$

Define

$$\Omega_\tau \equiv \{x \in \mathcal{L}(x^0) \mid \|x - \bar{x}\| \leqslant \tau \text{ for some } \bar{x} \in \Omega\}. \tag{A.4}$$

Lemma A.4. *Suppose Assumptions 3, 4 and 5 hold and let $\hat{x} \in \mathcal{P}$ be an accumulation point of $\{x^k\}$. Then either $\hat{x} \in \Omega$ or $\hat{x} \notin \Omega_\tau$.*

Proof. Suppose for contradiction that there exists an accumulation point $\hat{x}$ of $\{x^k\}$ such that $\hat{x} \in \Omega_\tau$ and $\hat{x} \notin \Omega$. Clearly, $\hat{x} \in \mathcal{P}$ and $f(\hat{x}) = f(\bar{x})$. Then, by the definition of Ω and the fact that $\hat{x} \notin \Omega$, there exists an index $l \in N$ such that $\hat{x}_l > 0$. Since $\hat{x} \in \Omega_\tau$, we know that there exists a $\tilde{x} \in \Omega$ such that $\|\hat{x} - \tilde{x}\| \leqslant \tau$. Taking $x = \hat{x}$ and $z = \tilde{x}$ in (A.3) and noting that $s(\tilde{x}) = \bar{s}$, we obtain

$$\|s(\hat{x}) - s(\tilde{x})\| = \|s(\hat{x}) - \bar{s}\| \leqslant \min_{i \in N} \frac{|\bar{s}_i|}{2} \leqslant \tfrac{1}{2}|\bar{s}_l|.$$

Hence, $|s_l(\hat{x}) - \bar{s}_l| \leqslant |\bar{s}_l|/2$, which yields $s_l(\hat{x}) \neq 0$, and hence, $\hat{x}_l s_l(\hat{x}) \neq 0$. By Theorem 3.11, we must have $\hat{x}_j s_j(\hat{x}) = 0$ for all j, contradicting the earlier conclusion that $\hat{x}_l s_l(\hat{x}) \neq 0$. □

Lemma A.5. *Suppose Assumptions 3, 4 and 5 hold. Then any accumulation point of $\{x^k\}$ is in Ω.*

Proof. Assume for contradiction that $\{x^k\}$ has an accumulation point not in Ω. Then, in view of Lemma A.4, this accumulation is not in Ω_τ. Since $\{x^k\}$ has accumulation points both in Ω and outside the closed set Ω_τ, it is easy to see that there exists a subsequence $\{x^k\}_{k \in K}$ such that $x^k \in \Omega_\tau$ and $x^{k+1} \notin \Omega_\tau$ for all $k \in K$. Let $\hat{x}$ and $\tilde{x}$ be accumulation points of $\{x^k\}_{k \in K}$ and $\{x^{k+1}\}_{k \in K}$, respectively. Obviously, $\hat{x} \in \Omega_\tau$ and $\tilde{x} \notin \Omega$. By definition of Ω, we conclude that $\tilde{x}_N \neq 0$. In view of Lemma A.4 and the fact that $\hat{x} \in \Omega_\tau$, we must have $\hat{x} \in \Omega$, and hence, $\hat{x}_N = 0$. Using this fact and letting $k \in K$ tend to ∞ in Proposition 2.2 (b), we obtain $0 \leqslant \tilde{x}_N \leqslant (1 + \bar{\beta})\hat{x}_N = 0$. But this contradicts the earlier conclusion that $\tilde{x}_N \neq 0$. □

We are now ready to prove Lemma 3.12.

