A strong combinatorial relaxation for the problem of Min-Time Coverage in Constricted Environments

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***Abstract*—In a recent work, we introduced a new set of problems in the area of networked multiple mobile robotic systems that concern the time-optimal execution of certain coverage tasks taking place in constricted environments. That work provided the detailed problem definitions, positioned these problems in the context of the corresponding literature, formulated them as Mixed Integer Programs (MIPs), and established their NP-hardness. The current work introduces a strong combinatorial relaxation for these problems, and further establishes that the feasible solutions of these relaxations can be converted to feasible solutions for the original MIPs with equal or better objective values than the objective values of the converted solutions. The new results also enable the development of more efficient solution methods and heuristic approaches for the considered problems; this potential is addressed in the last part of the paper.**

***Index Terms*—Networked mobile robotic systems; multi-robot coordination; coverage problems; combinatorial scheduling; combinatorial relaxations of mixed integer programs.**

I. Introduction

In the recent years, there has been an increasing interest in employing teams of mobile robots – or, more formally, *multiple mobile robot systems (MMRS)* – for the execution of various tasks that are deemed to be too dangerous, physically challenging or very tedious for the human element [1]. Among the most celebrated examples of such MMRS applications are some search-and-rescue operations where teams of robots have performed successfully various reconnaissance tasks in environments that are too hazardous for the human respondents [2], [3]. Currently, there is also extensive interest in the employment of MMRS for the support of more routine functions evolving in operational environments that are quite stable and well-managed. Some characteristic examples of such applications are: (i) the employment of fleets of mobile robots as the primary material handling devices in many industrial and warehousing facilities [4], [5]; (ii) the delivery by drones or mobile robots of groceries and take-out orders in some residential or rural areas [6], [7]; (iii) the surveillance of public spaces, like public squares and commercial malls, with strategically (re-)positioned cameras that are mounted on mobile robots [8]; and (iv) various other patrolling and data-gathering functions where the robots must visit or monitor persistently some critical locations [9], [10], [11], [12].

In this work, we focus on a class of MMRS applications that concern the employment of mobile robotic teams for the support of routine monitoring and inspection operations in subterranean or other physically constricted structures that are not easily or safely accessible by the human element. Some examples of such environments are (a) the water supply, sewage and other underground utility networks in modern urban areas, (b) mines and other (e.g., archeological) excavation sites, and also (c) the pipeline networks that are used for the transport of oil, gas and other similar commodities over long distances.

During the past decade, these MMRS applications have received extensive attention by the robotics community [13], [14], [15], [16], [17], [18]. Furthermore, these applications have been the focus of a major DARPA challenge known as the "Subterranean – or SubT – Challenge" [19]. This research activity has provided the technological capability that enables (i) stable, flexible and safe motion of the deployed robots in the spatially constricted, and potentially adversarial, corridors (i.e., tunnels, pipes, etc.) that support the robotic traffic, and (ii) reliable communication mechanisms among the robots and the command-&-control (C-&-C) center that manages the overall operation.

But the corresponding literature has also recognized that the effective use of this emerging technological capability in the targeted MMRS applications requires the development of a methodological base that will model, analyze and control the execution of the involved tasks for certain notions of *correctness* and *efficiency* that are defined by various operational specifications and performance metrics. A first set of results in this direction has been provided in our recent work of [20]. More specifically, in [20], we have provided: (i) a systematic introduction of the operational requirements of the considered MMRS; (ii) a formal characterization of the traffic management problem that is induced by these requirements, and the detailed positioning of this problem in the existing MMRS literature; (iii) a complete analytical representation of this problem in the form of some mixed integer programming (MIP) formulations [21]; and (iv) a worst-case complexity analysis of the decision problems that underlie these formulations.[1]

The problems formulated in [20] are hard combinatorial optimization problems [22]. Hence, there is a further need for the development of additional methodology for these problems that will enable a more effective management of their computational complexity. Such methodology can either seek to take advantage of any special structure that might be present in the considered problems, or to develop suboptimal solution methods able to establish a satisfactory trade-off between (i)

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[1] An overview of the results of [20] that are necessary for the developments of this work, is provided in the next section.

the involved computational complexity and (ii) the operational efficiency of the derived solutions. The results presented in this paper constitute a significant contribution in this direction.

More specifically, the main contributions of this work can be summarized as follows:

1) First, we develop some *strong combinatorial relaxations* for the MIP formulations of [20]. These relaxations are MIP formulations themselves that (i) are derived from the original MIPs of [20] by relaxing the integrality requirement of a large part of – but not all – the integer variables that are employed in those formulations, and (ii) have the same optimal objective value with the original MIPs.
2) We also show that we can restrict the relaxed MIPs to a certain subset of their feasible solutions that (i) preserves optimality, and (ii) possesses the following additional property: Any feasible solution in this subset can be algorithmically converted to a feasible solution for the original MIP with an objective value that is no worse than the objective value of the starting solution. Furthermore, the corresponding algorithm has polynomial complexity with respect to any parsimonious representation of the underlying problem instance.
3) Items #1 and #2 above also suggest an alternative method for solving the problems that were introduced in [20]. We detail this method, and we demonstrate and assess the computational gains that can result from it through a numerical experiment that is presented in the last part of the paper.
4) We also show that the developments of items #1 and #2 enable the testing for traffic plans which attain pre-specified visitation times for the target zones, by formulating and solving a linear program [21]. This capability can be especially useful in the development of some heuristic approaches for the considered problems. However, the complete development of such heuristic methods is a nontrivial issue, and it will be thoroughly addressed in a sequel paper.

The rest of the current manuscript is organized as follows: In the next section we review the basic characterizations of the coverage problems that are considered in [20], and the MIP formulations for these problems that are provided in that work. Section III introduces the considered relaxations for the original MIPs of [20], and establishes some additional properties for these relaxations that are useful for establishing their strong nature. The strong nature of the considered relaxations is established in Section IV. Furthermore, the closing part of Section IV details a new solution method for the considered problems that is derived from the theoretical developments that are presented in its earlier parts. Section V reports a numerical experiment that demonstrates and assesses the computational gains resulting from this new method. Finally, Section VI concludes the paper and suggests some directions for future work. Also, due to the imposed page limit for this work, the technical proofs of some results with a more supportive role to the main results of the paper, and some details for the experiment of Section V, are provided in a companion electronic supplement that is accessible through the publications webpage of the second author.

II. The considered coverage problems and their MIP formulations

In this section we review the min-time coverage problems that are considered in this work and their MIP formulations, based on the corresponding developments of [20]. The provided exposition of this material is the minimum necessary for the presentation of the main results of this paper. The reader is referred to [20] for a more expansive treatment of this material.

The considered coverage tasks and the corresponding traffic-management problems: A fleet of mobile robots must be used to inspect a set of locations of an underground guidepath network that constitutes a *tree*. As discussed in the introductory section, some examples of such a guidepath network might be a water supply network, a sewage network, or a network of tunnels in an underground mine [13], [14].

The robots are initially located at a C&C-center of the entire facility, which defines the root of the tree, and the targeted locations are its leaves. Furthermore, the tunnels are narrow and the robots have limited sensing and maneuvering capability. Therefore, for safety reasons, the robots must be separated through the imposition of a zoning scheme that splits the underlying guidepath network into zones of unit buffering capacity – i.e., each zone cannot accommodate more than one robot at any time. The robots are granted access to these zones through a traffic coordinator.[2]

The robots possess wireless communication capability, but their communication range is severely limited by their operational environment. Since these wireless communication links are the only way for each robot to communicate with its operational environment, the robot motion must be coordinated in a way that, at any time point, the active links among the robots define a multi-hop communication network connecting the robots to each other and to the C&C-center. This connectivity can be ensured by (i) defining the imposed zones in a way that any pair of robots occupying neighboring zones will communicate with each other in a stable manner, and (ii) further requiring that, at any time point, a zone cannot be occupied unless its parent zone in the underlying tree is also occupied.

Finally, following standard practice in the formal study of the traffic dynamics generated by zoning schemes similar to those considered in this work, we further assume that zones are defined so that they have uniform traversal time. Then, picking this traversal time as the time unit, we can study the resulting traffic dynamics in discrete time.[3]

In view of the above description, the considered MMRS can be formally represented by a tuple $\mathcal{M} = \langle \mathcal{R}, \mathcal{T} \rangle$, where $\mathcal{R}$ is

[2]Similar zoning schemes have been used extensively in automated unit-load industrial material handling systems [4], and more recently they have provided a safety control mechanism in other mobile robotic applications, as well [23], [1].

[3]Nonuniform traversal times for the system zones can be easily introduced into our model. But this feature would overload the employed notation and complicate some details in the pursued analysis, without adding anything substantial to the main results and insights that are presented in this work.

the set of the robots and $\mathcal{T}$ is a rooted tree representing the tunnel system. The node set V of $\mathcal{T}$ represents the zones of the tunnel system, and the edge set E represents the *neighboring relation* among the zones; two zones are neighbors if they share a common boundary that allows robots to transition from one to the other.

The root node of $\mathcal{T}$ – i.e., the initial location of all robots and the point of command and control for the entire system – is denoted by o. The set of the leaf nodes of $\mathcal{T}$ is denoted by L. As already stated, each zone $v \in L$ must be visited by some robot for inspection purposes, and the inspection of a leaf zone is carried out by the visiting robot in the time interval corresponding to a discrete period.

The set of neighbors of a zone $v \in V$ is denoted by $\mathcal{N}(v)$, and for any zone $v \neq o$, $p(v)$ denotes the parent of v in $\mathcal{T}$. Let $z(r,t)$ denote the zone $v \in V$ occupied by robot r at period t. Then, $z(r,t+1) \in \{z(r,t)\} \cup \mathcal{N}(z(r,t))$; i.e., robot r can either remain in the same zone at period $t+1$, or move to a neighboring zone $v' \in \mathcal{N}(v)$. Furthermore, at any period t, a zone $v \neq o$ cannot contain more than one robot.

On the other hand, at any period t, a group of robots can coordinate their advancement over a path of neighboring zones; i.e., for a group of robots $r_1, r_2, \ldots, r_n$ with $z(r_i,t) \in \mathcal{N}(z(r_{i-1},t))$, for $i = 2,\ldots,n$, we allow $z(r_i, t+1) = z(r_{i+1}, t)$, $i = 1,\ldots,n-1$, provided that robot r_n moves itself to a free zone or to the root zone o at period $t+1$. We characterize such a string of robot moves as a robot *flow* occurring at time t, and we shall denote it by $f(z(r_1), z(r_n); t)$. The net effect of this flow is the transfer of a robot from zone $z(r_1,t)$ to zone $z(r_n,t+1)$. Also, the traffic dynamics that were described in the previous paragraphs imply that two flows $f(v_o, v_d; t)$ and $f(v'_o, v'_d; t)$ are conflicting if the supporting paths of these two flows have a common internal node $v \neq o$.

Robots can reverse the direction of their motion within their zone. This assumption is reasonable when considering the types of robots that are used in the considered applications, and furthermore, it is necessary due to the tree structure of the underlying tunnel system.

