An electronic supplement to the manuscript "A strong combinatorial relaxation for the problem of Min-Time Coverage in Constricted Environments"

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***Abstract*—This document constitutes an "electronic supplement" to the manuscript entitled "A strong combinatorial relaxation for the problem of Min-Time Coverage in Constricted Environments", that is co-authored by the same authors, providing the proofs of some supporting results in the analytical developments of the main manuscript, and some further details on the organization of the numerical experiment that is presented in Section V of that document.**

Proof of Proposition 2

First we notice that, under the working assumptions, flow plan F^* is focused. Hence, since $x_{v,t} < x_{v,t-1}$ at period t, there exists at least one leaf node v' of tree $\mathcal{T}$ that is a descendant of node v and has $C(v'; F^*) < t$. Let $L' \subset L$ be the set of all these nodes, and $\tau \equiv \max\{C(v'; F^*) : v' \in L'\}$. We prove the result of Proposition 2 by considering two types of descendants of node v in $\mathcal{T}$.

Case 1: First we focus on descendants of node v that are on a path $\pi(v, v'),\ v' \in L'$, and have all their descendant leaf nodes in the set L' (i.e., visited by period t). We claim that all these nodes are empty of any fluid at period t. Indeed, if this claim is not true, then there exists a node v'' in the considered class of nodes with $x_{v'',t} > 0$ and either being a leaf node of $\mathcal{T}$ or having all its descendants empty. Hence, its fluid could have been used to cover at least part of the needs that are addressed by the fluid amount $(x_{v,t-1} - x_{v,t})$ drawn from node v. Furthermore, since all the descendant leaf nodes of v'' are in L', the fluid drawn from node v'' need not be replaced in the future periods of plan F^*. Therefore, the considered modification of flow plan F^* will result in a flow plan $\tilde{F}$ with equal performance to plan F^* but incurring less variation of the nodal fluid concentrations, something that contradicts the structural minimality of F^*.

Case 2: Next, we consider the case where every descendant node v'' of node v that belongs in the class addressed in Case 1 has been depleted from its fluid by period t, but there are still non-empty nodes, other than node v, in the subtree $\mathcal{T}'$ of $\mathcal{T}$ that is rooted at node v and consists of all the paths $\pi(v, v')$ leading to nodes v' with $C(v'; F^*) > t$. Consider the plan $\tilde{F}$ that, at period t, substitutes for the fluid $(x_{v,t-1} - x_{v,t})$ by drawing from some nodes of the subtree $\mathcal{T}'$ that are furthest away from node v. The feasibility of the plan F^* implies that such substitution is feasible with respect to Constraints 6–9 of the combinatorial relaxation. Furthermore, this substitution does not alter the total amount of fluid contained in the subtree $\mathcal{T}'$ at period t, and furthermore, it does not compromise the performance of plan F^* with respect to the visitation times of the leaf nodes of $\mathcal{T}'$, since the fluid that is provided by plan F^* for the restoration of the fluid in node v to the unit value, in some period(s) $t' > t$, can be forwarded immediately to the nodes from which this fluid was drawn by plan $\tilde{F}$. Plan $\tilde{F}$ is unfocused, and therefore, does not belong in the set of plans that satisfy the working assumptions of Proposition 2. But by using arguments similar to those employed in the proof of Proposition 1 (also c.f. reference [25] in the main document), it is possible to construct another flow plan $\hat{F}$ that will redirect the fluid to be drawn from the subtree $\mathcal{T}'$ by plan $\tilde{F}$, directly to the destination node(s) of the fluid $(x_{v,t-1} - x_{v,t})$ that is drawn by plan F^* from node v at period t. Flow plan $\hat{F}$ is focused, and, in fact, it would result in a smaller value for the cost defined by Eq. 21 in the main document than plan F^*. This fact contradicts the structural minimality of plan F^*, and proves Proposition 2 for this case, as well.

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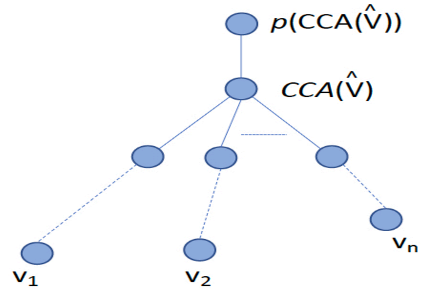

Fig. 1: A maximal set of leaf nodes, $\hat{V} = \{v_1, v_2, \ldots, v_n\}$, having the same closest common ancestor, denoted by $CCA(\hat{V})$.