Proof of Lemma 3.12. By Assumption 4, we know that $s(x)$ is a continuous function of x over $\mathcal{P}$. This fact together with the boundedness of $\{x^k\}$ implies that the sequence $\{s(x^k)\}$ is bounded. Hence, it suffices to show that $\bar{s}$ is the only accumulation point of $\{s(x^k)\}$. Indeed, let $\hat{s}$ be an accumulation point of $\{s(x^k)\}$. Clearly, there exists an accumulation point $\hat{x}$ of $\{x^k\}$ such that $\hat{s} = s(\hat{x})$. Using Lemma A.5, we conclude that $\hat{x} \in \Omega$. It then follows from Lemma A.3 that $\hat{s} = s(\hat{x}) = \bar{s}$.

Appendix B

In this appendix, we establish some properties of the class of functions considered in Section 4.

Lemma B.1. *Let $U \subseteq \mathbb{R}^l$ be an open and connected set and let $\phi_1 : U \mapsto \mathbb{R}$ be a twice continuously differentiable function such that $\nabla^2\phi_1(u) = 0$ for all $u \in U$. Then, there exist unique $r \in \mathbb{R}^l$ and $\alpha \in \mathbb{R}$ such that $\phi_1(u) = r^T u + \alpha$ for all $u \in U$.*

Lemma B.2. *Let $U \subseteq \mathbb{R}^l$ be an open and connected set and let $\phi_1 : U \mapsto \mathbb{R}$ be a continuously differentiable function such that $\nabla\phi_2(u) = 0$ for all $u \in U$. Then, ϕ_2 is a constant function on U.*

Lemma B.3. *Let $g : R^l \mapsto \mathbb{R}$ be a twice continuously differentiable function and let $C \subseteq \mathbb{R}^l$ be a relatively open convex set. Assume that for all $x \in C$, the matrix $\nabla^2 g(x) \geqslant 0$ and that Null $(\nabla^2 g(x)) = \mathcal{N}$, where $\mathcal{N}$ is a subspace independent of x. Then, there exist a full row rank matrix $E \in R^{l_1 \times l}$ such that the set $E_C \equiv \{Ex \mid x \in C\}$ is open, a vector $c \in \mathbb{R}^l$ and a twice continuously differentiable mapping $g_1 : E_C \mapsto \mathbb{R}$ such that*

$$g(x) = g_1(Ex) + c^T x \text{ and } \nabla^2 g_1(Ex) > 0, \qquad \forall x \in C.$$

Proof. We divide the proof into two parts: we first prove the lemma under the assumption that C is open and then use this conclusion to prove the lemma under the assumption that C is relatively open. Assume then that C is open. Let $P = [P_1, P_2] \in \mathbb{R}^{l \times l}$ be an orthogonal matrix such that the columns of $P_1 \in \mathbb{R}^{l \times l_1}$ form a basis for $\mathcal{N}^\perp$ and the columns of $P_2 \in \mathbb{R}^{l \times l_2}$ form a basis for $\mathcal{N}$. Consider the function $\phi : \mathbb{R}^l \mapsto \mathbb{R}$ defined by $\phi(y) = g(Py)$. Letting $D \equiv \{y \mid Py \in C\}$, we have

$$\nabla^2\phi(y) = P^T \nabla^2 g(Py) P = \begin{bmatrix} P_1^T \nabla^2 g(Py) P_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \forall y \in D.$$

Decomposing y accordingly into $y = (y_1, y_2)$, where $y_1 \in \mathbb{R}^{l_1}$and $y_2 \in \mathbb{R}^{l_2}$, we claim that for all $y = (y_1, y_2) \in D$, there hold:

