

# Performance Measures in Dynamic Parameter Design

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## 1. Introduction

Dynamic parameter design introduced by Taguchi (1987) is one of the most important tool in quality engineering. It is also known as parameter design in signal-response systems (Miller and Wu, 1996; Wu and Hamada, 2000, chapter 11). The name suggests that the interest lies in a signal-response relationship rather than a single value of the response. Taguchi's approach to dynamic parameter design can be described as follows. Let  $Y$  be the response and  $M$  the signal factor. There exists an ideal relationship between the signal and the response given by  $Y = \beta_1 M$ . In reality due to the presence of noise factors, deviations occur from this ideal function. Therefore a more realistic statistical model is

$$Y = \beta M + \varepsilon, \quad (1)$$

where  $E(\varepsilon) = 0$ ,  $Var(\varepsilon) = \sigma_0^2$ , and  $\varepsilon$  is a random error caused by the noise factors. The  $\beta$  and  $\sigma_0^2$  are functions of the control factors  $X$ . The objective of dynamic parameter design is to choose an  $X$  to make (1) as close to the ideal relationship as possible.

Taguchi introduced the following signal-to-noise (SN) ratio for evaluating the performance of the system,

$$SN = \frac{\beta^2}{\sigma_0^2}, \quad (2)$$

which will be referred to as *Taguchi's SN ratio*. In (2)  $\beta$  and  $\sigma_0^2$  are estimated for each control run as

$$\hat{\beta}_i = \frac{\sum_{j=1}^J \sum_{k=1}^K M_j Y_{ijk}}{K \sum_{j=1}^J M_j^2} \quad \text{and} \quad \hat{\sigma}_{0i}^2 = \frac{1}{JK} \sum_{j=1}^J \sum_{k=1}^K (Y_{ijk} - \hat{\beta}_i M_j)^2, \quad (3)$$

where  $Y_{ijk}$  is the response value at control run  $i$ , signal level  $j$ , and noise level  $k$ . The SN ratio for the  $i^{\text{th}}$  control run can be estimated as  $\hat{SN}_i = \hat{\beta}_i^2 / \hat{\sigma}_{0i}^2$ . Then an  $X$  is sought that maximizes the SN ratio. Next an adjustment parameter is used to adjust  $\beta$  to its ideal value. The justification for the SN ratio was that when an adjustment parameter is used to adjust the value of  $\beta$  to  $\beta_1$ ,  $\sigma_0^2$  will change to  $\sigma_0^2 (\beta_1 / \beta)^2$  and therefore we should minimize  $\sigma_0^2 / \beta^2$  rather than  $\sigma_0^2$ .

In statistical literature there was some skepticism of using SN ratio indiscriminately to all problems. Miller and Wu (1996) classified the dynamic parameter design into measurement systems and multiple target systems. A third class of dynamic parameter design is on the optimization with functional response. Miller and Wu proved that SN ratio is a meaningful performance measure in measurement systems with linear calibration equation, but criticized its use in multiple target systems. Joseph and Wu (2002) gave a general formulation for multiple target systems and showed that  $\sigma_0^2$  is the right performance measure under model (1). They derived a *modified version* of the SN ratio based on a different statistical model. Interestingly this SN ratio can be justified even without using the notion of an ideal function. On the other hand, no reasonable justification is available for the use of SN ratio in the functional response problem (Nair, Taam, and Ye, 2002). Performance measures and SN ratios for parameter design with control systems are developed in Joseph (2002).

In this article we will describe the application of SN ratio in the analysis of the multiple target systems. The exposition is mostly based on the work of Joseph and Wu (2002). We will derive an SN ratio as a performance measure independent of adjustment under some modeling assumptions. We will also present two modeling approaches known as response modeling and performance measure modeling for the analysis of dynamic parameter design experiments. This will be illustrated with a real experiment.

## 2. Signal-to-Noise Ratio

Let  $X$  be the set of control factors and  $Z$  the set of noise factors. Assume  $Y$  and  $M$  to be nonnegative variables taking values in  $[0, \infty)$ . Let

$$Y = f(X, Z, M)$$

be the signal-response relationship. If this relationship passes through the origin, then as shown in Joseph and Wu (2002), we can approximate it as

$$Y = \beta(X, Z)M .$$

Let  $N$  denote the observable set of noise factors and  $U$  the unobservable set of noise factors. So  $Z = \{N, U\}$ . The observable noise factors are systematically varied in the experiment. Let  $E_U[\beta(X, N, U) | N] = \beta(X, N)$  and  $Var_U[\beta(X, N, U) | N] = \sigma^2(X, N)$ .