Proof of Proposition 3

The first part of Proposition 3 is clearly true for any subtree of $\mathcal{T}$ that constitutes a simple path to some leaf node, without any branching in its internal nodes. For subtrees containing branching nodes, we establish the first part of Proposition 3 through an inductive argument that proceeds from the branching nodes of tree $\mathcal{T}$ that are closest to its leaves, towards the root node o.

Hence, for the base case of this induction, consider a subtree $\mathcal{T}'$ of $\mathcal{T}$ that possesses the structure depicted in Figure 1 in this document. The attainment of the target level of one unit of fluid by any leaf node v_i in this subtree, requires the provision of $l(CCA(\hat{V}), v_i)$ units of fluid from node $CCA(\hat{V})$ towards the corresponding path $\pi(CCA(\hat{V}), v_i)$. Furthermore, since the considered plan F^* is focused, it does not incur the provision of any additional flow to path $\pi(CCA(\hat{V}), v_i)$. Hence, Proposition 3 holds for node $CCA(\hat{V})$.

For the induction step, consider an internal node $v \in V$ of $\mathcal{T}$ and suppose that Proposition 3 holds for all the subtrees of $\mathcal{T}$ that are rooted to the children of v; i.e., there is a structurally minimal, optimal flow plan $\hat{F}$ with $\sum_{t=1}^{\bar{T}} \sum_{v'' \in V: p(v'')=v'} u_{v',v'',t} \in \mathbb{Z}^+$ for every node v' in these subtrees. In particular, this inclusion is true for every child $\hat{v}$ of node v. However, in general, it is possible that, in flow plan $\hat{F}$, node v provides only a fraction of the flow $\sum_{t=1}^{\bar{T}} \sum_{v'' \in V: p(v'')=\hat{v}} u_{\hat{v},v'',t}$ that is conveyed from a child $\hat{v}$ of v towards its own children, and the remaining part of this total flow results from fluid transfers among the subtrees emanating from node $\hat{v}$, through node $\hat{v}$ itself.

Next, consider the case where at least one of these internal transfers involves the conveyance of a fractional amount of fluid, $\tilde{f}$, from some subtree emanating from node $\hat{v}$, let's say $\mathcal{T}'$, towards the other subtrees emanating from $\hat{v}$. Constraints 6–9 imply the existence of a flow plan $\tilde{F}$ that (a) is obtained from flow plan $\hat{F}$ by (i) augmenting the fractional amount of fluid $\tilde{f}$ provided by subtree $\mathcal{T}'$ to an integral amount of fluid, equal to the $\lceil \tilde{f} \rceil$, and (ii) reducing accordingly the total flow $\sum_{t=1}^{\bar{T}} u_{v,\hat{v},t}$ conveyed by node v to node $\hat{v}$, and (b) has $C(\tilde{v}; \tilde{F}) = C(\tilde{v}; F^*)$ for each leaf node $\tilde{v}$ in subtree $\mathcal{T}'$. The last equality is true because Constraint 9 implies that for any leaf node $\tilde{v}$ of $\mathcal{T}'$, and any valid flow plan F, at period $C(\tilde{v}; F)$, the total amount of fluid deployed on the path $\pi(\hat{v}, \tilde{v})$ is equal to $l(\hat{v}, \tilde{v})$, which is integral. Hence, the availability of an extra fractional amount of fluid in tree $\mathcal{T}'$ cannot expedite the attainment of the target fluid concentration of 1.0 by any leaf node $\tilde{v}$ of $\mathcal{T}'$.

Also, by drawing the additional fluid to be extracted from tree $\mathcal{T}'$ according to the logic that was delineated in the proof of Proposition 2, we can ensure that the resulting flow plan $\tilde{F}$ remains structurally minimal.

The above adjustment can be applied to every fractional internal transfer taking place in any subtree of $\mathcal{T}$ rooted to a child $\hat{v}$ of node v, through node $\hat{v}$. In this way, we can obtain a structurally minimal, optimal flow plan F^* where the total flow conveyed from node v to each of its children is integer, and the first part of Proposition 3 has been proven.

The second part of Proposition 3 is an immediate consequence of the arguments that were provided in the earlier part of this proof.

Proof of Lemma 1

We prove the result of Lemma 1 through an induction on the elements of the set $V \setminus \{o\}$ that proceeds from the children of the root node o towards the leaf nodes.