$$\phi(y) = g_1(y_1) + c_2^T y_2 \text{ and } \nabla^2 g_1(y_1) > 0, \tag{B.1}$$

where $c_2 \in \mathbb{R}^{l_2}$ and g_1 is a twice continuously differentiable function defined on the open set $D_1 \equiv \{y_1 \mid y \in D\} \subseteq \mathbb{R}^{l_1}$ and taking values in $\mathbb{R}$. Indeed, let $y_1 \in D_1$ be given and define $D_{y_1} \equiv \{y_2 \mid (y_1, y_2) \in D\}$. Clearly, D_{y_1} is a nonempty open convex set. Moreover, letting $\phi_{y_1}(y_2) = \phi(y_1, y_2)$ for all $y_2 \in D_{y_1}$, we have that $\nabla^2\phi_{y_1}(y_2) = \nabla^2_{y_2 y_2}\phi(y_1, y_2) = 0$, for all $y_2 \in D_{y_1}$. By Lemma B.1, there exist unique $c_2(y_1) \in \mathbb{R}^{l_2}$ and $g_1(y_1) \in \mathbb{R}$ such that

$$\phi_{y_1}(y_2) = g_1(y_1) + c_2(y_1)^T y_2, \qquad \forall y_2 \in D_{y_1}, y_1 \in D_1. \tag{B.2}$$

Differentiating this relation with respect to y_2, we obtain $\nabla_{y_2}\phi(y_1,y_2) = \nabla_{y_2}\phi_{y_1}(y_2) = c_2(y_1)$ for all $y_2 \in D_{y_1}$ and $y_1 \in D_1$, or equivalently, for all $(y_1,y_2) \in D$. Since any point $(y_1,y_2) \in D$ has a rectangle neighborhood contained in D, it follows that $\nabla_{y_2}\phi(y_1',y_2) = c_2(y_1')$ for any y_1' sufficiently close to y_1. Hence, we conclude that c_2 is continuously differentiable on D_1 and that $\nabla c_2(y_1) = \nabla_{y_1y_2}\phi(y_1,y_2) = 0$ for all $y_1 \in D_1$. By Lemma B.2, it follows that c_2 is a constant function on D_1. Hence, the first relation in (B.1) follows and hence, $g_1(y_1)$ is twice continuously differentiable on D_1. Differentiating this relation with respect to y_1 twice, we see that $\nabla^2 g_1(y_1) = \nabla_{y_1y_1}\phi(y_1,y_2) > 0$ for any $y \in D$. Letting $x = Py$ and observing that $x \in C \Leftrightarrow y \in D$, it follows from relation (B.1) that

$$g(x) = g(Py) = \phi(y) = \phi(P^{\mathrm{T}}x) = g_1(P_1^{\mathrm{T}}x) + c_2^{\mathrm{T}}P_2^{\mathrm{T}}x = g_1(Ex) + c^{\mathrm{T}}x, \quad \forall x \in C,$$

where $E \equiv P_1^{\mathrm{T}}$ and $c \equiv P_2c_2$. Hence the result follows when C is open.

Assume now that C is relatively open. Fix $x^0 \in C$, let aff (C) denote the affine hull of C and let $H \in \mathbb{R}^{l\times k}$ be a matrix whose columns form a basis for the subspace aff $(C) - x^0$. Define the mapping $\Pi : \mathbb{R}^k \mapsto \text{aff } C$ by $\Pi(u) \equiv x^0 + Hu$. Clearly, Π is an isomorphism and $\tilde{C} \equiv \Pi^{-1}(C) \subseteq \mathbb{R}^k$ is an open convex set. Consider the function $\tilde{g}(u) : \tilde{C} \mapsto \mathbb{R}$ defined by $\tilde{g}(u) \equiv g(x^0+Hu)$. We have $\nabla^2\tilde{g}(u) = H^{\mathrm{T}}\nabla^2 g(x^0+Hu)H \geqslant 0$ for every $u \in \tilde{C}$. Using this fact and the assumption that Null $(\nabla^2 g(x)) = \mathcal{N}$ for every $x \in C$, we obtain

$$\nabla^2\tilde{g}(u)d = 0 \Leftrightarrow d^{\mathrm{T}}\nabla^2\tilde{g}(u)d = 0 \Leftrightarrow (Hd)^{\mathrm{T}}\nabla^2 g(x^0+Hu)(Hd) = 0 \Leftrightarrow Hd \in \mathcal{N}, \quad \forall u \in \tilde{C},$$