Finally, as observed in the earlier part of this section, the communication connectivity among the robots and the system controller is established by stipulating that, for every zone $v \neq o$ and every period t,

$$\exists\ r \in \mathcal{R} : z(r,t) = v \implies \exists\ r' \in \mathcal{R} : z(r',t) = p(v) \quad (1)$$

The above requirement implies that for every zone v occupied by a robot in period t, all the zones in the path connecting zone v to the root zone o in tree $\mathcal{T}$ are also occupied by a robot in period t. Furthermore, the root zone o is always occupied by at least one robot.

We want to determine a plan that will advance the robots $r \in \mathcal{R}$ in a way that is consistent with the above assumptions regarding the robot capabilities and the zone allocation protocol, and at the end of its execution, each leaf zone $v \in L$ will have been visited by some robot.[4] Let $\mathcal{P}$ denote the set of feasible plans, and for every $P \in \mathcal{P}$ and $v \in L$, let $C(v;P)$ denote the first period that plan P places a robot in zone v. We are especially interested in plans P^* such that

$$P^* = \arg\min_{P\in\mathcal{P}} \max_{v \in L} C(v;P) \quad (2)$$

or

$$P^* = \arg\min_{P\in\mathcal{P}} \sum_{v \in L} C(v;P) \quad (3)$$

Each of Eqs (2) and (3) defines a combinatorial optimization – or (traffic-)scheduling – problem. The traffic-scheduling problem defined by Eq. (2) is characterized as the *Makespan-minimization* problem, or the *M-problem*, and the traffic-scheduling problem defined by Eq. (3) is characterized as the *Total Visitation Time-minimization* problem, or the *TVT-problem.* In [20] it is shown that these two problems are in a Pareto optimal relationship [24], [25]; i.e., there are MMRS where the sets of optimal plans for these two problems have no common element. Hence, these two problems require separate treatments.

MIP formulations of the M- and TVT-problems: Next, we consider the mathematical programming (MP) formulation for the M- and TVT-problems of Eqs (2) and (3) that were developed in [20]. In the subsequent discussion, $\bar{T}$ denotes an upper bound for the completion time of an optimal plan P^* for each problem; one way to obtain such an upper bound is by considering the completion time of the plan P that tries to reach one leaf zone at a time, while scanning the tree $\mathcal{T}$ in a depth-first sense.

The decision variables employed by the MP formulations of [20] are as follows:

- State variables
  - $x_{v,t},\ v \in V,\ t \in \{0,1,\ldots,\bar{T}\}$: a nonnegative integer variable indicating the number of robots in zone v at period t.
- Control variables
  - $u_{v,v',t},\ v \in V,\ v' \in \mathcal{N}(v),\ t \in \{1,\ldots,\bar{T}\}$: a nonnegative integer variable representing the number of robots moving from zone v to neighboring zone v' at period t.
- Auxiliary variables
  - $y_{v,t},\ v \in L,\ t \in \{1,\ldots,\bar{T}\}$: a binary variable for testing whether leaf zone v has been visited by period t.[5]

[4] In more technical terms, a plan is a sequence of distributions, $\mathcal{D}_t, t = 0,1\ldots$, of the system robots to the various zones of the underlying tunnel system. The distribution $\mathcal{D}_{t+1}$, for period $t+1$, is obtained from the distribution $\mathcal{D}_t$ by relocating a number of robots from their zones at period t to some neighboring zone, while abiding to the introduced assumptions about the maneuvering capabilities of the robots and the zone allocation protocol. This characterization is specified further through the MIP formulations of the considered problems that are provided in the second part of this section.

[5] Actually, the values of the variables $y_{v,t}$ is part of the "informational state" that drives the underlying decision making process at period t. We have included these variables in the set of the auxiliary variables since their values are determined by the values of the corresponding variable sets $\{x_{v,q}, q = 1,\ldots,t\}$.

- $s_t,\ t \in \{1,\dots,\bar{T}\}$: a binary variable for testing whether the entire leaf-node visitation task has been completed by period t.

The technological constraints employed in the MP formulations for the M- and TVT-problems in [20] are as follows:

$$x_{o,0} = |\mathcal{R}| \tag{4}$$

$$\forall v \in V \setminus \{o\},\quad x_{v,0} = 0 \tag{5}$$

$$\forall v \in V,\ \forall t \in \{1,\dots,\bar{T}\},$$
$$x_{v,t} = x_{v,t-1} + \sum_{v' \in \mathcal{N}(v)} \big(u_{v',v,t} - u_{v,v',t}\big) \tag{6}$$

$$\forall v \in V,\ \forall t \in \{1,\dots,\bar{T}\},\quad \sum_{v' \in \mathcal{N}(v)} u_{v,v',t} \le x_{v,t-1} \tag{7}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1,\dots,\bar{T}\},\quad x_{v,t} \le 1 \tag{8}$$

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1,\dots,\bar{T}\},\quad x_{v,t} \le x_{p(v),t} \tag{9}$$

$$\forall v \in L,\ \forall t \in \{1,\dots,\bar{T}\},\quad y_{v,t} \le \sum_{q \in \{1,\dots,t\}} x_{v,q} \tag{10}$$

$$\forall v \in L,\ \forall t \in \{1,\dots,\bar{T}\},\quad s_t \le y_{v,t} \tag{11}$$

Constraints (4) and (5) define the initial distribution of the robots by means of the state variables $x_{v,0},\ v \in V$. Constraint (6) expresses the evolution of the robot distribution to the system zones at period t, based on the control decisions that are expressed by the variables $u_{v,v',t}$. Constraint (7) stipulates that the control decisions at period t must be feasible with respect to the robot distribution over the system zones at period $t-1$. Constraint (8) enforces the buffering capacity of the zones $v \neq o$. Constraint (9) enforces the condition of Eq. (1). Constraint (10) forces the binary variable $y_{v,t}$ to zero if leaf zone v has not been visited by period t. Finally, Constraint (11) forces the binary variable s_t to zero if there is a leaf zone v that has not been visited by period t.

The M-problem can be expressed by the following formulation:

$$\max \sum_{t \in \{1,\dots,\bar{T}\}} s_t \tag{12}$$

s.t. Constraints (4) – (11) plus the sign restrictions for the problem variables specified during the introduction of these variables.

The TVT-problem can be expressed by the following formulation:

$$\max \sum_{v \in L} \sum_{t \in \{1,\dots,\bar{T}\}} y_{v,t} \tag{13}$$

s.t. Constraints (4) – (10) plus the sign restrictions for the problem variables specified during the introduction of these variables.

The above two formulations are Integer Programming (IP) formulations [21]. An optimal solution for each of these two formulations determines an optimal plan P^* for the corresponding scheduling problem through the quantities $[u_{v,v',t} - u_{v',v,t}]^+$ for every pair (v,v') of neighboring zones and period t.[6]

Furthermore, we can replace the original sign restrictions of the variables $x_{v,t}$, $y_{v,t}$ and s_t with the following constraints that relax the integrality requirements for these variables:

$$\forall v \in V,\ \forall t \in \{1,\dots,\bar{T}\},\quad x_{v,t} \ge 0 \tag{14}$$

$$\forall v \in L,\ \forall t \in \{1,\dots,\bar{T}\},\quad 0 \le y_{v,t} \le 1 \tag{15}$$

$$\forall t \in \{1,\dots,\bar{T}\},\quad 0 \le s_t \le 1 \tag{16}$$

Indeed, as long as we retain the integrality requirement for the variables $u_{v,v',t}$, Constraints (4)–(6) will ensure the integrality of the variables $x_{v,t}$, and this fact subsequently preserves the mechanism that establishes the correct setting of the variables $y_{v,t}$ and s_t in any optimal solution of the resulting formulation.

III. The combinatorial relaxations of the M- and the TVT-problems

The considered relaxations: The combinatorial relaxations of the M- and TVT-problem considered in this work are obtained from the linear-programming (LP) relaxations [21] of their MIP formulations that were introduced in the previous section, by posing the following additional requirement:[7]

$$\forall v \in L,\ \forall t \in \{1,\dots,\bar{T}\},$$
$$y_{v,t} = 1.0 \iff \big(y_{v,t-1} = 1.0 \ \vee\ x_{v,t} = 1.0\big) \tag{17}$$

This requirement is enforced in the MP formulations that define the considered relaxations, by (i) treating the variables $y_{v,t}$ as binary, and (ii) rewriting Constraint (10) by means of the following two constraints:

$$\forall v \in L,\quad y_{v,1} \le x_{v,1} \tag{18}$$

$$\forall v \in L,\ \forall t \in \{2,\dots,\bar{T}\},\quad y_{v,t} \le y_{v,t-1} + x_{v,t} \tag{19}$$

Hence, the considered relaxation for the M-problem is the MIP formulation defined by Constraints (4)–(9), (11), (12), (14), (16), (18), (19), the sign restriction regarding the binary nature of the variables $y_{v,t}$, and the substitution of the integrality requirement for the variables $u_{v,v',t}$ with the more relaxed constraint

$$\forall v \in V,\ \forall v' \in \mathcal{N}(v),\ \forall t \in \{1,\dots,\bar{T}\},\quad u_{v,v',t} \ge 0 \tag{20}$$

The MIP formulation of the considered relaxation for the TVT-problem is obtained from the MIP formulation for the combinatorial relaxation of the M-problem by (i) using the objective function of Eq. (13) instead of that of Eq. (12), and (ii) dropping Constraints (11) and (16).

[6] We remind the reader that $[x]^+ = \max\{x, 0\}$.

[7] This requirement was initially motivated by an effort to cope with the fact that the LP relaxation of the MIP for the M-problem tries to increase progressively the variables $y_{v,t},\ v \in L,\ t \in \{1\dots,\bar{T}\}$, to their upper limit of 1.0 (c.f. Eq. (15)) by aggregating the amount of fluid that is present at the corresponding node v over a number of past periods (c.f. Eq. (10)). Because of this effect, optimal solutions for the LP-relaxation tend to spread out the initially available fluid at the root node o of tree $\mathcal{T}$ to its leaf nodes rather evenly. The net result of these dynamics is that (i) the optimal objective value of the LP relaxation constitutes a poor-quality lower bound for the optimal objective value of the original MIP formulation, and (ii) an optimal solution of the LP relaxation offers no substantial guidance towards the development of an optimal solution for the original scheduling problem.

The integer variables in the above two MIP formulations are binary, and their number is equal to $|L| \cdot \bar{T}$. This number is much smaller than $|V|^2 \cdot \bar{T}$, which is the number of the integer variables in the MIP formulations of Section II. The computational advantage implied by these comparisons becomes especially important when considering the strong property of the relaxing MIPs with respect to the original MIP formulations, that was outlined in the introductory section. In the rest of this work, we formally establish this property, and we employ it to define a new solution method for the M- and TVT-problems.