For the base case, first notice that, in the input plan F, the flows $\sum_{\tau=1}^{t} u_{o,v,\tau}$, for every period t and node $v \in \mathcal{N}(o)$, are supported by a total fluid availability of $\xi_{o,t} := |\mathcal{R}| + \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{v',o,\tau}$. On the other hand, in the modified flow plan $\hat{F}$, the respective flows $\sum_{\tau=1}^{t} \hat{u}_{o,v,\tau}$ trace the quantities $\lfloor \sum_{\tau=1}^{t} u_{o,v,\tau} \rfloor$ based on a total fluid availability of $\hat{\xi}_{o,t} := |\mathcal{R}| - 1 + \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{v',o,\tau}$, at each period t. Furthermore, in flow plan F,

$$\forall t \in \{1, \ldots, M(F)\},$$
$$x_{o,t} = |\mathcal{R}| + \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{v',o,\tau} - \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{o,v',\tau} \tag{1}$$

Equation 1 together with the fact that $x_{o,t} > 0, \forall t$, imply that

$$\forall t \in \{1, \ldots, M(F)\}, \quad \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{o,v',\tau} - 1 < \hat{\xi}_{o,t} \tag{2}$$

From Equation 1 we also have that

$$\forall t \in \{1, \ldots, M(F)\},$$
$$x_{o,t} \geq 1 \implies \hat{\xi}_{o,t} \geq \sum_{\tau=1}^{t} \sum_{v' \in \mathcal{N}(o)} u_{o,v',\tau} \tag{3}$$

But then, for the children $v \in \mathcal{N}(o)$ of the root node o, Equation 31 in the main document is an implication of Equations 2, 3, and the aforementioned tracing of the quantities $\lfloor \sum_{\tau=1}^{t} u_{o,v,\tau} \rfloor$ by the respective cumulative allocations $\sum_{\tau=1}^{t} \hat{u}_{o,v,\tau}$, based on the corresponding deficits $d_{o,v,t}$ (c.f. Algorithm 2).

For the inductive step, consider a node $v \in V \setminus (\{o\} \cup \mathcal{N}(o))$, and suppose that Equation 31 in the main document holds for its parent node $p(v)$; i.e.,

$$\forall t \in \{1, \ldots, M(F)\}, \quad \sum_{\tau=1}^{t} \hat{u}_{p(p(v)),p(v),\tau} \geq \left\lfloor \sum_{\tau=1}^{t} u_{p(p(v)),p(v),\tau} \right\rfloor - \sum_{v' \in \pi(o,p(p(v)))} I_{\{x_{v',t} < 1\}} \tag{4}$$

From the flow balance equations that are observed by flow plan F, we also have:

$$\forall t \in \{1, \ldots, M(F)\},$$
$$\sum_{\tau=1}^{t} u_{p(p(v)),p(v),\tau} = x_{p(v),t} + \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} u_{p(v),v',\tau} \tag{5}$$

Equation 5 further implies that

$$\forall t \in \{1, \ldots, M(F)\}, \quad \left\lfloor \sum_{\tau=1}^{t} u_{p(p(v)),p(v),\tau} \right\rfloor \geq x_{p(v),t} + \left\lfloor \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} u_{p(v),v',\tau} \right\rfloor \tag{6}$$

From Equations 4 and 6, we have:

$$\forall t \in \{1, \ldots, M(F)\},$$
$$\sum_{\tau=1}^{t} \hat{u}_{p(p(v)),p(v),\tau} + \sum_{v' \in \pi(o,p(p(v)))} I_{\{x_{v',t} < 1\}} \geq x_{p(v),t} + \left\lfloor \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} u_{p(v),v',\tau} \right\rfloor \tag{7}$$

From the logic that computes the modified flow plan $\hat{F}$ in ADJUST, we have:

$$\forall t \in \{1, \ldots, M(F)\},$$
$$\sum_{\tau=1}^{t} \hat{u}_{p(p(v)),p(v),\tau} = \hat{x}_{p(v),t} + \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} \hat{u}_{p(v),v',\tau} \tag{8}$$

From Equations 7 and 8, we have:

$$\forall t \in \{1, \ldots, M(F)\},$$
$$\sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} \hat{u}_{p(v),v',\tau} + \left(\hat{x}_{p(v),t} - x_{p(v),t}\right) + \sum_{v' \in \pi(o,p(p(v)))} I_{\{x_{v',t} < 1\}} \geq \left\lfloor \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} u_{p(v),v',\tau} \right\rfloor \tag{9}$$