Then we can use the following model

$$Y = \beta(X, N)M + \varepsilon , \quad (4)$$

where  $E(\varepsilon | N) = 0$  and  $Var(\varepsilon | N) = \sigma^2(X, N)M^2$ . Consider the quality loss function  $L = c(Y - T)^2$ , where  $T$  is the customer intent. Then

$$E(L | N) = c[(\beta(X, N)M - T)^2 + \sigma^2(X, N)M^2] .$$

Let  $\beta(X) = E_N[\beta(X, N)]$  and  $\sigma^2(X) = Var_N[\beta(X, N)] + E_N[\sigma^2(X, N)]$ . Then

$$EL = E_N E(L | N) = c[(\beta(X)M - T)^2 + \sigma^2(X)M^2] .$$

For a given  $T$ , we can set the signal factor to minimize the loss. Solving for  $M$  from

$$\frac{\partial}{\partial M} EL = c[2(\beta(X)M - T)\beta(X) + 2\sigma^2(X)M] = 0 ,$$

we get

$$M^* = \frac{T\beta(X)}{\beta^2(X) + \sigma^2(X)}. \quad (5)$$

Thus the expected loss at this optimal signal setting is

$$EL^* = c\left[(\beta(X)M^* - T)^2 + \sigma^2(X)M^{*2}\right] = \frac{T^2\sigma^2(X)}{\beta^2(X) + \sigma^2(X)}.$$

Denote

$$SN(X) = \frac{\beta^2(X)}{\sigma^2(X)} = \frac{E_N^2[\beta(X, N)]}{Var_N[\beta(X, N)] + E_N[\sigma^2(X, N)]}, \quad (6)$$

which can be called a signal-to-noise ratio. Then  $EL^* = T^2 / (1 + SN(X))$ . Suppose  $W(T)$  is the distribution of the customer intent  $T$ . Then we want to find an  $X \in D$  to minimize  $EL^*$  averaged over  $W(T)$ , where  $D$  is the feasible region of  $X$ . Thus

$$\min_{X \in D} \int EL^* dW(T) = \frac{1}{1 + SN(X)} \int T^2 dW(T).$$

This is equivalent of maximizing  $SN(X)$ . Therefore  $SN(X)$  can be considered as a performance measure independent of adjustment (PerMIA) for signal-response systems. This is an extension of PerMIA introduced by Leon, Shoemaker, and Kacker (1987) for static parameter design. See Leon and Wu (1992) for a theory on PerMIA. Thus the optimization procedure for multiple target systems can be stated as

1. Find  $X^* \in D$  to maximize the signal-to-noise ratio

$$SN(X) = \frac{\beta^2(X)}{\sigma^2(X)} \quad (7)$$

2. Adjust  $M$ , depending on  $T$ , as

$$M = \frac{T\beta(X^*)}{\beta^2(X^*) + \sigma^2(X^*)} \quad (8)$$

The optimization in step 1 is sometimes carried out with some bounds on  $\beta(X)$  (see Joseph and Wu (2002) for details). The adjustment in (8) can be interpreted as a shrinkage procedure as it results in a lower mean value than the target. The adjustment step could be modified as  $M = T / \beta(X^*)$ , so that the mean will be on target. It is easy to show that the step 1 remains the same even under this *unbiased* adjustment strategy.

The above modeling approach is known as *response modeling*. Another approach to parameter design is the *performance measure modeling* (see Wu and Hamada, 2000, chapters 10 and 11). This is commonly used in Taguchi's approach. It can be formulated as follows. Absorbing the observable noise into the error term, (4) becomes

$$Y = \beta(X)M + \varepsilon, \quad (9)$$

where  $E(\varepsilon) = 0$  and  $Var(\varepsilon) = \sigma^2(X)M^2$ . Now for a given control run  $X_i$ , we can estimate  $\beta(X_i)$  and  $\sigma^2(X_i)$  using weighted least squares as

$$\hat{\beta}_i = \frac{1}{JK} \sum_{j=1}^J \sum_{k=1}^K \frac{Y_{ijk}}{M_j} \quad \text{and} \quad \hat{\sigma}_i^2 = \frac{1}{JK} \sum_{j=1}^J \sum_{k=1}^K \frac{(Y_{ij} - \hat{\beta}_i M_j)^2}{M_j^2}. \quad (10)$$