A flow-based interpretation of the considered relaxations: In the following discussions, it is useful to perceive the values of the variables $x_{v,t}$ as *an amount of fluid* located at node v at period t. The values of the entire set of variables $\{x_{v,t} : v \in V\}$, at any period $t \in \{0, 1, \ldots, \bar{T}\}$, defines a *fluid distribution* at period t. Also, the values of the set of variables $\{u_{v,v',t} : v \in V,\ v' \in \mathcal{N}(v)\}$, at any period $t \in \{0, 1, \ldots, \bar{T}\}$, defines a *flow* $F(t)$ at period t, and a sequence of flows $F = \langle F(1), F(2), \ldots, F(\bar{T})\rangle$ defines a *flow plan*. The set of flow plans that constitute feasible solutions for the considered relaxations will be denoted by $\mathcal{F}$.

By analogy with the corresponding definitions for the original M- and TVT-problems, we consider the fluid amount of 1.0 as a "target" fluid level that must be attained by any leaf node $v \in L$. Also, for any feasible flow plan F and any leaf node $v \in L$, we set

$$C(v; F) \equiv \min\left\{t \in \{1, \ldots, \bar{T}\} : x_{v,t} = 1.0\right\}.$$

Finally, in the sequel, the unique path connecting any nodal pair $\{v_1, v_2\}$ of tree $\mathcal{T}$ is denoted by $\pi(v_1, v_2)$, and the length $l(v_1, v_2)$ of path $\pi(v_1, v_2)$ is defined by the number of edges in it. Since tree $\mathcal{T}$ is undirected, $\pi(v_1, v_2) \equiv \pi(v_2, v_1)$ and $l(v_1, v_2) = l(v_2, v_1)$. Also, a single node can be considered as a path of zero length.

Some useful properties of the considered relaxations: In the remaining part of this section, we establish some properties for flow-plan set $\mathcal{F}$ of the considered relaxations that will be useful for establishing the strong nature of these relaxations in Section IV. The next definition is necessary for the formal statement of the first of these properties.

Definition 1: A flow plan $F \in \mathcal{F}$ for the combinatorial relaxation of the M- or TVT-problem considered in this work is characterized as *focused* if it satisfies the following condition: For every internal node v of the corresponding tree $\mathcal{T}$, and every period $t \in \{1, \ldots, \bar{T}\}$ such that $x_{v,t} > x_{v,t-1}$,

1) $\mathcal{K} \triangleq \{v' \in L : v \in \pi(o, v')\ \wedge\ C(v'; F) \geq t\} \neq \emptyset$;
2) $\forall \tau \in \{t, \ldots, \min_{v' \in \mathcal{K}} C(v'; F)\},\ x_{v,\tau} \geq x_{v,t}$

A flow $F(t)$ involving fluid transfer that violates the above condition, is characterized as *unfocused*.

The significance of focused plans is revealed by the following proposition.

Proposition 1: For the combinatorial relaxation of the M- or TVT-problem considered in this section, there always exists an optimal flow plan F^* that is focused.

In [26], we have established a result similar to that of Proposition 1 for the plan set $\mathcal{P}$, and the integral flows defined by the variable sets $\{u_{v,v',t} : v \in V,\ v' \in \mathcal{N}(v)\}$, that correspond to the original M- and TVT-problems. The arguments establishing that earlier result extend straightforwardly to the result of Proposition 1, when realizing that any flow $F(t)$ in a feasible flow plan F for the considered relaxations, can be perceived as a superposition of a number of more pointed flows that transfer certain amounts of fluid between some nodal pairs (v, v') of the underlying tree $\mathcal{T}$. Hence, for the sake of brevity, we omit a formal proof of Proposition 1, referring the interested reader to the corresponding proof of Proposition 2 in [26].

The generation of a focused flow plan F^* for the considered relaxations can be ensured by adding the term

$$-c \cdot \sum_{v \in V} \sum_{t=1}^{\bar{T}} \delta_{v,t} \tag{21}$$

to the objective function of the corresponding MIPs, and the constraints

$$\forall v \in V,\ \forall t \in \{1, \ldots, \bar{T}\},\ \delta_{v,t} \geq x_{v,t} - x_{v,t-1} \tag{22}$$

$$\forall v \in V,\ \forall t \in \{1, \ldots, \bar{T}\},\ \delta_{v,t} \geq x_{v,t-1} - x_{v,t} \tag{23}$$

In the derived solutions, $\delta_{v,t} = |x_{v,t} - x_{v,t-1}|$, and the cost c associated with the variables $\delta_{v,t}$ penalizes unnecessary fluctuations of the fluid content of the nodes $v \in V$. The value of c must be chosen sufficiently small so that it does not compromise the generated solution with respect to the original objective of the formulation. Also, Equations (21)–(23) induce the following notion of "structural minimality" for the flow plans $F \in \mathcal{F}$.

Definition 2: A flow plan $F \in \mathcal{F}$ is *structurally minimal* if it is optimal with respect to the objective function of Equation (21), within the set of flow plans that have the same value with F for the primary objective functions of Equations (2) and (3).

From a more conceptual standpoint, the result of Proposition 1 implies that there is no essential need to use the zones of tree $\mathcal{T}$ as "temporary buffers" for the fluid transfers that are performed by an optimal flow plan F^*. Furthermore, for a more thorough understanding of the notion of a focused flow plan, we emphasize that Condition (2) of Definition 1 requires that the considered node v in this definition must not have its fluid level reduced until some descendant leaf node v' of v – i.e., a leaf node v' in the set $\mathcal{K}$ of Definition 1 – reaches its target fluid level of one unit; but the fluid content of v can be reduced after that period, even if there are additional leaf nodes in $\mathcal{K}$ that have not attained their target fluid level of one unit.[8] The next proposition establishes some additional conditions that must hold in the case of such an early fluid withdrawal from some internal node v of $\mathcal{T}$.

Proposition 2: Consider a structurally minimal, optimal flow plan $F^* \in \mathcal{F}$, and further suppose that, in plan F^*, $x_{v,t} < x_{v,t-1}$ for node $v \in V \setminus L$, while there is some descendant

[8]The work of [26] provides a concrete example of a fluid withdrawal from an internal node v of $\mathcal{T}$ before all the descendant leaf nodes of node v in $\mathcal{T}$ have attained their target level of one unit of fluid, in the context of the discrete robotic flows that are investigated in that paper.

leaf node $\tilde{v}$ of v in tree $\mathcal{T}$ with $C(\tilde{v};F^*) > t$. Then, for all nodes $\hat{v} \in V$ with $p(\hat{v}) = v$, $x_{\hat{v},t} = 0$.

The proof of Proposition 2 is provided in the companion electronic supplement of the paper. Furthermore, when combined with Constraint (9), Proposition 2 implies that the entire subtree that emanates from the considered node v is empty of any fluid at period t.

We conclude this section with the next proposition which is also necessary for establishing the main results of the next section. The proof of this proposition is provided in the companion electronic supplement of the paper.

Proposition 3: For the combinatorial relaxations of the M- and TVT-problems considered in this section, there is a structurally minimal, optimal flow plan $F^* \in \mathcal{F}$ where[9]

$$\forall v \in V \setminus L, \;\; \sum_{t=1}^{\bar{T}} \sum_{v' \in V: p(v')=v} u_{v,v',t} \;\in\; \mathbb{Z}^+ \qquad (24)$$

Furthermore, given an instance of the considered relaxations and a structurally minimal, optimal flow plan F^* that violates the integrality condition of Equation (24), we can obtain another structurally minimal, optimal flow plan $\tilde{F}^*$ that satisfies the integrality condition of Equation (24), by (i) setting $\bar{T} = \max_{v \in L} C(v; F^*)$, and (ii) solving the following LP:

$$\min \sum_{v \in V} \sum_{t=1}^{\bar{T}} \delta_{v,t} \qquad (25)$$

s.t. Constraints (4)–(9), (14), (20), (22)–(23), and the additional constraints

$$\forall v \in L, \quad x_{v,C(v;F^*)} = 1.0 \qquad (26)$$

$$\forall v \in V \setminus L, \; \forall v' : p(v') = v, \; \sum_{t=1}^{\bar{T}} u_{v,v',t} = \left\lfloor \sum_{t=1}^{\bar{T}} u^*_{v,v',t} \right\rfloor \qquad (27)$$

IV. The Strong Nature of the Considered Relaxations

Preamble: This section establishes the following two important properties for the combinatorial relaxations of the M- and TVT-problems that were introduced in the previous section:

1) For any given problem instance $\mathcal{M} = \langle \mathcal{R}, \mathcal{T} \rangle$ of the M- or the TVT-problem, the optimal value of the corresponding combinatorial relaxation of Section III is equal to the optimal value of the original MIP of Section II.
2) Let $F \in \mathcal{F}$ denote a feasible flow plan for the combinatorial relaxation of item #1, and further assume that F is structurally minimal and satisfies the integrality property of Equation (24). Then, F can be converted to a feasible plan $P \in \mathcal{P}$ for the given M- or TVT-problem instance with an objective value that is no worse than the objective value of flow plan F. Furthermore, this conversion can be performed in time polynomial with respect to $|V|$, i.e., the number of nodes of the underlying guidepath network $\mathcal{T}$.

[9] $\mathbb{Z}^+$ denotes the set of strictly positive integers.

The aforementioned results are developed in two major steps:

I) In the first step it is shown that any feasible flow plan F for the considered relaxations can be represented as a static flow F_S for a transshipment problem, $TSH(F)$, that is induced by F and constitutes an "unfolding" of the dynamics of this flow plan on a spatiotemporal network $N_S(F)$.

Transshipment problems are a well-defined class of problems in graph theory and operations research [22], [27]. Furthermore, it is well known that under some integrality conditions for the data elements that define an instance of these problems, the corresponding set of the feasible static flows constitutes a polytope with its extreme points being integral flows. In the $TSH(F)$ context, an integral static flow $\tilde{F}$ can be directly translated into a plan $\tilde{P}$ for the underlying M- or TVT-problem instance, with an objective value equal to the objective value of the problem-defining flow F.

II) In view of the last remark in the previous paragraph, the second step in the subsequent developments establishes that for any feasible flow plan F of the considered relaxations that is structurally minimal and satisfies the integrality property of Equation (24) (i.e., it possesses the additional properties specified in item #2 at the beginning of this section), it is possible to derive, in polynomial time with respect to $|V|$, another feasible flow plan $\hat{F}$ with the following two properties: (i) The performance of flow plan $\hat{F}$ is no worse than the performance of F, and (ii) the corresponding transshipment problem instance, $TSH(\hat{F}) \equiv \widehat{TSH}(F)$, satisfies the integrality conditions for its defining data that guarantee the integrality of its extreme solutions.

The last part of the section also discusses the implications of the aforementioned developments for the design of more efficient solution methods and some heuristic algorithms for the M- and the TVT-problems. The computational power of some of these methods is further explored numerically in Section V.

A flow plan F as a static flow F_S for an induced transshipment problem $TSH(F)$: We start this part by providing a working definition of the transshipment problem that is adequate for the needs of the subsequent developments.