The deficit-tracing implemented by ADJUST at the internal nodes of tree $\mathcal{T}$ implies that $\hat{x}_{p(v),t} > x_{p(v),t}$ only at periods $t \in \{1, \ldots, M(F)\}$ for which $x_{p(v),t} < 1$, and in that case, $\hat{x}_{p(v),t} - x_{p(v),t} < 1$. But then, Equation 9 further implies that

$$\forall t \in \{1, \ldots, M(F)\},$$
$$\sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} \hat{u}_{p(v),v',\tau} + \sum_{v' \in \pi(o,p(v))} I_{\{x_{v',t} < 1\}} \geq \left\lfloor \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \sum_{\tau=1}^{t} u_{p(v),v',\tau} \right\rfloor \geq \sum_{\substack{v' \in \mathcal{N}(p(v)) \\ v' \neq p(p(v))}} \left\lfloor \sum_{\tau=1}^{t} u_{p(v),v',\tau} \right\rfloor \tag{10}$$

Next, suppose that the considered node v violates the inequality of Equation 31 in the main document; i.e.,

$$\exists t \in \{1, \ldots, M(F)\},$$
$$\sum_{\tau=1}^{t} \hat{u}_{p(v),v,\tau} \leq \left\lfloor \sum_{\tau=1}^{t} u_{p(v),v,\tau} \right\rfloor - \sum_{v' \in \pi(o,p(v))} I_{\{x_{v',t} < 1\}} - 1 \tag{11}$$

When combined with Equation 10, Equation 11 implies that the existence of another child $\tilde{v}$ of node $p(v)$ with

$$\sum_{\tau=1}^{t} \hat{u}_{p(v),\tilde{v},\tau} \geq \left\lfloor \sum_{\tau=1}^{t} u_{p(v),\tilde{v},\tau} \right\rfloor + 1 \tag{12}$$

But Equations 10–12 violate the logic of ADJUST that defines the flows $\hat{u}_{v,v',t}$ for nodes $v \neq o$, based on the corresponding deficits $d_{v,v',t}$. This contradiction establishes the result of Lemma 1 for the inductive step, and concludes the proof.

The algorithm generating the trees for the various problem instances in the numerical experiment of Section V

The algorithm presented in this section generates a tree with a specified number of nodes V.

The algorithm starts by generating a random number $p_0 \in [0, 1]$ which is used for determining the extent of branching that takes place at the different nodes of the constructed tree. It also initializes the node set V by inserting in it the root node 0, and the edge set E by setting it equal to the empty set.

Algorithm : Generation of a tree $\mathcal{T}$ with a specified number of nodes

Inputs:
  The number of nodes $|V|$.

Outputs:
  A tree instance $\mathcal{T} := \langle V, E \rangle$.

Initialization:
  Generate a random number $p_0 \in [0, 1]$;
  $V := \{0\}$; $E := \{\}$;
  $i := 0$; $j := 1$;

while $j < |V|$ **do**
  Generate $p \in [0, 1]$;
  if $p \leq p_0$ **then**
    $V := V \cup \{j\}$;
    $E := E \cup \{(i, j)\}$;
    $j := j + 1$;
  else
    if $i + 1 < j$ **then**
      $i := i + 1$;
    else /* $i + 1 = j$ */
      $V := V \cup \{j\}$;
      $E := E \cup \{(i, j)\}$;
      $i := j$;
      $j := j + 1$;
    end if
  end if
end while

The while-loop in the above algorithm adds the remaining $|V| - 1$ nodes in tree $\mathcal{T}$ by using the indices i and j as follows: At each iteration, the candidate node to enter tree $\mathcal{T}$ is node j, while index i specifies the parent node of the newly added node. Then, starting with $i = 0$, the algorithm keeps adding children to node i, one per iteration, as long as the random number $p \in [0, 1]$ that is generated at each of these iterations is less than or equal to p_0. Otherwise, i is increased to $i+1$, and if node $i+1$ has been added already in the generated tree $\mathcal{T}$, then the above procedure repeats itself on this node. If, on the other hand, node $i+1$ is not in the already generated tree, then it must be the next node to be added in the tree (i.e., $i+1 = j$) and in order to preserve the connectivity of the generated tree, it is added as a child to the current node i. Subsequently, the newly added node $j = i + 1$ becomes the new parent node, and j is also increased by one unit to properly represent the next node to enter tree $\mathcal{T}$. The entire process terminates when tree $\mathcal{T}$ has obtained the specified number of nodes $|V|$.

Closing the presentation of this algorithm, we also notice that the generated tree $\mathcal{T}$ is expanded in a "breadth-first" sense. Hence, for a fixed value of $|V|$, smaller values of the generated number p_0 will result in deeper trees and a smaller number of leaf nodes, while larger values of p_0 will have the opposite effects. By randomizing p_0, we enable all these possibilities.