Note that the estimation procedure in (10) is markedly different from that of (3). The signal-to-noise ratio can be estimated as  $\hat{SN}_i = \hat{\beta}_i^2 / \hat{\sigma}_i^2$ . This can be modeled (after a log transform) in terms of  $X$  and optimized. The above signal-to-noise ratio is different from (2) because the underlying models are different. The two models (1) and (9) are pictorially shown in Figure 1. In most cases as  $M$  reduces to 0, the variance also reduces to 0 and therefore model (9) is more reasonable than model (1).

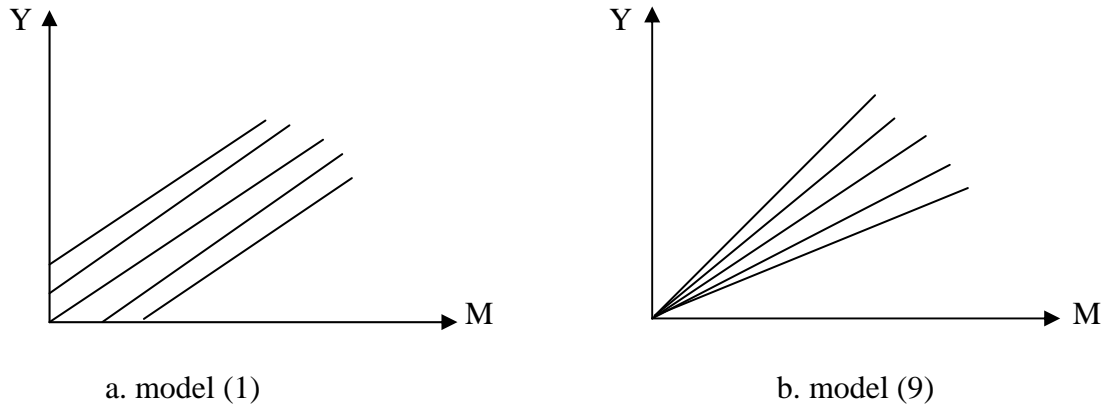


Figure 1. Signal-Response Systems

### 3. An Example

We will illustrate the approach using a push-pull cable actuator experiment reported in Byrne and Quinlan (1993). This experiment was also analyzed in Tsui (1999). There were 11 control factors and one noise factor in the experiment. The input displacement is the signal factor and the output displacement is the response. The data from an  $OA(12,2^{11})$  experiment is given in Table 1.

Assume a normal error in model (4). Then using maximum likelihood estimation we get,

$$\log \hat{\sigma}^2(X, N) = -5.5738 - .8372x_2 + .4875x_6, \quad (11)$$

$$\hat{\beta}(X, N) = .5631 + .0457x_1 + .0880x_2 + .0278x_7 + .0434x_8 - .0627x_{10}, \quad (12)$$

where the two levels of the variables are coded as  $-1$  and  $1$ . The proposed SN ratio in (6) becomes

$$\hat{SN}(X) = \frac{(.5631 + .0457x_1 + .0880x_2 + .0278x_7 + .0434x_8 - .0627x_{10})^2}{\exp(-5.5738 - .8372x_2 + .4875x_6)}.$$

Maximizing this with the variables restricted in the region  $D = \{X \mid -1 \leq x_i \leq 1, i = 1, \dots, 11\}$ , we get

$$x_1^* = 1, x_2^* = 1, x_6^* = -1, x_7^* = 1, x_8^* = 1, x_{10}^* = -1.$$

If we use performance measure modeling, then we will get the model

$$\hat{SN} = \log \frac{\hat{\beta}^2}{\hat{\sigma}^2} = 5.3488 + .5887x_2 + .5196x_7,$$

which on maximization gives  $x_2^* = 1, x_7^* = 1$ . The results of Taguchi's signal-to-noise ratio analysis from (2) is

$$SN_0 = \log \frac{\hat{\beta}^2}{\hat{\sigma}_0^2} = -.3296 + .5949x_2 + .5818x_7,$$

which gives the same conclusions as in the performance measure modeling.