Definition 3: For the needs of this work, a transshipment problem instance, TSH, is formally defined by: (a) network $N_S = (V_S, E_S)$; (b) two functions that are defined on the edge-set E_S of N_S and assign, respectively, directions and capacities to these edges; and (c) two additional functions that are defined on the node-set V_S of N_S and assign, respectively, nodal supplies and demands. The problem seeks the specification of a static flow F_S on N_S that satisfies the nodal demands by means of the available supplies while respecting the directions and the capacities of the edges of N_S.[10]

The feasibility of a transshipment problem instance depends on the adequacy of the available supplies with respect to the

[10] Usually, the complete specification of a transshipment problem also involves a set of unit-flow costs associated with the edges of N_S, and the sought flow F_S must minimize the incurred total cost. Such a transshipment problem is also known as a "min-cost" flow problem [27]. Since in this work we are only interested in feasible flows for the formulated transshipment problems, we have opted to ignore this problem aspect in the provided definition.

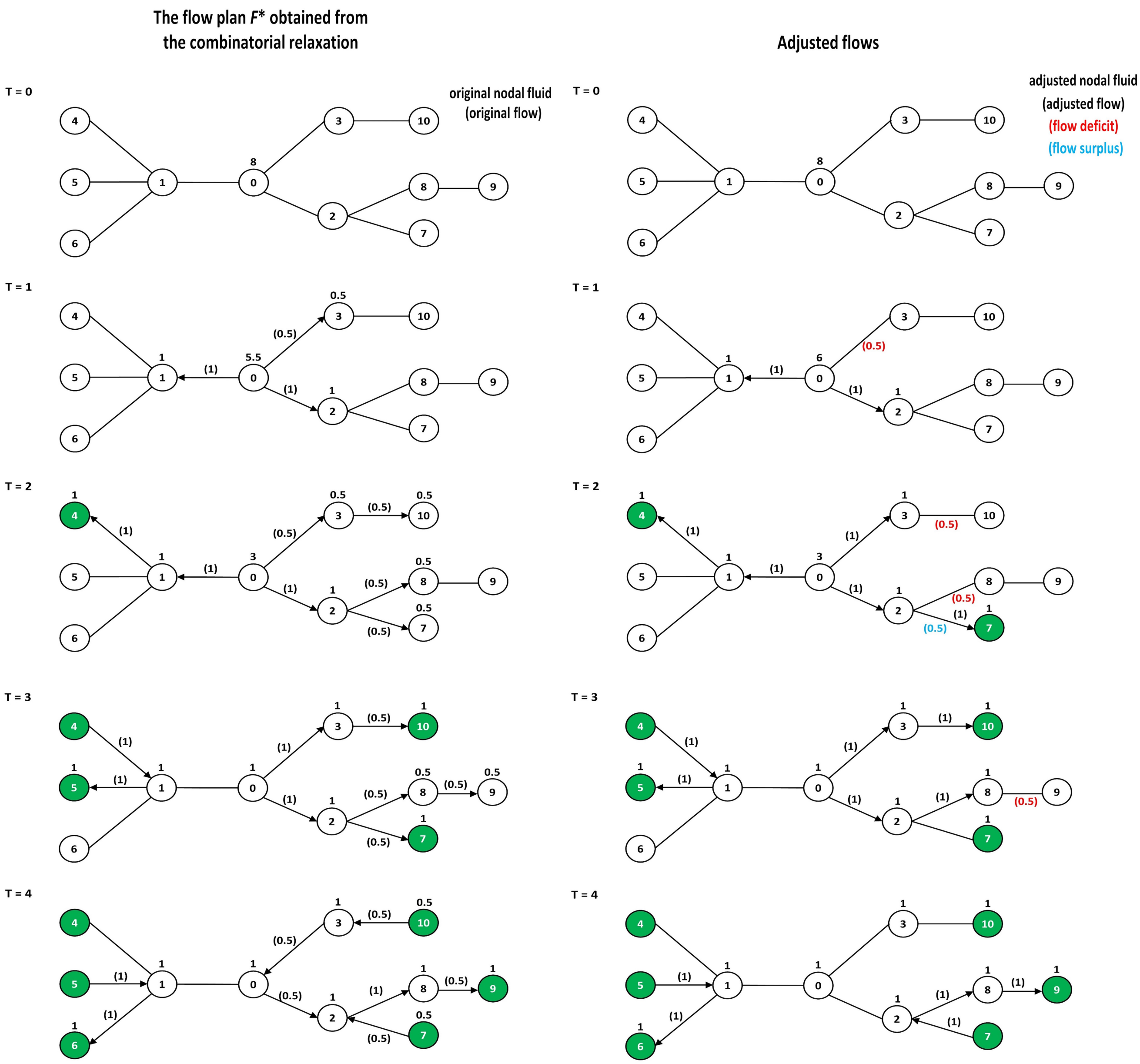

Fig. 1: The left part of this figure presents an optimal flow plan, F^*, for the combinatorial relaxation of the M-problem that is defined by the tree $\mathcal{T}$ and the robot availability that are depicted in the left-top part of the figure. The right part presents the modifications to the flow plan F^* that are effected by the procedure ADJUST which is presented in the later parts of this section.

posed demand, but also on the ability to match the available supply with the occurring demand under the connectivity and the edge-capacity constraints of the corresponding network N_S. For feasible transshipment problem instances, the set of feasible flows, F_S, constitutes a polytope representable by numbers of variables and constraints that are polynomial with respect to the size of the underlying network N_S. Furthermore, if the problem data are integral, all the extreme points of this polytope constitute *integral* flows F_S [27]; as already pointed out, this property is instrumental for the presented developments.

Next, we present (i) the transshipment problem $TSH(F)$ that is induced by a flow plan $F \in \mathcal{F}$ for the considered relaxations of the M- or TVT-problem, and (ii) the corresponding static flow F_S that constitutes a feasible solution for this problem, using a particular example that is provided in Figures 1 and 2.

The left part of Figure 1 depicts an optimal flow pan F^* for the combinatorial relaxation of the M-problem defined by (i) the tree $\mathcal{T}$ depicted at the top-left part of the figure, and (ii) a presumed availability of 8 robots at node o at period $t = 0$. The optimal makespan is 4, and the fluid content of each node

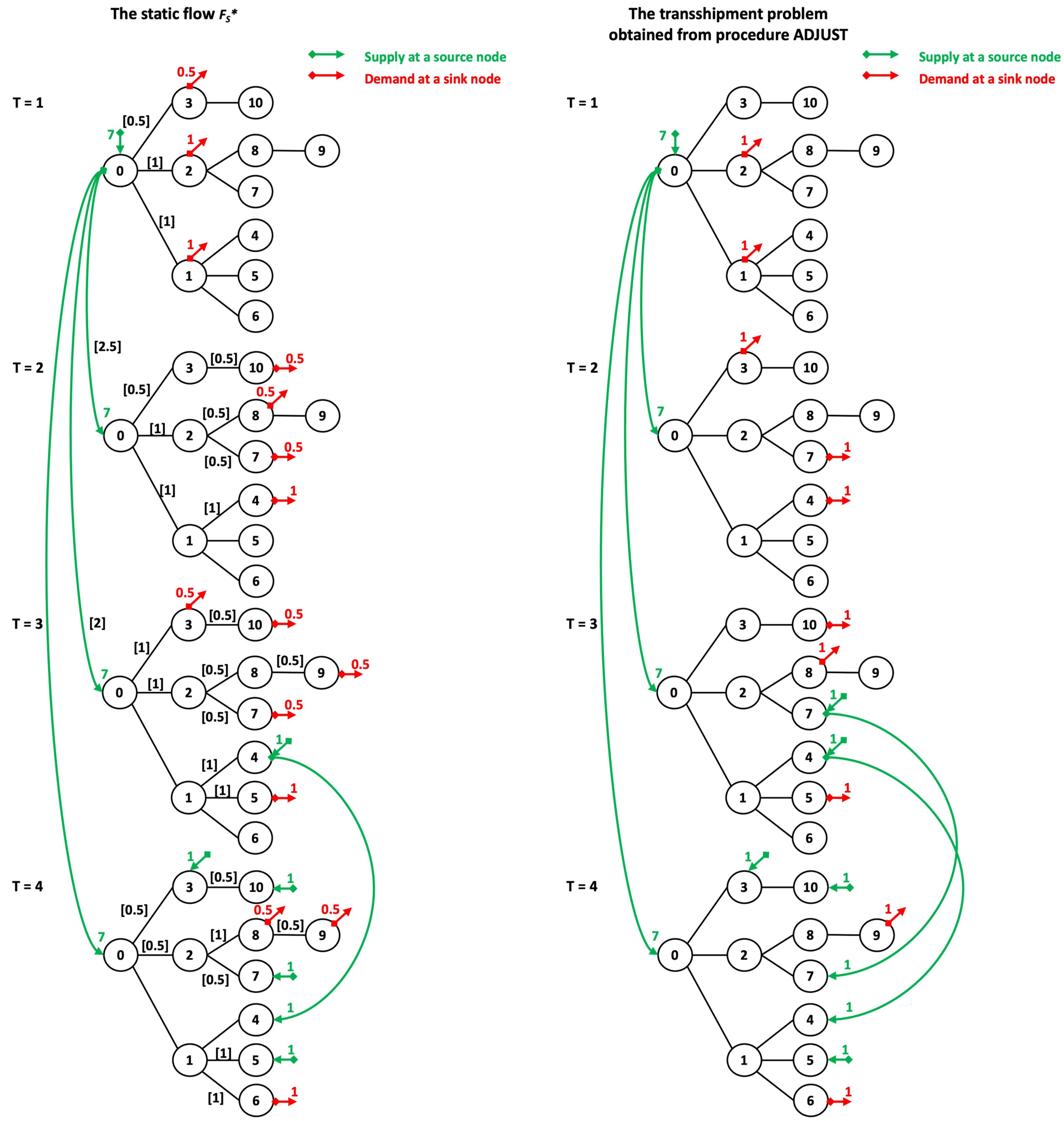

Fig. 2: The static flow F_S^* corresponding to the flow plan F^* that is depicted in the left part of Figure 1, and the transshipment problem $\widehat{TSH}(F^*)$ that is obtained from the application of procedure ADJUST to F^*.

v of $\mathcal{T}$, at each period $t \in \{0, \ldots, 4\}$, is reported next to the node; the absence of such a label implies a fluid content of zero. The values of the variables $u_{v,v',t}$, for every neighboring nodal pair $\{v, v'\}$ and every period t, is reported next to the corresponding edge; again, the absence of a reported value implies a value of 0.0.

The static flow F_S^* corresponding to the flow plan F^* is provided in the left part of Figure 2. The network N_S supporting this static flow consists of the replications of tree $\mathcal{T}$ for each period $t \in \{1, \ldots, 4\}$, together with some additional edges (the green ones) that enable the transfer of the available fluid among different periods. The replications of $\mathcal{T}$ are depicted in black in Figure 2. The edges of these trees are undirected and are implicitly assumed of unit capacity. The nodes of all these trees define the set of nodes, V_S, of the network N_S. A node of N_S corresponding to a node $v \in V$ in the replication of the tree $\mathcal{T}$ corresponding to period t, will be denoted by (v, t).