from which it follows that Null $(\nabla^2\tilde{g}(u))$ is independent of $u \in \tilde{C}$. Applying the first part to the function $\tilde{g}(\cdot)$ and the set $\tilde{C}$, we conclude the existence of a full row rank matrix $\tilde{E} \in \mathbb{R}^{k_1\times k}$, a twice continuously differentiable function $\tilde{g}_1 : \tilde{E}_{\tilde{C}} \to \mathbb{R}$ defined on the open set $\tilde{E}_{\tilde{C}} \equiv \{\tilde{E}u \mid u \in \tilde{C}\}$ and a vector $\tilde{c} \in \mathbb{R}^k$ such that

$$g(x^0+Hu) = \tilde{g}(u) = \tilde{g}_1(\tilde{E}u) + \tilde{c}^{\mathrm{T}}u = \tilde{g}_1(\tilde{E}\tilde{H}(x-x^0)) + \tilde{c}^{\mathrm{T}}\tilde{H}(x-x^0), \quad \forall u \in \tilde{C},$$

$$\nabla^2\tilde{g}_1(\tilde{E}u) = \nabla^2\tilde{g}_1(\tilde{E}\tilde{H}(x-x^0)) > 0, \qquad \forall u \in \tilde{C},$$

where $\tilde{H} \equiv (H^{\mathrm{T}}H)^{-1}H^{\mathrm{T}}$. Letting $E = \tilde{E}\tilde{H}$, $c = \tilde{H}^{\mathrm{T}}c$ and $g_1 : \tilde{E}_{\tilde{C}} + Ex^0 \subseteq \mathbb{R}^{k_1} \mapsto \mathbb{R}$ defined by $g_1(w) = \tilde{g}_1(w - Ex^0) - c^{\mathrm{T}}\tilde{H}x^0$ for all w, we have

$$g(x) = g_1(Ex) + c^{\mathrm{T}}x, \quad \nabla^2 g_1(Ex) > 0, \qquad \forall x \in C.$$

The result thus follows. □

References

[1] I. Adler, N.K. Karmarkar, M.G.C. Resende and G. Veiga, An implementation of Karmarkar's algorithm for linear programming, *Mathematical Programming* 44 (1989) 297–335. Errata in: *Mathematical Programming* 50 (1991) 415.

[2] I. Adler and R.D.C. Monteiro, Limiting behavior of the affine scaling continuous trajectories for linear programming problems, *Mathematical Programming* 50 (1991) 29–51.

[3] E.R. Barnes, A variation on Karmarkar's algorithm for solving linear programming problems, *Mathematical Programming* 36 (1986) 174–182.

[4] D.A. Bayer and J.C. Lagarias, The nonlinear geometry of linear programming, Part I: Affine and projective scaling trajectories, *Transactions of the American Mathematical Society* 314 (1989) 499–526.

[5] J. Bonnans and M. Bouhtou, The trust region affine interior point algorithm for convex and nonconvex quadratic programming, *RAIRO - Recherche Opérationnelle* 29 (1995) 195–217.

[6] I.I. Dikin, Iterative solution of problems of linear and quadratic programming, *Doklady Akademii Nauk SSSR* 174 (1967) 747–748. Translated in: *Soviet Mathematics Doklady* 8 (1967) 674–675.

[7] I.I. Dikin, On the convergence of an iterative process, *Upravlyaemye Sistemi* 12 (1974) 54–60. (In Russian.)

[8] I.I. Dikin, The convergence of dual variables, Technical Report, Siberian Energy Institute, Irkutsk, Russia, December 1991.

[9] I.I. Dikin and V.I. Zorkaltsev, *Iterative solutions of mathematical programming problems* (Nauka, Novosibirsk, 1980).

[10] R. Fletcher, *Practical Methods of Optimization* (John Wiley, New York, 2nd ed., 1987).