The response modeling approach is more informative and statistically valid. We see that the noise factor does not appear in (11) and (12). Therefore this was not a useful noise factor to experiment with. The variables  $x_2, x_6$  affect the variance and are therefore useful to make the cable actuator robust against unobserved noise variation. The variables  $x_1, x_7, x_8, x_{10}$  affect the sensitivity. Their levels are chosen to increase the sensitivity, thereby increasing the SN ratio. If a high sensitivity is undesirable, then some of these variables can be manipulated to achieve a desired sensitivity.

Table 1. Data of the push-pull cable actuator experiment

run	$x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11}$	N	M=8			M=16			M=24		
1	-----.	-1 +1	4.97 4.69	5.19 4.9	5.41 5.11	10.05 9.48	10.49 9.9	10.94 10.32	13.28 12.53	13.95 13.16	14.62 13.79
2	-----++++	-1 +1	4.0 3.84	4.27 4.1	4.54 4.35	7.2 6.9	7.74 7.42	8.27 7.93	8.56 8.21	9.37 8.98	10.18 9.76
3	--++++-----	-1 +1	4.46 4.4	4.65 4.48	4.84 4.66	9.04 8.7	9.41 9.06	9.79 9.42	11.78 11.34	12.35 11.89	12.91 12.43
4	-+-+--+--+-	-1 +1	5.0 4.89	5.14 5.03	5.28 5.16	9.59 9.38	9.86 9.65	10.14 9.92	13.66 13.36	14.07 13.76	14.48 14.17
5	-+-+--+--+--	-1 +1	6.0 5.75	6.27 6.01	6.54 6.27	12.1 11.6	12.64 12.12	13.18 12.64	18.2 17.45	19.01 18.23	19.82 19.0
6	-++++-+-+---	-1 +1	3.86 3.72	4.0 3.85	4.14 3.99	7.7 7.42	7.98 7.68	8.26 7.95	10.5 10.11	10.92 10.51	11.33 10.91
7	+--++-+--+--	-1 +1	4.04 3.96	4.23 4.14	4.42 4.32	7.29 7.13	7.66 7.49	8.03 7.86	8.72 8.53	9.28 9.08	9.84 9.62
8	+--+--+--+--	-1 +1	3.39 3.2	4.61 3.41	3.83 3.62	6.36 6.0	6.8 6.42	7.25 6.84	6.96 6.57	7.63 7.2	8.29 7.83
9	+---+++-+---	-1 +1	6.13 6.0	6.27 6.13	6.41 6.27	11.98 11.72	12.25 11.99	12.53 12.26	16.53 16.17	16.94 16.58	17.36 16.98
10	+++-----++--	-1 +1	6.54 6.27	6.76 6.48	6.98 6.69	13.18 12.64	13.62 13.06	14.06 13.48	19.82 19.0	20.48 19.64	21.14 20.27
11	++-+-+-----	-1 +1	4.89 4.71	5.08 4.89	5.27 5.07	10.28 9.9	10.65 10.26	11.03 10.62	15.15 14.58	15.71 15.12	16.27 15.66
12	++-+-+--+--+--	-1 +1	3.28 3.09	3.55 3.35	3.82 3.60	6.53 6.16	7.07 6.67	7.61 7.19	8.73 8.23	9.54 8.23	10.36 9.77

#### 4. Conclusions

In this article we have shown that the dynamic signal-to-noise ratio in (6) can be justified as a PerMIA under some modeling assumptions. The signal-to-noise ratio derived is different from Taguchi's proposal. We have also described two modeling approaches to parameter design. The response modeling is statistically more efficient than the performance measure modeling, which has a theoretical justification in Berube and Nair (1998).

Joseph and Wu (2002) described strategies for analyzing nonlinear signal-response systems. Taguchi treats non-linearity as an error and tries to minimize it in his signal-to-noise ratio optimization. However, some of the signal-response systems are inherently

nonlinear and can work well with a non-linear signal-response relationship. Therefore forcing such systems to behave like a linear system will lead to sub-optimal solutions. Interestingly the model in (4) can still be used to analyze a nonlinear signal-response relationship by separating the lack-of-fit term.

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