The small green arrows in Figure 2 represent the "nodal supply" for the defined static flow F_S^*, with the labels of these arrows reporting the corresponding volume of fluid. These nodal supplies are defined by the following elements in the input flow plan F^*: (i) The fluid corresponding to the presence of the eight robots at node o at period $t = 0$, after discounting for the permanent presence of a robot at the root node throughout the entire makespan $M(F)$. (ii) Any fluid concentration in any node $v \in V$ that, at some period t, is conveyed by plan F^* towards nodes outside of the subtree of $\mathcal{T}$ that is rooted in v. For structurally minimal flow plans F, nodal supplies of the second type result either from (a) the attainment of the target concentration level of one unit of fluid by some leaf node $v \in L$, or from (b) a fluid redirection along the lines that are considered in Proposition 2. The flow plan F^* depicted in Figure 1 is structurally minimal and it does not involve any fluid redirection along the lines discussed in

Proposition 2. Therefore, the nodal supplies appearing in nodes $(v,t) \in V_S$ with $v \neq o$ are due to the attainment of unit fluid concentration by some leaf node; this fact can be verified by juxtaposing (a) the distribution of the small green arrows at the nodes of the network N_S in Figure 2, with (b) the timing of the satisfaction of the leaf node requirements in Figure 1.

The directed green edges in network N_S spanning across replications of tree $\mathcal{T}$ that correspond to different periods, enable the consumption of the fluid supplies located at their tail nodes (v,t) in subsequent periods. The capacity of these green edges is equal to the available supplies at their tail nodes (v,t), and therefore, part or even the entire amount of these supplies can be consumed in some period $t' > t$.

Finally, the red arrows in Figure 2 define "nodal demand". In particular, a red arrow connected at some node (v,t) of N_S implies an increase, in the flow plan F^*, of the fluid content $x_{v,t}$ of node v at period t, from its value $x_{v,t-1}$ at period $t-1$. This increase is equal to the arrow label.

Since, the makespan $M(F)$ of any competitive flow plan F is no higher than $|V|$, the size of the network N_S of Figure 2 is polynomially related to the size of the input data of the underlying M- or TVT-problem instance. Also, it should be clear from the above discussion that the network N_S depicted in Figure 2, its edge capacities, and the nodal supplies and demands for the flow F_S^*, can be determined, by parsing the input plan F^*, in time polynomial with respect to the size of the input data.

The transshipment problem $\widehat{TSH}(F)$ and the procedure ADJUST: Next, we focus on flow plans $F \in \mathcal{F}$, for the considered relaxations of the M- and TVT-problem instances, that are structurally minimal and satisfy the integrality condition of Equation (24). For this class of flow plans, we provide a procedure that when inputted with a flow plan F from this class, it constructs a transshipment problem $\widehat{TSH}(F)$, which, in general, will be different from $TSH(F)$. Transshipment problem $\widehat{TSH}(F)$ is feasible, and its solutions induce flow plans $\hat{F}$ for the considered relaxation with a performance that is equal to – and in certain cases, even better than – the performance of the flow plan F. Furthermore, in the transshipment problem $\widehat{TSH}(F)$, all the supplies and demands are integral, and therefore, there exists an *integral* feasible static flow $\hat{F}_S$ for this problem that can be obtained efficiently by formulating and solving an LP (or by the application of even faster network-flow algorithms) [27]. The integrality of the static flow $\hat{F}_S$ implies that the flow plan $\hat{F}$ induced by this static flow for the underlying relaxation is also a feasible plan $\hat{P}$ for the original M- or TVT-problem instance. In particular, if the input flow plan F is an optimal flow plan for the combinatorial relaxation under consideration, the plan $\hat{P}$ derived by the presented procedure is an optimal plan for the original M- or TVT-problem instance. For reasons that will become clearer in the following, the presented procedure is named ADJUST.

From a more operational viewpoint, the basic ideas that underlie the construction of the transshipment problem $\widehat{TSH}(F)$ by ADJUST, are the following: ADJUST modifies the "forward" flows shipped by the input plan F from each node $v \in V$ towards its children, so that these flows transfer "unit parcels" of fluid among these nodes, while observing the Constraints (6)–(9) of the underlying MIP formulation. Hence, these modified flows are integral flows and they also result in an integral fluid content $x_{v,t}$ for every node $v \in V$ and period t. On the other hand, ADJUST assumes that the external supplies for these modified flows are the corresponding supplies that are provided by the original flow plan F. This assumption is justified in the following by showing that, thanks to the assumed properties of the input plan F, the "forward" flow plan $\hat{F}$ that results from the aforementioned modifications, attains, for each leaf node $v \in L$, a fluid concentration of one unit no later than flow plan F. Therefore, the fluid that is released by these target attainments, is as available – in terms of, both, volume and time – to support the needs of flow plan $\hat{F}$ as in the original flow plan F.[11] Finally, the aforementioned result regarding the timing of the attainment of the target fluid levels of one unit by the leaf nodes $v \in L$ under flow plan $\hat{F}$, also implies that flow plan $\hat{F}$ retains or even improves the performance of the original plan F in terms of the objective of the underlying MIP formulation.

Algorithm 1 Procedure ADJUST – INITIALIZATION

Inputs:

An instance $\langle \mathcal{R}, \mathcal{T} \rangle$ of the M- or TVT-problem
A feasible flow plan F with makespan $M(F)$

Outputs:

The network N_S, the edge capacities, and the nodal supply and demand of the transshipment problem $\widehat{TSH}(F)$

$I := \{1, \ldots, M(F)\}$;
$\forall t \in I$, ADD a replicate of $\mathcal{T}$ to N_S;
ADD a supply of $|\mathcal{R}|-1$ to node $(o,1)$ and directed edges $((o,1),(o,t))$ of capacity $|R|-1$, $\forall t \in I \setminus \{1\}$;
$\forall t \in I \cup \{0\}$, $\hat{x}_{o,t} := 1$; $\hat{x}_{v,t} := 0, \forall v \in V \backslash \{o\}$;
$\forall v \in V$, $\forall v' \in \mathcal{N}(v)$, $\forall t \in I$, $\hat{u}_{v,v',t} := 0$;
$\forall v \in V$, $\forall v' \in \mathcal{N}(v)$, $d_{v,v',0} := 0$;
$\widetilde{V}$:= ordered list of the nodes in V in increasing distance from root node o;

The detailed logic executed by ADJUST is presented in the pseudo-code of Algorithms 1 and 2. In the presented pseudo-code, the variables $u_{v,v',t}$ represent the flows of the input flow plan F, while the variables $\hat{u}_{v,v',t}$ represent the integral flows that define the modified plan $\hat{F}$ returned by ADJUST. Also, the variables $\hat{x}_{v,t}$ trace the nodal fluid concentrations that are maintained by flow plan $\hat{F}$. All the $\hat{u}_{v,v',t}$ variables and the variables $\hat{x}_{v,t}$ for $v \neq o$ are initialized to zero. On the other hand, for all t in the considered time horizon, the variables $\hat{x}_{o,t}$ are set equal to one, since any feasible plan must always maintain a robot at the root node o. During its initializing

[11] A particular issue that needs some further attention, concerns the fluid supplies in flow plan F that result from the redirection of fluid from some node $v \in V$ while there are descendant nodes of v that have not attained their target fluid levels of one unit (c.f. Proposition 2 and the discussion provided in its proof). This case is addressed explicitly by the presented procedure, in a way that guarantees the integrality and the feasibility of the resulting flows.

Algorithm 2 Procedure ADJUST – MAIN PART

for $t \in I$ **do**
  for $\forall v \in \widetilde{V}$ **do**
    if $v = o$ **then**
      $\hat{a}_{o,t} := |\mathcal{R}| - \hat{x}_{o,t} + \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{v',o,\tau} - \sum_{\tau=1}^{t-1} \sum_{v' \in \mathcal{N}(o)} \hat{u}_{o,v',\tau}$;
      if $\hat{a}_{o,t} \geq 1$ **then**
        $\forall v' \in \mathcal{N}(o)$,
          $d_{o,v',t} := d_{o,v',t-1} + u_{o,v',t} - \hat{u}_{o,v',t-1}$;
        $\widetilde{\mathcal{N}}(o) :=$ ordered list of $\mathcal{N}(o)$ in decreasing deficit;
        while $d_{o,v',t} \geq 1 \ \wedge \ \hat{a}_{o,t} \geq 1$ **do**
          $v' := HD(\widetilde{\mathcal{N}}(o))$; $\widetilde{\mathcal{N}}(o) := \widetilde{\mathcal{N}}(o) \backslash v'$;
          $\hat{u}_{o,v',t} := 1$; $\hat{a}_{o,t} := \hat{a}_{o,t} - 1$;
        end while
      end if
    else if $v \in V \backslash \big(L \cup \{o\}\big)$ **then**
      $\hat{a}_{v,t} := \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(v) \backslash p(v)} u_{v',v,\tau} - \hat{x}_{v,t-1} - \sum_{\tau=1}^{t} u_{v,p(v),\tau} + \sum_{\tau=1}^{t} \hat{u}_{p(v),v,\tau} - \sum_{\tau=1}^{t-1} \sum_{v' \in \mathcal{N}(v) \backslash p(v)} \hat{u}_{v,v',\tau}$;
      if $0 \leq \hat{a}_{v,t} < 1$ **then** $\hat{x}_{v,t} := \hat{x}_{v,t-1}$;
      else if $\hat{a}_{v,t} \geq 1$ **then**
        if $\hat{x}_{v,t-1} = 0$ **then**
          $\hat{x}_{v,t} = 1$;
          ADD a unit demand for node (v,t) in $\widehat{TSH}(F)$;
        else /* $\hat{x}_{v,t-1} = 1$ */
          $\forall v' \in \mathcal{N}(v) \backslash p(v)$,
            $d_{v,v',t} := d_{v,v',t-1} + u_{v,v',t} - \hat{u}_{v,v',t-1}$;
          $\widetilde{v} := \arg\max\{d_{v,v',t} : v' \in \mathcal{N}(v) \backslash p(v)\}$
          $\hat{u}_{v,\widetilde{v},t} := 1$;
        end if
      else /* $\hat{a}_{v,t} < 0$ */
        ADD a unit supply to node (v,t) and directed edges $\big((v,t),(v,\tau)\big)$, $\tau = t+1, \ldots, M(F)$, of unit capacity, in $\widehat{TSH}(F)$;
        $\hat{x}_{v,t} := 0$; $d_{p(v),v,t} := -\hat{a}_{v,t}$;
      end if
    else /* $v \in L$ */
      $\hat{a}_{v,t} := \sum_{\tau=1}^{t} \hat{u}_{p(v),v,\tau}$;
      if $\hat{a}_{v,t} = 1$ **then**
        ADD a unit demand to node (v,t) in $\widehat{TSH}(F)$; $v' := v$; $\widetilde{V} := \widetilde{V} \backslash v'$;
        ADD a unit supply to node $(v', t+1)$, and directed edges $\big((v',t+1),(v',\tau)\big)$, $\tau = t+2, \ldots, M(F)$, of unit capacity, in $\widehat{TSH}(F)$;
        while $\big\{\mathcal{N}(p(v') \backslash \{v', p\big(p(v')\big)\}\big\} \cap \widetilde{V} = \emptyset$ **do**
          $v' := p(v)$; $\widetilde{V} := \widetilde{V} \backslash v'$;
          ADD a unit supply to node $(v', t+1)$, and directed edges $\big((v',t+1),(v',\tau)\big)$, $\tau = t+2, \ldots, M(F)$, of unit capacity, in $\widehat{TSH}(F)$;
        end while
      end if
    end if
  end for
end for

