Lecture 3

M 3-manifold

Thm:

$M \not\cong S^3$ then

$M = M_1 \# \dots \# M_n$ where M_i are prime

$\Rightarrow \pi_1(M) = \pi_1(M_1) * \dots * \pi_1(M_n)$

Prop:

$G * H$ LO iff G and H are LO

Proof: $(\Rightarrow)$ obvious since G, H subgp of $G * H$

$(\Leftarrow)$ there is a map

$$G * H \longrightarrow G \times H$$

induced by $g \longmapsto (g, id_H)$

$h \longmapsto (id_G, h)$

get short exact sequence

$$1 \longrightarrow K \longrightarrow G * H \longrightarrow G \times H \longrightarrow 1$$

K is free (basis is $\{ [g,h] \mid g \in G \setminus \{id\}, h \in H \setminus \{id\} \}$)

so K LO and $G \times H$ LO (lexicographically)

$\Rightarrow G * H$ has a lex. left-order

2nd proof (constructive) (Dicks and Sunic)

define $\tau : G * H \to \mathbb{Z}$ as follows

fix $P_G \subset G$, $P_H \subset H$ positive cone

$$w = a_1 a_2 \dots a_n$$

$$\eta(w) = \begin{cases} 0 & \text{if } a_1, a_n \in G \text{ or } a_1, a_n \in H \\ 1 & \text{if } a_1 \in G, a_n \in H \\ -1 & \text{if } a_1 \in H, a_n \in G \end{cases}$$

$$\tau(w) = \left| \{ i \mid a_i \in P_G \cup P_H \} \right| - \left| \{ i \mid a_i \in P_G^{-1} \cup P_H^{-1} \} \right| + \eta(w)$$

$P = \{ w \in G * H \mid \tau(w) > 0 \}$ is a pos. cone ✓

So $\pi_1(M)$ is LO iff $\pi_1(M_i)$ is LO for all prime factors M_i.

In fact, prime reducible manifold ($S^1 \times S^2$) contribute $\mathbb{Z}$ to free product $\pi_1(M) = \pi_1(M_1) * \dots * \pi_1(M_n)$

so it is enough to LO $\pi_1(M)$ where M is irreducible

Need:

Thm: Suppose M orientable and irreducible and $\tilde{M} \to M$ is a cover space then $\tilde{M}$ is irreducible

(Hatcher 3-mfd notes 3.15)

Thm (Scott) Suppose M is a 3-mfd with finitely generated fundamental group then $\exists$ compact $N \subset \operatorname{int}(M)$ such that

$$i_* : \pi_1(N) \to \pi_1(M)$$

is an isomorphism

Thm (Boyer-Rolfsen-Wiest):

Suppose M irred., $M \not\cong S^3$ then $\pi_1(M)$ is LO iff there is a surjective homo $\pi_1(M) \to L$ onto a nontrivial LO group L

trivial group is not LO from now on, but in 1st 2 lectures it was

Proof: (Burns–Hale)

let $H \subset \pi_1(M)$ be a finite gen subgroup

Two cases

1) H is finite index

   then $\phi : \pi_1(L) \twoheadrightarrow L$ use $\phi|_H$

2) H is infinite index

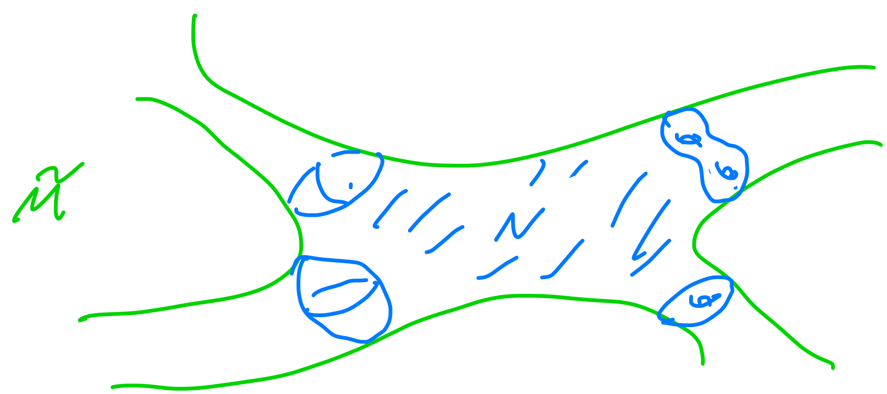
   there is a covering space $p : \tilde{M} \to M$

