Lecture 2

def^{n}: a circular order of a set S is

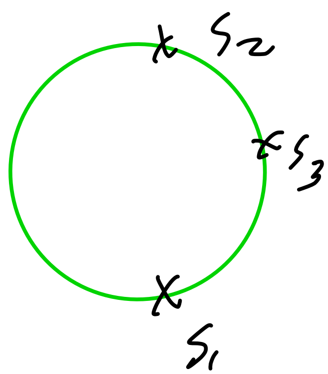
$$c : S^3 \to \{0, \pm 1\}$$

s.t.

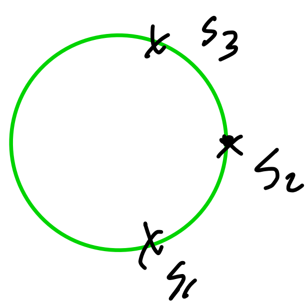
(i) $c^{-1}(0) = \{(s_1, s_2, s_3) \mid s_i = s_j \text{ for } i \neq j\}$

(ii) $c(s_2, s_3, s_4) - c(s_1, s_3, s_4) + c(s_1, s_2, s_4) - c(s_1, s_2, s_3) = 0$

idea:



$c(s_1, s_2, s_3) = -1$



$c(s_1, s_2, s_3) = +1$

condition (ii) says when add s_4 there are restrictions a group is circular ordered (CO) if $\exists\, c : G^3 \to \{0, \pm 1\}$ s.t.

(i), (ii) above and

(iii) $c(g_1, g_2, g_3) = c(hg_1, hg_2, hg_3)$ $\quad h, g_1, g_2, g_3 \in G$

example: S^1, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$

example: G is LO with order $<$

define $c(g_1, g_2, g_3) = \operatorname{sign} \sigma$

$\sigma \in S_3$ is unique permutation s.t.

$$g_{\sigma(1)} < g_{\sigma(2)} < g_{\sigma(3)}$$

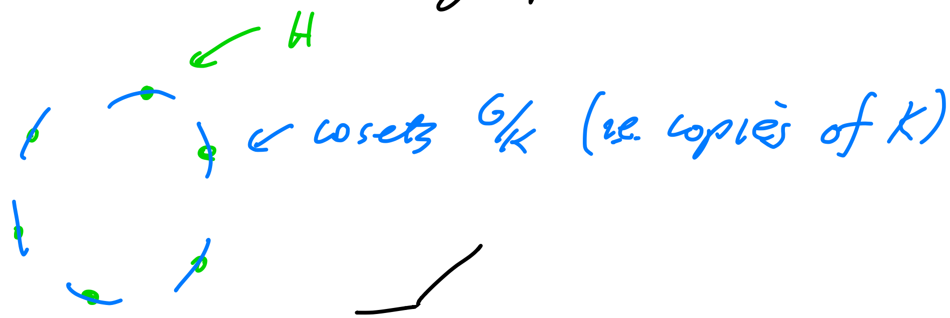
Prop:

if K is LO and H is CO and

$$1 \to K \to G \to H \to 1$$

then G admits a lexicographic CO

Pf:



Thm: a countable group G is CO iff $\exists\ G \hookrightarrow \mathrm{Homeo}_+(S^1)$

Pf: ($\Rightarrow$) copy the LO proof

($\Leftarrow$) just need to CO group $\mathrm{Homeo}_+(S^1)$

choose $p \in S^1$, and consider the "short exact sequence"

$$1 \to \underset{\substack{\text{LO since it acts}\\ \text{on } S^1 \setminus \{p\} \simeq \mathbb{R}}}{\mathrm{Stab}(p)} \to \mathrm{Homeo}_+(S^1) \to \underset{\text{CO}}{\mathrm{Homeo}_+(S^1) / \mathrm{Stab}(p)} \to 1$$

example: $PSL(2,\mathbb{R})$ is CO

e.g. $PSL(2,\mathbb{R})$ is isometries of $\mathbb{H}$

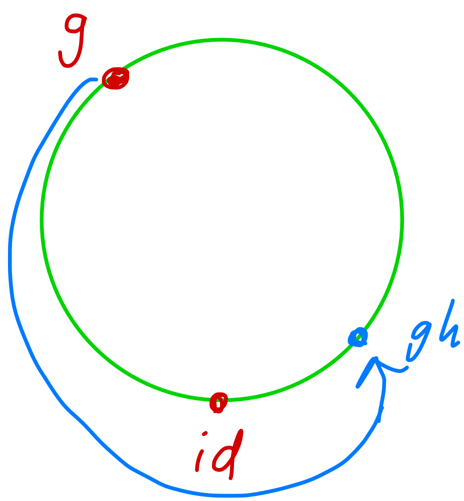
defn: a CO of G is a function $f: G^2 \to \mathbb{Z}$ s.t.