phase, ADJUST also adds to the constructed network N_S a replicate of the tree $\mathcal{T}$ for each period $t \in \{1, \ldots, M(F)\}$, and attaches a supply of $|\mathcal{R}|-1$ to node $(o,1)$ of N_S, together with the directed edges that enable the potential consumption of this supply at subsequent periods. Finally, during the initializing phase, ADJUST also computes an ordered list, $\tilde{V}$, of the elements of V, where the nodes are ordered in increasing distance from the root note o. This list is used in the main part of the procedure for computing the flows $\hat{u}_{v,v',t}$ and the nodal fluid concentrations $\hat{x}_{v,t}$, at each period t, by proceeding from the root node towards the leaves of $\mathcal{T}$. Furthermore, list $\tilde{V}$ is dynamically updated at each period t so that the only nodes considered at this period are the nodes with descendant leaf nodes that have not met yet their target of a unit-flow concentration.

The main part of ADJUST consists of a double iteration, with the external iteration running over the considered time horizon $\{1, \ldots, M(F)\}$, and the inner iteration processing each remaining node in $\tilde{V}$ in order to determine the corresponding flows $\hat{u}_{v,v',t}$ and the nodal fluid concentration $\hat{x}_{v,t}$ for the running period t.

More specifically, for each period $t \in \{1, \ldots, M(F)\}$, ADJUST first computes the flows $\hat{u}_{o,v',t}$, for each child v' of o, according to the following logic: First, it computes the fluid availability $\hat{a}_{o,t}$ at node o at period t, based on: (i) the initial supply defined by the $|\mathcal{R}|-1$ robots at period 0; (ii) the total flow returned to node o via its children by the original plan F, over the time interval $\{1, \ldots, t\}$; and (iii) the total flow that has been shipped from node o to its children by the modified plan $\hat{F}$, over the time interval $\{1, \ldots, t-1\}$; i.e., the availability $\hat{a}_{o,t}$ accounts for the entire inflow to node o up to period t according to flow plan F and the total outflow from this node up to period $t-1$ according to the modified plan $\hat{F}$. If $\hat{a}_{o,t} \geq 1$, then ADJUST (i) computes, for every child $v' \in \mathcal{N}(o)$, a “deficit”

$$d_{o,v',t} = \sum_{\tau=1}^{t} u_{o,v',\tau} - \sum_{\tau=1}^{t-1} \hat{u}_{o,v',\tau} \ , \tag{28}$$

(ii) orders these children in decreasing deficit, and (iii) scans the constructed ordered list shipping a unit of flow to each encountered child v' until either it encounters a child v' with $d_{o,v',t} < 1$ or the remaining fluid is not adequate for an integral shipment.

The processing of the internal nodes $v \in V \setminus \big(L \cup \{o\}\big)$, for any period $t \in \{1, \ldots, M(F)\}$, is also driven by the computation of a fluid availability $\hat{a}_{v,t}$. In this case, $\hat{a}_{v,t}$ is determined by: (i) the total flow returned to node o via its children by the original plan F, over the time interval $\{1, \ldots, t\}$, minus any flow that is shipped from node v to its parent node v' by flow plan F over the same time interval; (ii) the fluid stored at node v by the modified flow plan $\hat{F}$ during the past $t-1$ periods; and (iii) the total outflow from node v towards its children over the time interval $\{1, \ldots, t-1\}$ according to the modified plan $\hat{F}$. A fluid availability of $\hat{a}_{v,t} \in [0,1)$ maintains the fluid content of node v at its previous level, and it does not incur any outflow from node v towards its children in the considered period t. A fluid availability of $\hat{a}_{v,t} \geq 1$ leads to the storage of a unit of fluid at node v, if its current content is zero. Also, this increase of the fluid content of v by one unit is represented in the constructed problem $\widehat{TSH}(F)$ by the attachment of a unit of demand to node (v,t). If node v already possesses a unit of fluid, an availability $\hat{a}_{v,t} \geq 1.0$ leads to the shipment of a unit of fluid to the child $\tilde{v}$ of v that presents the highest current deficit $d_{v,\tilde{v},t}$; the involved deficits are computed as in the case of the root node o. Finally, a negative fluid availability $\hat{a}_{v,t}$ signifies a drawing of fluid from node v towards its parent node $p(v)$. The working assumptions regarding the input plan F imply that the conditions and the arguments presented in the proof of Proposition 2 hold; i.e., the entire subtree emanating from node v is empty of any fluid, and it must also hold that

$$\hat{x}_{v,t-1} = 1.0 \tag{29}$$

$$\hat{x}_{v,t} = \hat{x}_{v,t-1} - \Bigg(\sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(v) \setminus p(v)} u_{v',v,\tau} - \sum_{\tau=1}^{t} u_{v,p(v),\tau} + \sum_{\tau=1}^{t} \hat{u}_{p(v),v,\tau} - \sum_{\tau=1}^{t-1} \sum_{v' \in \mathcal{N}(v) \setminus p(v)} \hat{u}_{v,v',\tau} \Bigg) = -\hat{a}_{v,t} \tag{30}$$

ADJUST (i) turns the unit of fluid located at node v at the beginning of period t to a unit of supply attached to the node (v,t) of the constructed network N_S, (ii) sets $\hat{x}_{v,t}$ to zero, and (iii) it accounts for any remaining fluid at node v at the end of period t by expressing this amount as a deficit $d_{p(v),v,t}$;[12] this deficit will be taken into consideration whenever a new unit of fluid becomes available at node $p(v)$ during the subsequent periods.

The processing of a leaf node $v \in L \cap \tilde{V}$, for any period t, also starts with the estimation of a fluid availability $\hat{a}_{v,t}$. In this case, $\hat{a}_{v,t}$ is defined by the total flow shipped to node v by its parent node $p(v)$ during the interval $\{1,\ldots,t\}$, according to the modified flow plan $\hat{F}$. From the description of the processing of the nodes $v' \in V \setminus L$, we know that the aforementioned flow occurs in single units of fluid. As soon as the considered leaf node v receives such a unit flow, it reaches its target. Hence, this unit amount of fluid is turned into a unit of supply that becomes available in the subsequent periods $t' \in \{t+1,\ldots,M(F)\}$. In addition, Constraint (9) implies that, at period t, every node on the path $\pi(o,v)$ possesses a unit of fluid, as well. Any node v' on the path $\pi(o,v)$ that has no further descendants with unattained target fluid levels, is removed from list $\tilde{V}$, and its fluid is turned into a unit of supply that will become available at period $t+1$, as well. Finally, ADJUST traces the initial increase of the fluid level at the considered node v from zero to one during period t, by attaching a unit of demand to the corresponding node (v,t) of the network N_S.

The right part of Figure 1 depicts the flow plan $\hat{F}^*$ that is computed by the procedure ADJUST when inputted with the flow plan F^* depicted in the left part of Figure 1. This figure also reports the flow deficits for the various nodes that are maintained by the procedure during this computation. The transshipment problem $\widehat{TSH}(F^*)$ that is eventually constructed by ADJUST in this case, is depicted in the right part of Figure 2.

Analysis: Next we establish the properties of the modified flow $\hat{F}$ and the transshipment problem $\widehat{TSH}(F)$ developed by ADJUST, that were claimed in the preamble of this section. The following lemma has a central role in the establishment of these properties.

Lemma 1: Consider the application of procedure ADJUST on a feasible flow plan F for the combinatorial relaxation of some M- or TVT-problem instance, that is structurally minimal and satisfies the integrality condition of Equation (24). Then, it holds that[13]

$$\forall v \in V \setminus \{o\},\ \forall t \in \{1,\ldots,M(F)\},\quad \sum_{\tau=1}^{t} \hat{u}_{p(v),v,\tau} \ \geq \left\lfloor \sum_{\tau=1}^{t} u_{p(v),v,\tau} \right\rfloor - \sum_{v' \in \pi(o,p(v))} I_{\{x_{v',t} < 1\}} \tag{31}$$

The proof of Lemma 1 is provided in the companion electronic supplement of the paper. Its significance is revealed in the proof of the following proposition.

Proposition 4: Consider the application of the procedure ADJUST on a feasible flow plan F for the combinatorial relaxation of some M- or TVT-problem instance, that is structurally minimal and satisfies the integrality condition of Equation (24). Then, the modified flow plan $\hat{F}$ developed by ADJUST is feasible, and

$$\forall v \in L,\quad C(v;\hat{F}) \ \leq\ C(v;F) \tag{32}$$

Proof: First we notice that Constraint (9) implies that, for any leaf node $v \in L$, at period $C(v;F)$, all nodes v' on the path $\pi(o,v)$ have $x_{v',C(v;F)} = 1.0$. But then, Equation (31) implies that

$$\sum_{\tau=1}^{C(v;F)} \hat{u}_{p(v),v,\tau} \ \geq\ 1.0$$

and establishes Equation (32).

Next we argue the feasibility of the flow plan $\hat{F}$. The "forward" flows that ship fluid from any node v towards its children, satisfy Constraints (6)–(9) by construction. Furthermore, the satisfaction of the integrality condition of Equation (24) by the input plan F ensures that the "forward" total flows $\sum_{\tau=1}^{M(F)} u_{v,v'\tau}$, for any nodal pair $v \in V,\ v' \in \mathcal{N}(v) \setminus \{p(v)\}$, are attainable under the integrality requirement for the corresponding flows $\sum_{\tau=1}^{M(F)} \hat{u}_{v,v'\tau}$ that is forced by ADJUST.

In order to fully establish the feasibility of the flow plan $\hat{F}$, it remains to show that flow plan $\hat{F}$ is also compatible with the fluid availability that is defined by the original plan F, since it is this availability that is used in the computation of the function $\hat{a}_{v,t},\ v \in V \setminus L,\ t \in \{1,\ldots,M(F)\}$. From the provided description of ADJUST (c.f. also the first part of the proof of Lemma 1), it is clear that ADJUST accounts correctly the fluid availability that is defined by the initial

[12] We also notice, for completeness, that under the working assumptions, $d_{p(v),v,t-1} = 0.0$.