[11] D.M. Gay, Computing optimal locally constrained steps, *SIAM Journal on Scientific and Statistical Computing* 2 (1981) 186–197.

[12] C.C. Gonzaga, Convergence of the large step primal affine-scaling algorithm for primal nondegenerate linear programs, Technical Report ES-230/90, Dept. of Systems Engineering and Computer Science, COPPE Federal University of Rio de Janeiro, 21941 Rio de Janeiro, RJ, Brazil, September 1990.

[13] C.C. Gonzaga, An interior trust region method for linearly constrained optimization, *COAL Newsletter* 19 (1991) 55–66.

[14] C.C. Gonzaga and L.A. Carlos, A primal affine-scaling algorithm for linearly constrained convex programs, Technical Report ES-238/90, Dept. of Systems Engineering and Computer Science, COPPE Federal University of Rio de Janeiro, 21941 Rio de Janeiro, RJ, Brazil, December 1990.

[15] C.-G. Han, P. Pardalos and Y. Ye, On the solution of indefinite quadratic problems using an interior-point algorithm, *Informatica* 3 (1992) 474–496.

[16] A.P. Kamath, N.K. Karmarkar, K.G. Ramakrishnan and M.G.C. Resende, A continuous approach to inductive inference, *Mathematical Programming* 57 (1992) 215–238.

[17] N.K. Karmarkar, A new polynomial-time algorithm for linear programming, *Combinatorica* 4 (1984) 373–395.

[18] N.K. Karmarkar, M.G.C. Resende and K.G. Ramakrishnan, An interior point algorithm to solve computationally difficult set covering problems, *Mathematical Programming* 52 (1991) 597–618.

[19] K. Levenberg, A method for the solution of certain nonlinear problems in least squares, *Quart. Appl. Math.* 2 (1944) 164–168.

[20] D.W. Marquardt, An algorithm for least squares estimation of nonlinear parameters, *SIAM Journal on Applied Mathematics* 11 (1963) 431–441.

[21] W.F. Mascarenhas, The affine scaling algorithm fails for $\lambda = 0.999$, technical report, *SIAM Journal on Optimization* 7 (1997) 34–46.

[22] N. Megiddo and M. Shub, Boundary behavior of interior point algorithms in linear programming, *Mathematics of Operations Research* 14 (1989) 97–114.

[23] C.L. Monma and A.J. Morton, Computational experience with the dual affine variant of Karmarkar's method for linear programming, *Operations Research Letters* 6 (1987) 261–267.

[24] R.D.C. Monteiro, Convergence and boundary behavior of the projective scaling trajectories for linear programming, *Mathematics of Operations Research* 16 (1991) 842–858.

[25] R.D.C. Monteiro and T. Tsuchiya, Global convergence of the affine scaling algorithm for convex quadratic programming, Technical Report, School of Industrial Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA, March 1995. To appear in *SIAM Journal on Optimization*.

[26] R.D.C. Monteiro, T. Tsuchiya and Y. Wang, A simplified global convergence proof of the affine scaling algorithm, *Annals of Operations Research* 47 (1993) 443–482.

[27] R.D.C. Monteiro and S. Wright, Superlinear primal-dual affine scaling algorithms for LCP, *Mathematical Programming* 69 (1995) 311–333.

[28] J.J. Moré, The Levenberg-Marquardt algorithm: Implementation and theory, in: *Proceedings of the Dundee Conference of Numerical Analysis*, G.A. Watson, ed. (Springer-Verlag, 1978).