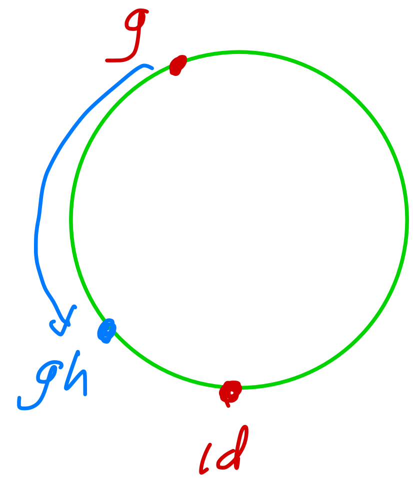
(i) $f(id, g) = f(g, id) = 0 \quad \forall g \in G$

(ii) $f(h,k) - f(gh,k) + f(g,hk) - f(g,h) = 0 \quad \forall g,h,k \in G$

(iii) $f(g,h) \in \{0,1\}$

(iv) $f(g, g^{-1}) = 1 \quad \forall g \in G \setminus \{id\}$





then $f(g,h) = 1$ 	then $f(g,h) = 0$

$\exists$ a bijection between c's and f's

$$c \longmapsto f^c(g,h) = \begin{cases} 0 & \text{if } g = id \text{ or } h = id \\ 1 & gh = id \text{ with } g \neq id \\ \frac{1}{2}(1 - c(id, g, gh)) & \text{otherwise} \end{cases}$$

also formula $f \longmapsto c^f$ (see notes)

2 objects : $H^2(G;\mathbb{Z}) \rightleftarrows \mathcal{E}(G;\mathbb{Z})$

= equivalence classes of $\mathbb{Z}$-central extensions

What is $H^2(G;\mathbb{Z})$?

$$Z^2(G;\mathbb{Z}) = \{ f : G^2 \to \mathbb{Z} \mid f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2) = 0$$
$$f(id,g) = f(g,id) = 0 \}$$

$$B^2(G;\mathbb{Z}) = \{ f : G^2 \to \mathbb{Z} \mid \exists h : G \to \mathbb{Z} \text{ s.t. } f(g_1,g_2) = h(g_1) + h(g_2) - h(g_1g_2)$$
$$\text{and } h(id) = 0 \}$$

$$H^2(G;\mathbb{Z}) = Z^2(G;\mathbb{Z}) / B^2(G;\mathbb{Z})$$

for $\mathcal{E}(G;\mathbb{Z})$

Suppose we have central extensions

$$\begin{tikzcd} & & H_1 \arrow[dd, dashed, "\phi"] \arrow[dr, "q_1"] & & \\ 0 \arrow[r] & \mathbb{Z} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & G \arrow[r] & 0 \\ & & H_2 \arrow[ur, "q_2"'] & & \end{tikzcd}$$

these are equivalent if $\exists \phi : H_1 \to H_2$ s.t. this commutes

the correspondence: $H^2(G;\mathbb{Z}) \leftrightarrow \mathcal{E}(G;\mathbb{Z})$

given $f : G^2 \to \mathbb{Z}$

define $\tilde{G}_f$ by taking $G \times \mathbb{Z}$ and equipping it with multiplication

$$(g,a)(h,b) = (gh, a+b+f(g,h))$$

the central copy of $\mathbb{Z}$ is $\langle (id, 1) \rangle$

the quotient to G is $(g,a) \mapsto g$

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start with $0 \to \mathbb{Z} \xrightarrow{i} H \xrightarrow{q} G \to 1$

choose $s : G \to H$ s.t. $s(id) = id$ and

$\varphi \circ s(g) = g \quad \forall g \in G$

define $f(g,h)$ to be the element

$s(g)\, s(h)\, s(gh)^{-1} \in i(\mathbb{Z})$

Claim: the bijection $H^2(G;\mathbb{Z}) \Leftrightarrow \mathcal{E}(G;\mathbb{Z})$

$id \longleftrightarrow G \times \mathbb{Z}$

Prop: If G is a group $f: G^2 \to \mathbb{Z}$ is CO then

$\tilde{G}_f$ is a LO group

Pf: as a set, $\tilde{G}_f = G \times \mathbb{Z}$ positive cone

Set $P = \{(g,a) \mid a \geq 0\} \setminus \{(id, 0)\}$

exercise: $(g,a)^{-1} = ?$

Cor: Suppose that G has a CO $f: G^2 \to \mathbb{Z}$ s.t.

$$[f] = id \in H^2(G;\mathbb{Z})$$

then G is LO

Pf: since $[f] = id \in H^2(G;\mathbb{Z})$,

then $\tilde{G}_f \cong G \times \mathbb{Z}$

LO

so G a subgroup of LO

$\therefore G$ is LO

Thm:

Suppose f is a CO of G and $[f] \in H^2(G;\mathbb{Z})$ has order k. Then $\exists H \leq G$, H is LO st.

$$G/H \cong \mathbb{Z}/k\mathbb{Z}$$

"Pf": Consider the cocycle $kf: G^2 \to \{0, k\}$

since f has order k, $[kf] = id \in H^2(G;\mathbb{Z})$

i.e. $\exists\, \eta: G \to \mathbb{Z}$ $\eta(id) = 0$ and

$$kf(g,h) = \eta(g) + \eta(h) - \eta(gh) \quad \forall g,h \in G$$

$G \xrightarrow{\eta} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/k\mathbb{Z}$ is a homomorphism

and the H we want is kernel

$$H \xrightarrow{\sim} \widetilde{G}_f$$

example: $G = \langle a, b \mid bababa^{-1}b^2a^{-1}, ababab^{-1}a^2b^{-1} \rangle$

Suppose $\exists f$

(i) $[f] = id \in H^2 \Rightarrow G$ LO but last time we saw this is not LO

(ii) $[f] \neq id \in H^2(G;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

now show this can't work