[13] For any predicate ϕ, $I_{\{\phi\}}$ is the corresponding indicator function; i.e., $I_{\{\phi\}}$ is equal to one if the predicate ϕ is true, and it is equal to zero otherwise.

presence of the $\mathcal{R}$ robots at the root node o. Next, we show that the fluid availability coming from nodal supplies other than the root node o in the ADJUST computation is indeed realizable under the modified plan $\hat{F}$.

We establish this result through an induction on the time-sequence of the realizations of these additional supplies in the input flow plan F. The working assumptions of Proposition 4 imply that the first occurrence of these additional supplies in flow plan F takes place when the first leaf node, let's say node v, attains the target level of one unit of fluid. For this case, Equation (32) implies that leaf node v has attained its target fluid level by period $C(v;F)$ in plan $\hat{F}$, as well. Hence, the fluid supply for flow plan $\hat{F}$ that is assumed in ADJUST due to this particular event in flow plan F, is valid.

If all the further nodal supplies generated by plan F are due to similar target attainments by some other leaf nodes $v' \in L$, then the validity of any particular supply in this sequence with respect to the modified flow plan $\hat{F}$ results from the validity of all the preceding supplies in the considered sequence and an argument similar to that in the previous paragraph.

In the remaining case, there will exist nodal supplies resulting from the redirection of the fluid that is available at some internal node v of tree $\mathcal{T}$ towards its parent node, in spite of the presence of leaf nodes that are descendants of node v and have not attained their target fluid levels of one unit. Consider the first occurrence of such a supply in plan F. The structural minimality of F together with Proposition 2 imply that, at the corresponding period t, every node v' on the path $\pi(o,v)$ has $x_{v',t} = 1.0$. Hence, Equation (31) implies that $\hat{x}_{v,t} = 1$. Furthermore, the remark following Proposition 2 in Section III implies that $\hat{x}_{v',t} = 0$, for every node $v' \neq v$ in the subtree of $\mathcal{T}$ that is rooted at node v. Hence, the unit of fluid that is located in node v at period t is converted to a unit of supply in the ADJUST computation, and the corresponding nodal variable $\hat{x}_{v,t}$ is set equal to zero in plan $\hat{F}$, without violating Constraint (9). ADJUST also considers the possibility of only a partial drawing of some fluid from node v at period t by accounting for any fluid remaining in node v as a positive deficit $d_{p(v),v,t}$ that will impact accordingly the distribution of any fluid that will become available at node $p(v)$ at the subsequent periods.

Finally, the compatibility of the flow plan $\hat{F}$ with any subsequent supplies resulting from similar fluid redirections at other internal nodes v', can be argued in the same manner with the case discussed above, since Proposition 2 implies that such redirections occur first at nodes that are located further from the root node o in the underlying tree $\mathcal{T}$; hence, all the descendants of any such node v' are empty at the period of the corresponding redirection. □

The next theorem is a direct implication of Proposition 4, and it constitutes a more formal statement of the second part of the results that are claimed in the preamble of this section.

Theorem 1: Consider the application of the procedure ADJUST on a feasible flow plan F for the combinatorial relaxation of some M- or TVT-problem instance $\mathcal{M} = \langle \mathcal{R}, \mathcal{T} \rangle$, that is structurally minimal and satisfies the integrality condition of Equation (24). Then, there exists a plan $P \in \mathcal{P}$ for the underlying M- or TVT-problem instance with an objective value that is no worse than the objective value of flow plan F. Furthermore, plan P can be computed from the input problem instance $\mathcal{M} = \langle \mathcal{R}, \mathcal{T} \rangle$ and the considered flow plan F in time polynomial with respect to $|V|$, i.e., the number of nodes in tree $\mathcal{T}$.

Proof: Consider the flow plan $\hat{F}$ that is obtained by applying ADJUST on flow plan F. Proposition 4 ensures that flow plan $\hat{F}$ is feasible for the underlying relaxation, and its objective value is no worse than the objective value of flow plan F. Furthermore, since flow plan $\hat{F}$ involves unit forward-flow transfers among the different nodes of tree $\mathcal{T}$, the transshipment problem $TSH(\hat{F}) = \widehat{TSH}(F)$, that is also constructed by ADJUST, has integral supplies, demands and edge capacities for the generated network N_S. Hence, the static flows that constitute extreme feasible solutions for $\widehat{TSH}(F)$ are integral, and we can obtain such an extreme feasible solution, $\tilde{F}_S$, by solving the corresponding min-cost flow problem with all edge costs of N_S set to zero. Finally, from the semantics that define $\widehat{TSH}(F)$, $\tilde{F}_S$ can be translated directly to a plan $P \in \mathcal{P}$ for the underlying M- or TVT-problem instance, and the objective value of P is equal to the objective value of $\hat{F}$.

Also, as remarked in the introduction of the transshipment problem $TSH(F)$, the size of the network N_S is polynomially related to $|V|$, and the set of its feasible solutions is a polytope representable by sets of variables and the constraints that are polynomially sized with respect to the size of the net N_S. Furthermore, the number of iterations resulting by the two for-loops in the main part of ADJUST is $O(|I| \cdot |V|) = O(|V|^2)$, and it is also clear that each of these iterations has polynomial complexity in $|V|$. Finally, the computation of the flow plan $\tilde{F}_S$ is a polynomial task with respect to the size of N_S, and therefore, with respect to $|V|$. Hence, the sought plan P can be obtained in time polynomial with respect to $|V|$. □

Some implications of Theorem 1: As pointed out in the introductory section, the results of Theorem 1 suggest an alternative algorithm for solving any given instance of the M- or the TVT-problem. This algorithm will solve first the corresponding relaxation of Section III, and subsequently it will convert the obtained optimal solution to a plan P for the original problem instance. The necessary steps for this approach are suggested by the proof of Theorem 1, and the complete procedure is formally stated in Algorithm 3. Since Steps 2–6 of Algorithm 3 can be executed in polynomial time with respect to the size of the underlying M- or TVT-problem instance, and the relaxed MIP that is solved in Step 1 involves a much smaller number of integer variables, Algorithm 3 is expected to provide significant computational advantage with respect to the direct solution of the original MIP formulations of Section II, especially for the harder problem instances. We demonstrate and assess this advantage through a numerical experiment that is presented in the next section.

The results of Theorem 1 also imply that, for any given M- or TVT-problem instance, we can test the existence of a plan P with pre-specified visitation times $C(v;P),\ v \in L$, by testing the feasibility of an LP that is defined by Constraints (4)–(9) and the additional constraints that impose the specified visitation times. These tests can be especially useful in the

Algorithm 3 The new solution method for the M- and TVT-problems

Inputs:
An instance $\langle \mathcal{R}, \mathcal{T} \rangle$ of the M- or TVT-problem
Outputs:
An optimal plan P^*

1) Solve the corresponding combinatorial relaxation that was introduced in Section III, to obtain a structurally minimal, optimal flow plan F^*;
2) Check whether flow plan F^* satisfies the integrality condition of Proposition 3, and if not, set $F^* := \bar{F}^*$, where the flow plan $\bar{F}^*$ must be obtained by formulating and solving the LP of Proposition 3;
3) Run F^* through procedure ADJUST to get the transshipment problem $\widehat{TSH}(F^*)$;
4) Use the LP formulation of the min-cost flow problem that is defined by $\widehat{TSH}(F^*)$ and zero unit-flow costs associated with every edge of the network N_S, or any other efficient algorithm for the min-cost flow problem [27], in order to get an integral feasible solution $\tilde{F}_S$ for this problem;
5) Convert the obtained static flow $\tilde{F}_S$ to a plan P^* for the input M- or TVT-problem;
6) Return P^*.

adaptation to the solution of the M- and TVT-problems of various heuristic methods borrowed from the general combinatorial optimization theory [28], [22]. The development of such heuristic methods is part of our ongoing investigations.

V. A numerical experiment assessing the relative efficacy of Algorithm 3

In order to demonstrate and assess the computational advantage of Algorithm 3 in the solution of the M- and TVT-problems of Section II versus the original MIP formulations of Section II, we conducted a numerical experiment that compared statistically, through a set of paired t-tests, the computation times and/or the MIP gaps attained by (i) the MIP formulations of Section II and (ii) the corresponding combinatorial relaxations of Section III. This comparison is motivated, and justified, by the fact that Step 1 in Algorithm 3 is, by far, the most computationally demanding task in the entire computation.

The particular instantiations of the M-and TVT-problems that were considered in this experiment, and the obtained results from the aforementioned comparisons, are tabulated in Table I. More specifically, column $|V|$ in Table I reports the number of nodes in the trees, $\mathcal{T}$, of the considered instantiations. For each $|V|$ value, we considered three possible levels for $|\mathcal{R}|$: (a) *low (L)*, where $|\mathcal{R}|$ was set equal to the depth of the corresponding tree $\mathcal{T}$ plus 10% of $|V|$; (b) *moderate (M)*, where $|\mathcal{R}|$ was set equal to the depth of the corresponding tree $\mathcal{T}$ plus 40% of $|V|$; and (c) *high (H)*, where $|\mathcal{R}|$ was set equal to $|V|$. Eventually, for each pair ($|V|$, $|\mathcal{R}|$-level) shown in Table I, we generated five replications by generating randomly the corresponding trees $\mathcal{T}$; the detailed procedure that generated these trees is provided in the companion electronic supplement.

For each generated replication, we formulated the MIPs of Section II for the corresponding M- and TVT-problem instance, and also the MIPs corresponding to their combinatorial relaxations that were presented in Section III.[14] Also, for an easier interpretation of the obtained results, the actual objective function used in the MIP formulations and their relaxations for the M-problem instances was

$$\min \left(\bar{T} + 1 - \sum_{t \in \{1,\ldots,\bar{T}\}} s_t \right)$$

and the objective function used in the MIP formulations and their relaxations for the TVT-problem instances was

$$\min \left(|L| \cdot (\bar{T}+1) - \sum_{v \in L} \sum_{t \in \{1,\ldots,\bar{T}\}} y_{v,t} \right)$$

We ran each of the two formulations corresponding to each generated problem instance through CPLEX, with a time budget of (i) one hour (3600 secs) for problem instances with up to 75 nodes for tree $\mathcal{T}$, and (ii) two hours (7200 secs) for the larger problem instances. For each replication, we registered (a) the involved computational time, and (b) the MIP gap returned by CPLEX upon its completion or termination; the reported MIP gaps are defined by the following equation:

$$\text{MIP gap} = 100 \times \left(1.0 - \frac{\text{best lower bound obtained by CPLEX}}{\text{best objective value obtained by CPLEX}} \right) \quad (33)$$

The observed statistics that were attained by the original MIP formulations of Section II, are reported in the columns of Table I labeled by "O", averaged over the five replications for each ($|V|$, $|\mathcal{R}|$-level)-pair. The corresponding statistics attained by the combinatorial relaxations of Section III are reported in the columns of Table I labeled by "PR".