   s.t. $p_*[\pi_1(\tilde{M})] = H$

   then $\tilde{M}$ is noncompact but has f.g. π_1

   $\Rightarrow$ $\exists$ compact $N \subset \mathrm{int}(M)$ with

   $\pi_1(N) \simeq H$ by Scott



then ∂N is non-empty

may contain S^2's

but M irreducible so S^2 bounds B^3

must be "outside" N

so add to N

get new core

so can assume no S^2's in ∂N

What can be said about $H_1(N;\mathbb{Z})$?

by standard doubling argument shows $|H_1(N;\mathbb{Z})| = \infty$

$$H = \pi_1(N) \longrightarrow H_1(N;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

quotient all but one $\mathbb{Z}$

Cor:

M irreducible and $|H_1(M;\mathbb{Z})| = \infty$

then $\pi_1(M)$ is LO

Proof: $\exists\ \pi_1(M) \to H_1(M) \to \mathbb{Z}$ ✓

So we have arrived at L-space conj

Cor: if M as above, $H \leq \pi_1(M)$ infinite index
then H is LO

Thm: M as above, and $\pi_1(M)$ infinite
then $\pi_1(M)$ is CO iff $\exists\ \pi_1(M) \to C$
where C is infinite CO

Proof: if so

$$1 \to K \to \pi_1(M) \to C \to 1$$

K: LO, C: CO

$\Rightarrow \pi_1(M)$ is CO ✓

Thm: M as above, with infinite π_1, then $\pi_1(M)$ is CO
iff
M admits a finite cyclic cover with LO fund. group

Proof: Suppose $H_1(M;\mathbb{Z})$ is infinite

$\Rightarrow \pi_1(M)$ is LO so $M \xrightarrow{id} M$ is a cover that works

Suppose $H_1(M;\mathbb{Z})$ is finite

then $H^2(\pi_1(M);\mathbb{Z}) \simeq H^2(M;\mathbb{Z})$

$\simeq H_1(M;\mathbb{Z})$ finite

if f is a CO of $\pi_1(M)$,

$[f] \in H^2(\pi_1(M),\mathbb{Z})$ has finite order

$\Rightarrow \exists H < \pi_1(M)$ finite index and LO ✓

Conjecture (CO L-space)

With M as above, M not a lens space then the following are equivalent:

(i) $\exists$ a finite cyclic cover that is not a lens space

(ii) $\exists$ a finite cyclic cover that has a taut foliation

(iii) π_1 is CO

M a Seifert fibered space and assume $H_1(M;\mathbb{Z}) = \{id\}$

in this case, imposing this restriction yields

$$\pi_1(M) = \langle \gamma_1, \dots, \gamma_n, h \mid h \text{ central } \gamma_i^{\alpha_i} = h^{\beta_i}, \gamma_1 \cdots \gamma_n = 1 \rangle$$

where $n \geq 3$, $\alpha_i \geq 2$ and relatively prime

Thm: with M as above, $\pi_1(M)$ is LO as long as it is not $\Sigma(2,3,5)$

triangle group

recall $\Delta(\alpha_1, \alpha_2, \alpha_3) = \langle x, y, z \mid x^{\alpha_1} = y^{\alpha_2} = z^{\alpha_3} = xyz = 1 \rangle$

where $\{\alpha_1, \alpha_2, \alpha_3\} \neq \{2, 3, 5\}$

are subgroups of $PSL(2;\mathbb{R})$ (group is reflections in edges of hyperbolic $\triangle$)

$\therefore$ group is CO

It is also an infinite group

define a homomorphism by

$\gamma_1 \mapsto x$, $\gamma_2 \mapsto y$, $\gamma_3 \mapsto z$

kill all other generators

$\Rightarrow$ $\pi_1(M)$ has infinite CO quotient

$\Rightarrow$ $\pi_1(M)$ is CO

f the CO of $\pi_1(M)$, then $[f] = id \in H^2(\pi_1(M);\mathbb{Z})$

$$H^2(\pi_1(M);\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = \{id\}$$

$\Rightarrow$ $\pi_1(M)$ is LO ✓