Orderable groups and 3-manifolds

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a group is LO if $\exists$ a strict total order $<$ of its elts s.t.

$$g<h \Rightarrow fg<fh \quad \forall f,g,h \in G$$

(RO same, just pick LO)

a group is BO if it's LO and $g<h \Rightarrow gf<hf$

a group is LO if $\exists P \subset G$ s.t.

(i) $P \cdot P \subset P$

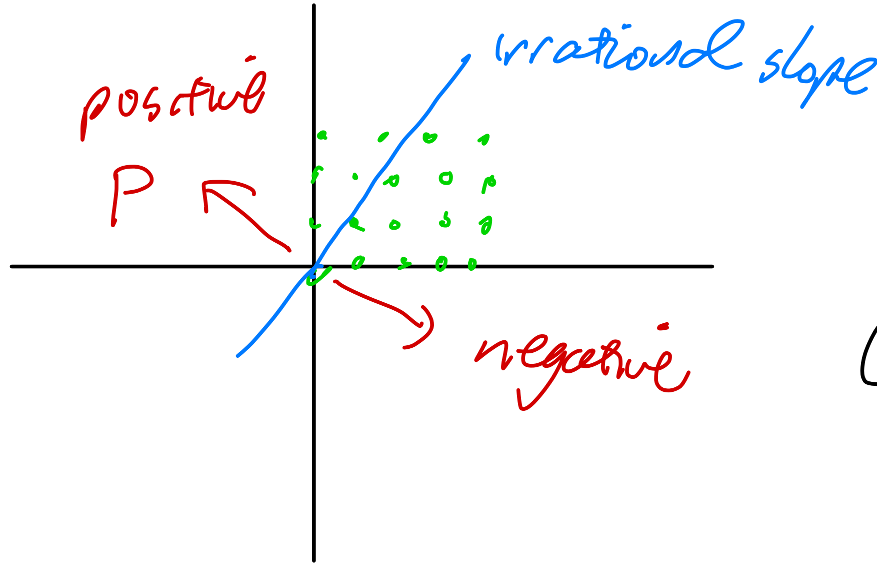
(ii) $P \sqcup P^{-1} = G \setminus \{id\}$ $\Big\} \Rightarrow$ LO

(iii) $gPg^{-1} \subset P$ $\forall g \in G$ then its BO

$$< \longmapsto \{g \in G : g > id\}$$

$$P \longmapsto g<h \Leftrightarrow g^{-1}h \in P$$

Ex: $\mathbb{Z}^2$



(x_1, x_2) normal to line

$(m,n) > (0,0)$ if $m x_1 + n y_2 > 0$

Ex: F = free gp on $\{x_1, x_2, \ldots\}$ (Magnus)

$$\Lambda = \mathbb{Z}[[X_1, X_2, \ldots]]$$

define $\mu : F \longrightarrow \mathbb{Z}[[X_1, X_2, \ldots]]$

$$\mu(x_i) = 1 + X_i \qquad \mu(x_i^{-1}) = 1 - X_i + X_i^2 - \ldots$$

lemma: μ is injective

lemma: The group of elements of Λ of the form $1 + O(1)$ is bi-orderable

$\therefore$ F biorderable

Prop: Suppose $P_K \subset K$, $P_H \subset H$ are positive cones, and

$$\{id\} \to K \xrightarrow{i} G \xrightarrow{q} H \to \{id\}$$ is a

short exact sequence

then $P = i(P_K) \cup q^{-1}(P_H)$ is a pos cone in G

ex: $G = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$

$$1 \to \langle y \rangle \to G \to \mathbb{Z} \to 0$$
$$\langle y \rangle \cong \mathbb{Z}$$

$\Rightarrow$ G is LO, not BO

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a group action of G on X is effective if $g \cdot x = x \ \forall x \in X$
$\Rightarrow g = id$

Thm: A group is LO iff it acts effectively on a totally ordered set (order-preserving)

Pf: ($\Rightarrow$) act on itself

($\Leftarrow$) Suppose G acts on $(X,<)$

Choose a well-order $\prec$ on X (Axiom of choice!)