For a more thorough assessment of the computational advantage that is attained by the considered relaxations versus the original MIP formulations, we also conducted a set of paired t-tests for each of the two statistics that were introduced in the previous paragraphs, Each paired t-test was performed on the corresponding performance data obtained for the five replications generated for each ($|V|$, $|\mathcal{R}|$-level)-pair and each of the two problem versions (i.e., M and TVT). Table I also reports the corresponding p-values of these t-tests.[15] More specifically, if CPLEX managed to complete its computation within the allocated time budget for all five replications and for both MIPs, the reported p-value is that concerning the t-test

[14] We also notice, for completeness, that the MIP formulations corresponding to TVT-problem instances, and their relaxing MIPs, included the additional constraints of Equation (22) in [26], that were established in that work as useful valid inequalities for the TVT-problem. Similarly, the MIPs corresponding to M-problem instances, and their relaxing MIPs, included the additional "symmetry-breaking" constraints that are suggested in the closing paragraph of Section 3 in [26].

[15] We remind the reader that the p-value corresponding to an observed value for a test statistic is the lowest level of significance for which the test-statistic value results in a rejection of the null hypothesis [29]. Equivalently, $100(1-p)$ is the highest level of confidence for accepting the alternative hypothesis that we want to verify through the test.

TABLE I: A comparison of the computation times, the MIP gaps and the inflation ratios attained by (i) the original MIP formulations of Section II and (ii) their combinatorial relaxations of Section III, for various instantiations of the M- and TVT-problems.

$\|V\|$	$\|\mathcal{R}\|$	**Makespan**							**Total Visitation Time**						
		The average of computation time (sec)		**The average of MIP gap (%)**		**The average of inflation ratios**		**p-value**	**The average of computation time (sec)**		**The average of MIP gap (%)**		**The average of inflation ratios**		**p-value**
		O	**PR**	**O**	**PR**	**O**	**PR**		**O**	**PR**	**O**	**PR**	**O**	**PR**	
10	L	0	0	0.00	0.00	1.00	1.00	0.08	0	0	0.00	0.00	1.00	1.00	0.11
	M	0	0	0.00	0.00	1.00	1.00	0.04	0	0	0.00	0.00	1.00	1.00	0.07
	H	0	0	0.00	0.00	1.00	1.00	0.25	0	0	0.00	0.00	1.00	1.00	0.11
20	L	**329**	**131**	0.00	0.00	1.00	1.00	0.12	**5**	**4**	0.00	0.00	1.00	1.00	0.09
	M	**720**	**35**	1.43	0.00	1.01	1.00	0.37	1	1	0.00	0.00	1.00	1.00	0.28
	H	**449**	**56**	0.00	0.00	1.00	1.00	0.19	1	1	0.00	0.00	1.00	1.00	0.25
30	L	3600	3600	25.59	20.95	1.34	1.26	0.06	**1043**	**872**	3.12	2.32	1.03	1.02	0.12
	M	**9**	**4**	0.00	0.00	1.00	1.00	0.08	**4**	**2**	0.00	0.00	1.00	1.00	0.04
	H	**10**	**5**	0.00	0.00	1.00	1.00	0.09	**3**	**2**	0.00	0.00	1.00	1.00	0.02
40	L	3600	3600	50.05	37.77	2.00	1.61	0.01	**3600**	**2759**	9.08	6.31	1.10	1.07	0.00
	M	2885	2885	21.15	16.80	1.27	1.20	0.02	1418	1148	2.36	1.76	1.02	1.02	0.37
	H	2883	2881	20.90	17.60	1.26	1.21	0.06	1399	1243	2.85	2.27	1.03	1.02	0.37
50	L	3600	3600	55.82	52.56	2.26	2.11	0.04	3600	3600	29.64	22.77	1.42	1.29	0.12
	M	3091	2992	24.30	18.85	1.32	1.23	0.03	**2938**	**2566**	8.29	4.35	1.09	1.05	0.05
	H	3043	2982	20.91	18.14	1.26	1.22	0.21	**2916**	**2305**	7.04	3.76	1.08	1.04	0.09
75	L	3600	3600	83.59	70.85	**6.09**	**3.43**	0.03	3600	3600	49.29	46.57	1.97	1.87	0.11
	M	**3600**	**2984**	42.28	29.78	1.73	1.42	0.03	**3600**	**2943**	18.98	13.52	1.23	1.16	0.11
	H	**3600**	**3026**	34.63	27.01	1.53	1.37	0.03	**3429**	**2752**	13.91	9.90	1.16	1.11	0.11
100	L	7200	7200	80.19	74.30	**5.05**	**3.89**	0.03	7200	7200	71.27	55.58	**3.48**	**2.25**	0.05
	M	7200	7200	52.68	45.68	2.11	1.84	0.01	7200	7200	25.47	19.60	1.34	1.24	0.14
	H	7200	7200	52.36	43.46	2.10	1.77	0.00	7200	7200	22.57	18.15	1.29	1.22	0.01
125	L	7200	7200	88.72	81.98	**8.87**	**5.55**	0.07	7200	7200	71.08	61.31	**3.46**	**2.58**	0.05
	M	7200	7200	68.88	55.13	**3.21**	**2.23**	0.08	7200	7200	34.19	20.61	1.52	1.26	0.12
	H	7200	7200	60.68	46.93	2.54	1.88	0.06	7200	7200	34.17	19.19	1.52	1.24	0.10
150	L	7200	7200	91.34	90.63	**11.55**	**10.67**	0.14	7200	7200	76.63	68.79	**4.28**	**3.20**	0.18
	M	7200	7200	78.03	68.15	**4.55**	**3.14**	0.28	7200	7200	50.51	25.81	2.02	1.35	0.10
	H	7200	7200	77.42	56.61	**4.43**	**2.30**	0.03	7200	7200	49.33	22.88	1.97	1.30	0.06

performed on the observed computational times; otherwise, the reported p-value is that concerning the t-test performed on the observed MIP gaps.

Finally, Table I also provides an alternative interpretation of the MIP gaps that are reported in it, which enables an easier understanding of the extent of the suboptimality of the obtained solutions for the corresponding MIPs. These results are provided under the title ”Average Inflation Ratios”, and they are obtained from the corresponding MIP gaps by rearranging Equation (33) as follows:

$$\frac{1}{1-\left(\frac{\text{MIP gap}}{100}\right)} = \left(\frac{\text{best objective value obtained by CPLEX}}{\text{best lower bound obtained by CPLEX}}\right) \quad (34)$$

The right hand-side of Equation (34) is the *“inflation ratio”* of the attained objective value with respect to the best obtained lower bound, and this ratio provides another surrogate measure for the suboptimality of the obtained objective value. As a more concrete example, Equation (34) implies that for a MIP gap of 50%, the inflation ratio is equal to 2.0; i.e., the attained objective value is at most two times higher than the optimal value. Similarly, a MIP gap of 75% implies that the attained objective value is at most four times higher than the optimal value. Naturally, the exact solution of a MIP formulation results in a MIP gap of 0% and an inflation ratio of 1.0. Finally, the relationship between the MIP gap and the inflation ratio defined by Equation (34) is highly nonlinear, with the inflation ratio arising fast to some very high values when the MIP gap takes values in the interval $(50, 100)$.

The perusal of Table I renders clear that, under the specified time budget, the MIPs corresponding to the combinatorial relaxations for, both, the M- and TVT-problems either (i) will derive optimal plans faster than the original MIP formulations for these problems, or (ii) will return suboptimal plans with stronger quality certificates than the corresponding plans obtained by the original MIP formulations. Table I highlights in boldface those cases where the aforementioned dominance of the relaxed MIPs is more prominent in the quoted values. Furthermore, it is worth-noticing that this dominance is especially evident, and useful, in the case of the harder problem instances where the original MIPs fail to provide solutions with low inflation ratios with respect to the optimal objective value.

We also notice, for completeness, that the obtained solutions for the relaxing MIPs must be post-processed through the remaining steps of Algorithm 3. But as already pointed out in the introduction of this experiment, the computational cost of these additional steps is much smaller than the solution of the MIP. This is especially true for the larger and the harder problem instances where, according to the previous remarks, this method is expected to provide the highest computational advantage.

Next, we provide some additional interesting observations on the results that are reported in Table I. It is evident from

Table I that low $|\mathcal{R}|$ levels relative to $|V|$ increase significantly the difficulty of, both, the M- and the TVT-problem. Characteristically, as $|V|$ increases, MIPs corresponding to problem instances with low $|\mathcal{R}|$ levels start exhausting the allocated time budget for their solution earlier than their counterparts with moderate or high $|\mathcal{R}|$ levels. An intuitive explanation of this fact is that the robot scarcity implied by the low $|\mathcal{R}|$ levels, gives rise to more critical choices regarding the pertinent use of this limited resource over the various parts of the underlying tree $\mathcal{T}$, compared to the case where the robots are in higher abundance and, therefore, many parts of $\mathcal{T}$ can be visited simultaneously.[16]

Also, the juxtaposition of the two major parts of Table I corresponding to the M- and the TVT-problem, suggests that the solution of the MIPs corresponding to the TVT-problem instances, and to their relaxations, is a computationally easier task than the solution of the counterparts of these MIPs for the respective M-problem instances. One possible factor contributing to this observation is the inclusion in the MIP formulations for the TVT-problem instances, and in their relaxations, of the additional constraints that are discussed in Footnote 14. But it is possible that the relative expediency that is observed in the solution of the MIPs corresponding to the TVT-problem instances and their relaxations, is also due to the particular objective function that is employed by these MIPs. A more profound understanding of this phenomenon is part of our future investigations.

Concluding the discussion of this experiment, we also notice that the generated MIPs were solved by CPLEX in Python, on a laptop with i7-8850H 2.6GHz CPU, 16 GB RAM, and running Mac OS.

VI. Conclusions

This paper has developed some strong combinatorial relaxations for the M- and TVT-problems that were introduced in [20]. Furthermore, the theoretical analysis of these relaxations provided additional insights and results for the M- and TVT-problems that can lead to new solution methods and pertinent heuristic algorithms for them. The last part of the paper detailed one such solution method, and demonstrated and assessed its computational advantage with respect to the original MIP formulations of [20] through a numerical experiment.

Our future work will seek the development of novel heuristic algorithms for the M- and the TVT-problems, by adapting to these problems some broader ideas and results coming from combinatorial optimization theory [28], [22]. These heuristic algorithms will be especially useful for dealing with the larger problem instances of the M- and the TVT-problems that are not easily tackled by the MIP-based methods that have been pursued in [20] and in this work. As remarked in the closing part of Section IV, the analytical results of this paper can provide important ingredients in the development of the sought heuristic algorithms, by supporting the effective and efficient execution of certain tests that must be performed by them.

[16]Of course, even in this case, this potential concurrency is still limited by the unit capacities of the imposed zoning scheme and the particular structure of the underlying tree $\mathcal{T}$ for each problem instance.

Another task in our future research concerns the extension of the investigation of the M-and TVT-problems to guidepath networks with more general structures than trees.[17] When it comes to the results of the current paper, it is interesting to see whether and/or how they can be extended to that broader problem setting. The corresponding investigation, and the potential challenges that might be encountered in it, will also lead to a more profound understanding of the dependency of the results developed in this work to the presumed tree structure of the underlying guidepath network.

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