[29] J.J. Moré, Notes on optimization software, in: *Nonlinear Optimization 1981*, M.J.D. Powell, ed. (Academic Press, 1982).
[30] J.J. Moré, Recent developments in algorithms and software for trust region methods, in: *Mathematical Programming Bonn 1982 - The State of the Art*, M.G.A. Bachem and B. Korte, eds. (Springer-Verlag, 1983) 259–287.
[31] J.J. Moré and D.C. Sorensen, Computing a trust region step, *SIAM Journal of Scientific and Statistical Computing* 4 (1983) 553–572.
[32] P.M. Pardalos and M.G.C. Resende, Interior point methods for global optimization, in: *Interior point methods in mathematical programming*, T. Terlaky, ed. (Kluwer Academic Publishers, 1996) 467–500.
[33] G.A. Schultz, R.S. Schnabel and R.H. Byrd, A family of trust-region-based algorithms for unconstrained minimization with strong global convergence properties, *SIAM Journal on Numerical Analysis* 22 (1985) 47–67.
[34] D.C. Sorensen, Trust region methods for unconstrained optimization, in: *Nonlinear Optimization 1981*, M.J.D. Powell, ed. (Academic Press, 1982).
[35] J. Sun, A convergence proof for an affine-scaling algorithm for convex quadratic programming without nondegeneracy assumptions, *Mathematical Programming* 60 (1993) 69–79.
[36] J. Sun, A convergence proof for a convex version of Dikin's algorithm, *Annals of Operations Research* 62 (1996) 357–374.
[37] P. Tseng and Z.Q. Luo, On the convergence of the affine-scaling algorithm, *Mathematical Programming* 56 (1992) 301–319.
[38] T. Tsuchiya, Global convergence of the affine-scaling methods for degenerate linear programming problems, *Mathematical Programming* 52 (1991) 377–404.
[39] T. Tsuchiya, Global convergence property of the affine scaling method for primal degenerate linear programming problems, *Mathematics of Operations Research* 17 (1992) 527–557.
[40] T. Tsuchiya, Global convergence of the affine scaling algorithm for primal degenerate strictly convex quadratic programming problems, *Annals of Operations Research* 47 (1993) 509–539.
[41] T. Tsuchiya and R.D.C. Monteiro, Superlinear convergence of the affine scaling algorithm, *Mathematical Programming* 75 (1996) 77–110.
[42] T. Tsuchiya and M. Muramatsu, Global convergence of a long-step affine scaling algorithm for degenerate linear programming problems, *SIAM Journal on Optimization* 5 (1995) 525–551.
[43] R.J. Vanderbei and J.C. Lagarias, I. I. Dikin's convergence result for the affine-scaling algorithm, in: *Mathematical Developments Arising from Linear Programming: Proceedings of a Joint Summer Research Conference held at Bowdoin College, Brunswick, Maine, USA, June/July 1988*, J.C. Lagarias and M.J. Todd, eds., Vol. 114 of Contemporary Mathematics (American Mathematical Society, Providence, RI, 1990) 109–119.
[44] R.J. Vanderbei, M.S. Meketon and B.A. Freedman, A modification of Karmarkar's linear programming algorithm, *Algorithmica* 1 (1986) 395–407.
[45] C. Witzgall, P.T. Boggs and P.D. Domich, On the convergence behavior of trajectories for linear programming, in: *Mathematical Developments Arising from Linear Programming: Proceedings of a Joint Summer Research Conference held at Bowdoin College, Brunswick, Maine, USA, June/July 1988*, J.C. Lagarias and M.J. Todd, eds., vol. 114 of Contemporary Mathematics (American Mathematical Society, Providence, RI, 1990) 161–187.
[46] Y. Ye, An extension of Karmarkar's projective algorithm and the trust region method for quadratic programming, in: *Progress in Mathematical Programming: Interior Point and Related Methods*, N. Megiddo, ed. (Springer, Berlin, 1989) 49–63.
[47] Y. Ye, On an affine scaling algorithm for nonconvex quadratic programming, *Mathematical Programming* 56 (1992) 285–300.
[48] Y. Ye and E. Tse, An extension of Karmarkar's projective algorithm for convex quadratic programming, *Mathematical Programming* 44 (1989) 157–180.