For each $g \in G \setminus \{id\}$ set

$$x_g = \min_{\prec} \{x \in X \mid g \cdot x \neq x\}$$

declare $g \in P \Leftrightarrow g \cdot x_g > x_g$

check P pos cone ✓

Thm: If G countable, then G LO $\Leftrightarrow \exists\, G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$

Pf: ($\Leftarrow$) last result

($\Rightarrow$) G LO

enumerate $G = \{id = g_0, g_1, \ldots\}$

build $t : G \to \mathbb{R}$

$t(g_0) = 0$

suppose $t(g_i)$ done for $i \leq k$

$$t(g_{k+1}) = \begin{cases} \max\{t(g_1), \dots, t(g_k)\}+1 & \text{if } g_{k+1} > g_i \ \forall i=1,\dots,k \\ \min\{ \quad " \quad " \quad \}-1 & \text{if } g_{k+1} < g_i \ \forall i=1,\dots,k \\ \dfrac{t(g_j)+t(g_i)}{2} & \text{if } g_j < g_{k+1} < g_i \\ & \text{and no other } g_l \text{ between } g_j, g_i \text{ for } l=1,\dots,k \end{cases}$$

define $\rho: G \to \mathrm{Homeo}_+(\mathbb{R})$ by

- $\rho(g)(t(h)) = t(gh)$
- extend to $\overline{t(G)}$ by continuity
- If gaps remain, extend affinely ✓

<u>Non LO?</u>

torsion groups are not LO

try to find more interesting

Consider $G = \langle a, b \mid babab a^{-1} b^2 a^{-1}, \; ababab^{-1} a^2 b^{-1} \rangle$

$\wr$

π_1 (Weeks manifold)

hyperbolic so torsion free

not LO! rewrite rel^{n}s to get $(ba)^2 = a^{-1} b a^{-2} b$

suppose $a > id$

<u>Case 1</u>: $b < id$

Case 2: $id < b < a$

Case 3: $id < a < b$

for Case 2: $a^{-1}b < id$

$$\underbrace{(ba)^2}_{\text{pos}} = (a^{-1}b)\, \underbrace{a^{-1}}_{\text{neg}}(a^{-1}b)$$

now try other cases

A property that semigroup $P \subset G$ can have:

(*) for every finite set $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$ $\exists\, \varepsilon_i = \pm 1$ st. $id \notin sg(P \setminus \{id\}, g_1^{\varepsilon_1}, \dots, g_n^{\varepsilon_n})$

semigroup gen by

<u>Thm</u>: Given a semigroup $Q \subset G$, there is a pos. cone $P \subset G$ with $Q \setminus \{id\} \subset P$ iff Q has (*)

<u>Proof</u>: ($\Rightarrow$) easy just choose ε_i so $g_i^{\varepsilon_i} > 0$

($\Leftarrow$) suppose Q has (*)

Observe: if $g \in G \setminus \{id\}$ then one of $sg(Q \setminus \{id\}, g)$ or

$sg(Q \setminus \{id\}, g^{-1})$ has $(*)$)

If not $\exists\, h_1, \dots, h_n$ and $f_1, \dots, f_m$ s.t.

$id \in sg(Q \setminus \{id\}, g, h_1^{\varepsilon_1}, \dots, h_n^{\varepsilon_n})$

no matter the ε_i's and

$id \in sg(Q \setminus \{id\}, g^{-1}, f_1^{\nu_1}, \dots, f_m^{\nu_m})$

no matter the ν_i's

$\Rightarrow id \in sg(Q \setminus \{id\}, g^{\varepsilon}, h_1^{\varepsilon_1}, \dots h_n^{\varepsilon_n}, f_1^{\nu_1}, \dots f_m^{\nu_m})$

no matter ε_i's, ν_i's so Q fails $(*)$

Set $M = \{$ semigroups $P \subset G$, with $Q \subset P$ satisfying $(*)\}$

M nonempty, ordered by inclusion, and chains have upperbounds by taking unions

$\Rightarrow \exists$ maximal $P \in M$

<u>Check</u> $P \setminus \{id\}$ is a pos. cone

$G \setminus \{id\} = P \cup P^{-1}$

$P \cap P^{-1} = \{id\}$ ✓

Cor: G is LO iff $\forall \{g_1, \dots, g_n\} \in G \setminus \{id\}$

$$\exists\, \varepsilon_i = \pm 1 \text{ s.t. } id \notin sg(g_1^{\varepsilon_1}, \dots, g_n^{\varepsilon_n})$$

Cor: A group is LO iff all f.g. subgroups are LO

Cor: All torsion free abelian groups are LO

Thm (Burns-Hale):

A group G is LO iff
$\forall$ f.g. $H \leq G$ $\exists$ a homom $H \to L$ (onto)
where L is nontrivial LO group

Proof: argue by induction that if the subgp
cond is satisfied, we can always
find the necessary ε_i's for
any $\{g_1, \dots, g_n\} \subset G \setminus \{id\}$

Can you do a BO version of any of this?

mostly Yes but no Burns-Hale

e.g. $G = \langle x, y \mid x y x^{-1} = y^{-1} \rangle$

$$1 \to \mathbb{Z} \to G \to \mathbb{Z} \to 1$$